New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.149

A way to obtain 2-uninorm on bounded lattice from uninorms defined on subintervals of bounded lattice

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Received: 31 January 2017, Accepted: 28 February 2017
Published online: 3 April 2017.

Abstract: In this paper, a way to obtain 2-uninorm on bounded lattice from U_1 disjunctive uninorm on $[0,k]$ and U_2 conjunctive uninorm on $[k,1]$ is presented. When the conditions disjunctive of U_1 or conjunctive of U_2 drop, it is showed that this method is invalid. Additionally, some properties of this construction method are investigated.

Keywords: Uninorm, 2-uninorm, bounded lattice, disjunctive, conjunctive.

1 Introduction

Uninorms have attracted great interest because of applications of uninorm like fuzzy logic, expert systems, neural networks, fuzzy system modelling [8,13]; after being defined on the unit interval $[0,1]$ by Yager and Rybalov [12]. Since bounded lattice case is more complex, uninorms on bounded lattices has been a challenging problem for many researchers [3,5,6,7,11]. Besides uninorm on $[0,1]$, 2-uninorms are defined and studied [1,2,4].

2-uninorms are special operators since they covers uninorms and nullnorms. Because of this reason some characterization of 2-uninorms on unit reel interval is done [2]. And also, some properties of 2-uninorms on unit reel interval are studied [1,4]. Despite being worked on unit reel interval, there is no work for 2-uninorms on bounded lattice.

In this study, a way to obtain 2-uninorm in $U_{k(e,f)}$ on bounded lattice from disjunctive uninorm $[0,k]$ and conjunctive uninorm on $[k,1]$ is presented. If the conditions of disjunctive of U_1 or conjunctive of U_2 are removed, an example is given to show that the proposition is invalid. Under this construction method, it is showed that k is absorbing element of U^2 and U^2 is neither disjunctive nor conjunctive 2-uninorm on L. Additionally it is obtained that even if U_1 and U_2 are idempotent, U^2 may not be idempotent 2-uninorm on L.

The paper is organized as follows. We shortly recall some basic notions and results in Section 2. In Section 3, we give a method to obtain 2-uninorm $U^2 \in U_{k(e,f)}$ on bounded lattice L using disjunctive uninorm on $[0,k]$ and conjunctive uninorm on $[k,1]$. Some properties of this construction method are also investigated in Section 3.

2 Notations, definitions and a review of previous results

A bounded lattice $(L,\leqslant)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, i.e., there exist two elements $1,0 \in L$ such that $0 \leqslant x \leqslant 1$, for all $x \in L$.

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Definition 1. [3] *Given a bounded lattice* $(L,\leq,0,1)$, *and* $a,b\in L$, *if* a *and* b *are incomparable, in this case we use the notation* $a\|b$.

Definition 2. [3] *Given a bounded lattice* $(L,\leq,0,1)$, *and* $a,b\in L$, $a\leq b$, *a subinterval* $[a,b]$ *of* L *is a sublattice of* L *defined as*

$$[a,b]=\{x\in L \mid a\leq x\leq\ b\}.$$

Similarly, $(a,b]=\{x\in L \mid a<x\leq\ b\}$, $[a,b)=\{x\in L \mid a\leq x<b\}$ *and* $(a,b)=\{x\in L \mid a<x<b\}$.

Definition 3. [11] *Let* $(L,\leq,0,1)$ *be a bounded lattice. An operation* $U:L^2\to L$ *is called a uninorm on* L, *if it is commutative, associative, increasing with respect to the both variables and has a neutral element* $e\in L$.

In this study, the notation $\mathscr{U}(e)$ *will be used for the set of all uninorms on* L *with neutral element* $e\in L$.

If $U(0,1)=0$, U *is called conjunctive uninorm and if* $U(0,1)=1$, U *is called disjunctive uninorm.*

If $U(x,x)=x$ *for all elements* $x\in L$, U *is called idempotent uninorm.*

Consider the set $\mathscr{U}$ of all uninorms on L with the following order. For $U,V\in\mathscr{U}$,

$$U\leqslant V \Longleftrightarrow U(x,y)\leqslant V(x,y) \text{ for all } (x,y)\in L^2.$$

Corollary 1. [11] *Let* $(L,\leq,0,1)$ *be a bounded lattice and* $e\in L\setminus\{0,1\}$. *Then the following uninorms* $U_{T_\wedge}:L^2\to L$ *and* $U_{S_\vee}:L^2\to L$, *respectively, are the greatest and the smallest uninorm on* L *with neutral element* e.

$$U_{T_\wedge}(x,y)=\begin{cases} x\wedge y, & if\ \ (x,y)\in[0,e]^2 \\ x\vee y, & if\ \ (x,y)\in[0,e]\times(e,1]\cup(e,1]\times[0,e] \\ y, & if\ x\in[0,e]\ \ , y\|e \\ x, & if\ y\in[0,e]\ \ , x\|e \\ 1, & otherwise, \end{cases}$$

$$U_{S_\vee}(x,y)=\begin{cases} x\vee y, & if\ \ (x,y)\in[e,1]^2 \\ x\wedge y, & if\ \ (x,y)\in[0,e)\times[e,1]\cup[e,1]\times[0,e) \\ y, & if\ x\in[e,1]\ \ , y\|e \\ x, & if\ y\in[e,1]\ \ , x\|e \\ 0, & otherwise. \end{cases}$$

Definition 4. [5] *An operation* T *(S) on a bounded lattice* L *is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element* 1 *(0). Let* $(L,\leq,0,1)$ *be a bounded lattice,* $U\in\mathscr{U}(e)$ *and* $e\in L$. *It is known that if it is* $e=1$, *uninorm* U *coincides t-norm and if it is* $e=0$, *uninorm* U *coincides t-conorm on* L.

Definition 5. [10] *Let* $(L,\leq,0,1)$ *be a bounded lattice. An operation* $V:L^2\to L$ *is called a nullnorm on* L, *if it is commutative, associative, increasing with respect to the both variables and there is an element* $a\in L$ *such that* $V(x,0)=x$ *for all* $x\leq a$, $V(x,1)=x$ *for all* $x\geq a$. *It can be easily obtained that* $V(x,a)=a$ *for all* $x\in L$. *So, the element* $a\in L$ *that provide* $V(x,a)=a$ *for all* $x\in L$ *is called (absorbing) zero element for operator* V *on* L.

Definition 6. [4] *Let* $(L,\leq,0,1)$ *be a bounded lattice. An operator* $F:L^2\to L$ *is called* 2*-uninorm if it is commutative, associative, increasing with respect to both variables and fulfilling*

$$\forall x\leq k\ F(e,x)=x \text{ and } \forall x\geq k\ F(f,x)=x,$$

where $e,k,f \in L$ with $0 \le e \le k \le f \le 1$. By $U_{k(e,f)}$ we denote the class of all 2-uninorms on bounded lattice L. Conjunctive, disjunctive or idempotent 2-uninorm can be defined as defined for uninorms.

3 A way to obtain 2-uninorm on bounded lattice

In this section, a method has been proposed for generating 2-uninorm $U^2 \in U_{k(e,f)}$ on bounded lattice L using U_1 disjunctive uninorm on $[0,k]$ and U_2 conjunctive uninorm on $[k,1]$. Even if one of conditions U_1 disjunctive uninorm on $[0,k]$ and U_2 conjunctive uninorm on $[k,1]$ is removed, an example is given to show that the proposition may be invalid.

Proposition 1. *Let $(L,\le,0,1)$ be a bounded lattice, $U_1:[0,k]^2 \to [0,k]$ be a disjunctive uninorm with neutral element e and $U_2:[k,1]^2 \to [k,1]$ be a conjunctive uninorm with neutral element f. Then, $U_1(x,k)=k$ for all $x \in [0,k]$ and $U_2(y,k)=k$ for all $y \in [k,1]$.*

Proof. Let $(L,\le,0,1)$ be a bounded lattice. Since $U_1:[0,k]^2 \to [0,k]$ be a disjunctive uninorm with neutral element e, $U_1(0,k)=k$. Then,

$$k = U_1(0,k) \le U_1(x,k) \le U_1(1,k) = k$$

for all $x \in [0,k]$. Then, it is obtained that $U_1(x,k)=k$ for all $x \in [0,k]$. Since $U_2:[k,1]^2 \to [k,1]$ be a conjunctive uninorm with neutral element f, $U_2(k,1)=k$. Then,

$$k = U_2(k,k) \le U_2(y,k) \le U_2(1,k) = k$$

for all $y \in [k,1]$. Then, it is obtained that $U_2(y,k)=k$ for all $y \in [k,1]$.

Theorem 1. *Let $(L,\le,0,1)$ be a bounded lattice, $U_1:[0,k]^2 \to [0,k]$ be a disjunctive uninorm with neutral element e and $U_2:[k,1]^2 \to [k,1]$ be a conjunctive uninorm with neutral element f. Then, the function $U^2:L^2 \to L$ given by*

$$U^2(x,y) = \begin{cases} U_1(x,y), & \text{if } (x,y) \in [0,k]^2 \\ U_2(x,y), & \text{if } (x,y) \in [k,1]^2 \\ k, & \text{otherwise,} \end{cases} \tag{1}$$

is 2-uninorm in $U_{k(e,f)}$.

Proof. (i) Monotonicity: We prove that if $x \leqslant y$ then for all $z \in L$, $U^2(x,z) \leqslant U^2(y,z)$. The proof is split into all possible cases.

Let $x \leqslant k$.

1.1. $y \leqslant k$,

1.1.1. $z \leqslant k$,

$$U^2(x,z) = U_1(x,z) \leqslant U_1(y,z) = U^2(y,z)$$

1.1.2. $z \ge k$ or $z \| k$,

$$U^2(x,z) = k = U^2(y,z)$$

1.2. $y \ge k$,

1.2.1. $z \leqslant k$,

$$U^2(x,z) = U_1(x,z) \leqslant U_1(k,k) = k = U^2(y,z)$$

1.2.2. $z \ge k$,

$$U^2(x,z) = k = U_2(y,z) = U^2(y,z).$$

1.2.3. $z||k$,

$$U^2(x,z) = k = U^2(y,z).$$

1.3. $y||k$,

1.3.1. $z \leqslant k$,

$$U^2(x,z) = U_1(x,z) \leqslant U_1(k,k) = k = U^2(y,z).$$

1.3.2. $z \geq k$ or $z||k$,

$$U^2(x,z) = k = U^2(y,z).$$

2. Let $x \geq k$ Then $y \geq k$.

2.1. $y \geq k$,

2.1.1. $z \leq k$ or $z||k$,

$$U^2(x,z) = k = U^2(y,z).$$

2.1.2. $z \geq k$,

$$U^2(x,z) = U_2(x,z) \leqslant U_2(k,k) = k = U^2(y,z)$$

3. Let $x||k$. Then $y \geq k$ or $y||k$.

3.1. $y \geq k$,

3.1.1. $z \leq k$ or $z||k$,

$$U^2(x,z) = k = U^2(y,z)$$

3.1.2. $z \geq k$,

$$U^2(x,z) = k = U_2(k,k) \leq U_2(y,z) = U^2(y,z)$$

3.2. $y||k$,

3.2.1. $z \in L$,

$$U^2(x,z) = k = U^2(y,z)$$

(ii) Associativity. We demonstrate that $U^2\left(x, U^2(y,z)\right) = U^2\left(U^2(x,y), z\right)$ for all $x,y,z \in L$. Again the proof is split into all possible cases considering the relationships of the elements x,y,z and k.

1. Let $x \leqslant k$.

1.1. $y \leqslant k$,

1.1.1. $z \leqslant k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x, U_1(y,z)) = U_1(x, U_1(y,z)) = U_1(U_1(x,y), z) = U_1\left(U^2(x,y), z\right) = U^2\left(U^2(x,y), z\right)$$

1.1.2. $z \geq k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = U_1(x,k) = k = U^2(U_1(x,y), z) = U^2\left(U^2(x,y), z\right)$$

1.1.3. $z||k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = U_1(x,k) = k = U_1(U_1(x,y), z) = U^2(U_1(x,y), z) = U^2\left(U^2(x,y), z\right)$$

1.2. $y \geq k$,

1.2.1. $z \leqslant k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = U_1(x,k) = k = U_1(k,z) = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

1.2.2. $z \geq k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,U_2(y,z))=k=U_2(k,z)=U^2(k,z)=U^2\left(U^2(x,y),z\right)$$

1.2.3. $z \| k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,k)=U_1(x,k)=k=U^2(k,z)=U^2\left(U^2(x,y),z\right)$$

1.3. $y \| k$,
1.3.1. $z \leqslant k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,k)=U_1(x,k)=k=U_1(k,z)=U^2(k,z)=U^2\left(U^2(x,y),z\right)$$

1.3.2. $z \geq k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,k)=U_1(x,k)=k=U_2(k,z)=U^2(k,z)=U^2\left(U^2(x,y),z\right)$$

1.3.3. $z \| k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,k)=U_1(x,k)=k=U^2(k,z)=U^2\left(U^2(x,y),z\right)$$

2. Let $x \geq k$.
2.1. $y \leqslant k$,
2.1.1. $z \leqslant k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,U_1(y,z))=k=U_1(k,z)=U^2(k,z)=U^2\left(U^2(x,y),z\right)$$

2.1.2. $z \geq k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,k)=U_2(x,k)=k=U_2(k,z)=U^2(k,z)=U^2\left(U^2(x,y),z\right)$$

2.1.3. $z \| k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,k)=U_2(x,k)=k=U^2(k,z)=U^2\left(U^2(x,y),z\right)$$

2.2. $y \geq k$,
2.2.1. $z \leqslant k$ or $z \| k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,k)=U_2(x,k)=k=U^2(U_2(x,y),z)=U^2\left(U^2(x,y),z\right)$$

2.2.2. $z \geq k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,U_2(y,z))=U_2(x,U_2(y,z))=U_2(U_2(x,y),z)=U_2\left(U^2(x,y),z\right)=U^2\left(U^2(x,y),z\right)$$

2.3. $y \| k$,
2.3.1 $z \leqslant k$,

$$U^2\left(x,U^2(y,z)\right)=U^2(x,k)=U_2(x,k)=k=U_1(k,z)=U^2(k,z)=U^2\left(U^2(x,y),z\right)$$

2.3.2. $z \geq k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = U_2(x,k) = k = U_2(k,z) = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

2.3.3 $z \| k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = U_2(x,k) = k = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

3. Let $x \| k$.
3.1. $y \leqslant k$,
3.1.1. $z \leqslant k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x, U_1(y,z)) = k = U_1(k,z) = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

3.1.2. $z \geq k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = k = U_2(k,z) = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

3.1.3. $z \| k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = k = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

3.2. $y \geq k$,
3.2.1. $z \leqslant k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = k = U_1(k,z) = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

3.2.2. $z \geq k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x, U_2(y,z)) = k = U_2(k,z) = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

3.2.3. $z \| k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = k = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

3.3. $y \| k$,
3.3.1. $z \leqslant k$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = k = U_1(k,z) = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

3.3.2. $1 > z > e$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = k = U_2(k,z) = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

3.3.3. $z \| e$,

$$U^2\left(x, U^2(y,z)\right) = U^2(x,k) = k = U^2(k,z) = U^2\left(U^2(x,y), z\right)$$

It is trivial to see the commutativity and the fact that $U^2 \in U_{k(e,f)}$.

Remark. If U_1 is not disjunctive or U_2 is not conjunctive, (3) may not produce a uninorm L. Consider the lattice $(L,\leq,0,1)$ whose lattice diagram is displayed in Fig 1.

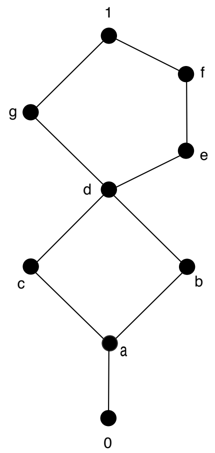

Fig. 1: $(L,\leq)$.

The function U_1 on $[0,d]$ as follows.

U_1	0	a	b	c	d
0	0	0	b	c	d
a	0	a	b	c	d
b	b	b	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

Table 1: The uninorm U_1 on $[0,d]$.

and the function U_2 on $[d,1]$ as follows.

U_2	d	e	f	g	1
d	d	d	d	g	1
e	d	d	e	g	1
f	d	e	f	g	1
g	g	g	g	1	1
1	1	1	1	1	1

Table 2: The uninorm U_2 on $[d,1]$.

It is clear that the function U_1 is an uninorm on $[0,d]$ with neutral element a and U_2 is an uninorm on $[d,1]$ with neutral element f. It can be seen that U_1 is a disjunctive uninorm and U_2 is not a conjunctive uninorm. U^2 is obtained from (1) as follows.

U	0	a	b	c	d	e	f	g	1
0	0	0	b	c	d	d	d	d	d
a	0	a	b	c	d	d	d	d	d
b	b	b	b	d	d	d	d	d	d
c	c	c	d	c	d	d	d	d	d
d	d	d	d	d	d	d	d	g	1
e	d	d	d	d	d	d	e	g	1
f	d	d	d	d	d	e	f	g	1
g	d	d	d	d	g	g	g	1	1
1	d	d	d	d	1	1	1	1	1

Table 3: U^2 on L.

Since U_2 dont satisfies the conditions of Theorem 2, U^2 is not an 2-uninorm on L since

$$U^2(U^2(g,b),c) = U^2(d,c) = d \neq g = U^2(g,d) = U^2(g,U^2(b,c)).$$

Corollary 2. *(1) in theorem 2 produce an 2-uninorm such that neither conjunctive nor disjunctive 2-uninorm in $U_{k(e,f)}$.*

Proposition 2. *Let $(L,\leq,0,1)$ be a bounded lattice, $U^2 \in U_{k(e,f)}$ such that satisfied conditions of theorem 2. Then, k is absorbing element of U^2.*

Proof. Let $(L,\leq,0,1)$ be a bounded lattice, $U^2 \in U_{k(e,f)}$. Then, $U^2(k,k) = k$ for $k \in L$ since $k = U^2(e,k) \leq U^2(k,k) \leq U^2(f,k) = k$. It is satisfied that one of $x \leq k$, $x \geq k$ or $x \| k$ for all $x \in L$. Let $x \leq k$. Then,

$$k = U_1(0,k) \leq U_1(x,k) = U^2(x,k) \leq U^2(k,k) = k.$$

So, $U^2(x,k) = k$ for $x \in L$ such that $x \leq k$. Let $x \geq k$. Then,

$$k = U^2(k,k) \leq U^2(x,k) = U_2(x,k) \leq U_2(1,k) = k.$$

So, $U^2(x,k) = k$ for $x \in L$ such that $x \geq k$. Let $x \| k$. Then, $U^2(x,k) = k$ for $x \in L$ such that $x \| k$ using (1). Then, it is obtained that $U^2(x,k) = k$ for $x \in L$.

Corollary 3. *Let $(L,\leq,0,1)$ be a bounded lattice such that there is at least one element such that incomporable element with k, $U_1 : [0,k]^2 \to [0,k]$ be a disjunctive uninorm with neutral element e and $U_2 : [k,1]^2 \to [k,1]$ be a conjunctive uninorm with neutral element f. Then, the function $U^2 : L^2 \to L$ as mentioned in theorem 2 is not idempotent 2-uninorm on L even if U_1 is idempotent uninorm on $[0,k]$ and U_2 is idempotent uninorm on $[k,1]$.*

Consider the set $U_{k(e,f)}$ of all 2-uninorms on L with the following order:
For $G,H \in U_{k(e,f)}$,

$$G \leqslant H \Longleftrightarrow G(x,y) \leqslant H(x,y) \text{ for all } (x,y) \in L^2.$$

Proposition 3. *Let $(L,\leq,0,1)$ be a bounded lattice. Let take $U_1 = U_{T_\wedge}$ on $[0,k]$ with neutral element e and $U_2 = U_{S_\vee}$ on $[k,1]$ with neutral element f. In these constraints, let call 2-uninorm in (1) as U^*. Then, U^* satisfies neither $U^* \leq F$ nor $U^* \geq F$ for every $F \in U_{k(e,f)}$ such that $U^* \downarrow [0,k] \neq F \downarrow [0,k]$ and $U^* \downarrow [k,1] \neq F \downarrow [k,1]$. So, if $U^* \downarrow [0,k] \neq F \downarrow [0,k]$ and $U^* \downarrow [k,1] \neq F \downarrow [k,1]$, U^* is incomporable with F for $F \in U_{k(e,f)}$.*

Proof. Let assume that $U^* \leq F$. Then, $U^*(x,y) \leq F(x,y)$ for all $(x,y) \in L^2$. So, it has to be satisfied that $U^*(x,y) = U_{T_\wedge}(x,y) \leq F(x,y)$ for all $(x,y) \in [0,k]^2$. It is known that if $F \in U_{k(e,f)}$, $F \downarrow [0,k]$ is an uninorm on $[0,k]$ with neutral element e. This contradict to $U^*(x,y) = U_{T_\wedge}$ is greatest uninorm on $[0,k]$. Let assume that $U^* \geq F$. Then, $U^*(x,y) \leq F(x,y)$ for all

$(x,y) \in L^2$. So, it has to be satisfied that $U^*(x,y) = U_{S_\vee}(x,y) \geq F(x,y)$ for all $(x,y) \in [k,1]^2$. It is known that if $F \in U_{k(e,f)}$, $F \downarrow [k,1]$ is an uninorm on $[k,1]$ with neutral element f. This contradict to $U^*(x,y) = U_{S_\vee}$ is smallest uninorm on $[0,k]$. So, if $U^* \downarrow [0,k] \neq F \downarrow [0,k]$ and $U^* \downarrow [k,1] \neq F \downarrow [k,1]$, U^* is incomporable with F for $F \in U_{k(e,f)}$.

4 Conclusion

2-Uninorms is generalization of both uninorms and nullnorms. Considering this, it is very important to study 2-uninorms on bounded lattices. 2-uninorms have been characterizated on $[0,1]$ unit reel interval as the point of discontinuity [1]. 2-uninorms have not been studied on bounded lattice yet in our best knowledge. In this paper, if there are U_1 disjunctive uninorm $[0,k]$ and U_2 conjunctive uninorm on $[k,1]$, it is showed that there is way to obtain 2-uninorms on bounded lattices such that $U^2 \in U_{k(e,f)}$. Moreover, this construction method gives a way to get 2-uninorms on bounded lattice such that neither conjunctive nor disjunctive. Additionally, it is showed that 2-uninorms obtained by this method does not have to be idempotent even if U_1 and U_2 are.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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