Ordering based 2-uninorm on bounded lattice

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Abstract: In this paper, an order induced by 2-uninorm on bounded lattices is given and some properties of the order are discussed. By defining such an order on bounded lattice, the T-partial order, S-partial order and V-partial order are extended to a more general form.

Keywords: Uninorm, 2-uninorm, bounded lattice, partial order.

1 Introduction

Uninorms on the unit interval [0, 1] by Yager and Rybalov [15]. Because of applications of uninorm like fuzzy logic, expert systems, neural networks, fuzzy system modelling [6, 16], it is attracted interest. The generalization of uninorm to a complete lattice has been an challenging problem for many researchers [2,4,10,14]. The order from logical operation has gained interest [11,12,13] in recent years.

In [11], a partial order defined by means of t-norms bounded lattices has been introduced. This partial order \( \preceq_T \) is called a T-partial order on \( L \).

Another interesting problems is to get an order generated by uninorms on bounded lattices since uninorms are generalization of t-norms and t-conorms. Hliněná et al. has introduced pre-order based on uninorm [8]. After this work, ordering based on uninorms is studied [5]. In the same paper the order obtained by 2-uninorms is introduced without proof and also on chain.

In this paper, we define an order induced by 2-uninorm on bounded lattices. Since uninorms are an combination of t-norms, t-conorms, also order from 2-uninorm contains T-partial order, S-partial order and V-partial order (V is a nullnorm) on bounded lattice with this order, the notion of ordering from 2-uninorm has importance. The paper is organized as follows: We shortly recall some basic notions and results in Section 2. In Section 3, we give an order \( \preceq_{U^2} \) induced by a 2-uninorm \( U^2 \) on bounded lattice \( L \). Some properties of order of 2-uninorm are investigated. Further, this generalization is extended to the n-uninorms on bounded lattice.

2 Notations, definitions and a review of previous results

A bounded lattice \((L, \leq)\) is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, i.e., there exist two elements 1, 0 \( \in L \) such that \( 0 \leq x \leq 1 \), for all \( x \in L \).
Definition 1. [2] Given a bounded lattice \((L, \leq, 0, 1)\), and a, b \in L, if a and b are incomparable, in this case we use the notation \(a \| b\).

Definition 2. [2] Given a bounded lattice \((L, \leq, 0, 1)\), and a, b \in L, a \leq b, a subinterval \([a, b]\) of L is a sublattice of L defined as

\[\{a, b\} = \{x \in L \mid a \leq x \leq b\}.\]

Similarly, \((a, b) = \{x \in L \mid a < x \leq b\}\) and \((a, b) = \{x \in L \mid a < x < b\}\).

Definition 3. [10] Let \((L, \leq, 0, 1)\) be a bounded lattice. An operation \(U : L^2 \to L\) is called a uninorm on L, if it is commutative, associative, increasing with respect to the both variables and has a neutral element \(e \in L\).

In this study, the notation \(\mathcal{U}(e)\) will be used for the set of all uninorms on L with neutral element \(e \in L\). Moreover, if \(U(0, 1) = 0\), \(U\) is called conjunctive uninorm and if \(U(0, 1) = 1\) \(U\) is called disjunctive uninorm.

Definition 4. [11] An operation \(T(S)\) on a bounded lattice \(L\) is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element \(1 (0)\).

Definition 5.[11, 12] A t-norm \(T\) (or a t-conorm \(S\)) on a bounded lattice \(L\) is divisible if the following condition holds. For all \(x, y \in L\) with \(x \leq y\) there is \(z \in L\) such that \(x = T(y, z)\) (or \(y = S(x, z)\)).

Definition 6. [9] Let \((L, \leq, 0, 1)\) be a bounded lattice. A commutative, associative, non-decreasing in each variable function \(V : L^2 \to L\) is called a nullnorm if there is an element \(a \in L\) such that \(V(x, 0) = x\) for all \(x \leq a\), \(V(x, 1) = x\) for all \(x \geq a\). It can be easily obtained that \(V(x, a) = a\) for all \(x \in L\). So \(a \in L\) is the zero element for \(V\).

Definition 7. [11] Let \(L\) be a bounded lattice, \(T\) be a t-norm on \(L\). The order defined by

\[x \preceq y \iff T(\ell, y) = x\text{ for some } \ell \in L\]

is called a \(T\)-partial order (triangular order) for t-norm \(T\).

Similarly, the notion \(S\)-partial order can be defined as follows:

Definition 8. Let \(L\) be a bounded lattice, \(S\) be a t-conorm on \(L\). The order defined by is called a \(S\)-partial order for t-conorm \(S\).

\[x \preceq y \iff S(\ell, x) = y\text{ for some } \ell \in L\]

is called a \(S\)-partial order for t-conorm \(S\).

Note that many properties satisfied for \(T\)-partial order are also satisfied for \(S\)-partial order.

Definition 9. [5] Let \((L, \leq, 0, 1)\) be a bounded lattice and \(U \in \mathcal{U}(e)\). Define the following relation, for \(x, y \in L\), as

\[x \preceq_U y :\iff \begin{cases} \text{if } x, y \in [0, e] \text{ and there exist } k \in [0, e] \text{ such that } U(k, y) = x \text{ or,} \\ \text{if } x, y \in [e, 1] \text{ and there exist } \ell \in [e, 1] \text{ such that } U(x, \ell) = y \text{ or,} \\ \text{if } (x, y) \in L^* \text{ and } x \leq y, \end{cases}\]

where \(L_e = \{x \in L \mid x \parallel e\}\) and \(L^* = [0, e] \times [e, 1] \cup [0, e] \times I_e \cup [e, 1] \times [0, e] \cup [e, 1] \times L_e \cup L_e \times [0, e] \cup I_e \times [e, 1] \cup I_e \times I_e\). Here, note that the notation \(x \parallel y\) denotes that \(x\) and \(y\) are incomparable.

Proposition 1. [5] The relation \(\preceq_U\) defined in (1) is a partial order on \(L\).
3 Ordering based 2-uninorm on bounded lattice

A 2-uninorm (introduced by Akella [1]) is an operation which is increasing, associative and commutative on the unit interval with an absorbing element separating two subintervals having their own neutral elements. Since 2-uninorms is generalization of both nullnorms and uninorms, the operator is important. In this section, the order obtained from 2-uninorms on bounded lattice is defined and the proof is given. By this way, we give more general form \( \preceq_U \) of the order \( \preceq_U \) given in (1) on bounded lattice.

**Definition 10.** [3] Let \((L, \leq, 0, 1)\) be a bounded lattice. An operator \(F : L^2 \rightarrow L\) is called 2-uninorm if it is commutative, associative, increasing with respect to both variables and fulfilling

\[
\forall x \leq k F(e, x) = x \quad \text{and} \quad \forall x \geq k F(f, x) = x,
\]

where \(e, k, f \in L\) with \(0 \leq e \leq k \leq f \leq 1\). By \(U_{k(e,f)}\) we denote the class of all 2-uninorms on bounded lattice \(L\).

**Definition 11.** Let \(U^2 \in U_{k(e,f)}\). Define the following relation: For every \(x, y \in L\),

\[
x \preceq_{U^2} y \iff \begin{cases} 
\exists \ell \leq e \text{ such that } U^2(\ell, y) = x, \text{ when } x, y \in [0, e], \text{ or,} \\
\exists m \in [e, k] \text{ such that } U^2(x, m) = y, \text{ when } x, y \in [e, k] \text{ or,} \\
\exists n \in [k, f] \text{ such that } U^2(y, n) = x, \text{ when } x, y \in [k, f] \text{ or,} \\
\exists p \in [f, 1] \text{ such that } U^2(x, p) = y, \text{ when } x, y \in [f, 1] \text{ or,} \\
x \leq y, \text{ otherwise.}
\end{cases}
\]

(2)

**Proposition 2.** The relation \(\preceq_{U^2}\) defined in (2) is a partial order on bounded lattice \(L\).

**Proof.** (1) If \(x \in [0, e]\) or \(x \in [e, k]\), \(x \preceq_{U^2} x\) since \(U^2(x, e) = x\). If \(x \in [k, f]\) or \(x \in [f, 1]\), \(x \preceq_{U^2} x\) since \(U^2(x, f) = x\). Otherwise, since \(x \leq x\), we have that \(x \preceq_{U^2} x\). So, the relation \(\preceq_{U^2}\) satisfies the reflexivity.

(2) Let \(x \preceq_{U^2} y\) and \(y \preceq_{U^2} x\) for elements \(x, y \in L\). Let \(x, y \in [0, e]\) \((x, y \in [k, f])\). Since \(x \preceq_{U^2} y\) and \(y \preceq_{U^2} x\), there exist elements \(\ell_1, \ell_2 \in [0, e]\) \((n_1, n_2 \in [k, f])\) such that

\[U^2(\ell_1, y) = x \text{ and } U^2(\ell_2, x) = y \quad (U^2(n_1, y) = x \text{ and } U^2(n_2, x) = y)\]

By using the monotonicity of \(U^2\), we have that

\[x = U^2(\ell_1, y) \leq U^2(e, y) = y \quad (x = U^2(n_1, y) \leq U^2(f, y) = y)\]

and

\[y = U^2(\ell_2, x) \leq U^2(e, x) = x \quad (y = U^2(n_2, x) \leq U^2(f, x) = x)\]

Thus, \(x = y\). Let \(x, y \in [e, k]\) \((x, y \in [f, 1])\). Since \(x \preceq_{U^2} y\) and \(y \preceq_{U^2} x\), there exist elements \(m_1, m_2 \in [e, k]\) \((p_1, p_2 \in [f, 1])\) such that

\[U^2(m_1, x) = y \text{ and } U^2(m_2, y) = x \quad (U^2(p_1, x) = y \text{ and } U^2(p_2, y) = x)\]

By using the monotonicity of \(U^2\), we have that

\[x = U^2(m_2, y) \geq U^2(e, y) = y \quad (x = U^2(p_2, y) \geq U^2(f, y) = y)\]

and

\[y = U^2(m_1, x) \geq U^2(e, x) = x \quad (y = U^2(p_1, x) \geq U^2(f, x) = x)\]

Thus, \(x = y\). Otherwise, since \(x \preceq_{U^2} y\) and \(y \preceq_{U^2} x\), it is obtained that \(x \leq y\) and \(y \leq x\), whence \(x = y\). So, the antisymmetry property holds.
Let $x \preceq_{U^2} y$ and $y \preceq_{U^2} z$ for elements $x, y, z \in L$.

Possible cases are as follows. 3.1. $x \in [0, e]$

3.1.1. $y \in [0, e]$

3.1.1.1. $z \in [0, e]$

Since $x \preceq_{U^2} y$ and $y \preceq_{U^2} z$, there exist $\ell_1, \ell_2$ of $[0, e]$ such that $U^2(y, \ell_1) = x$ and $U^2(\ell_2, z) = y$. Then,

$$x = U^2(\ell_1, y) = U^2(\ell_1, U^2(\ell_2, z)) = U^2(U^2(\ell_1, \ell_2), z).$$

Since $U^2(\ell_1, \ell_2) \preceq e$, it is obtained that $x \preceq_{U^2} z$.

3.1.2. $z \in [0, e]$.

Since $y \preceq_{U^2} z$, it is clear that $y \preceq z$. Also, since $x \preceq_{U^2} y$, there exists an element $\ell \preceq e$ such that $U^2(\ell, y) = x$. It follows $x \preceq_{U^2} z$ from $x = U^2(\ell, y) \preceq U^2(\ell, e) = y \preceq z$, it is obtained that $x \preceq_{U^2} z$.

3.1.2. $y \in [0, e]$.

Since $y \subseteq [0, e]$, it must be that $z \subseteq [0, e]$. The other hand, we have that $x \preceq y$.

3.1.2.1. $y$ and $z$ be in one of interval $[e, k], [k, f]$ or $[f, 1]$ at the same time.

Let $y, z \in [e, k]$. Since $y \preceq_{U^2} z$, there exist $m \in [e, k]$ such that $U^2(y, m) = z$. It must be that $x \preceq_{U^2} z$ from $x \preceq y \preceq U^2(y, e) \preceq U^2(y, m) = z$.

Same proof can be done for other cases.

3.1.2.2. $y$ and $z$ don’t be in one of interval $[e, k], [k, f]$ or $[f, 1]$ at the same time.

In this case, we have that $y \preceq z$ from $y \preceq_{U^2} z$. Since $x \preceq y$ and $y \preceq z$, it is obtained that $x \preceq z$. Thus $x \preceq_{U^2} z$.

3.2. $x \in [e, k]$.

Let $y \in [0, e]$. Since $x \preceq_{U^2} y$, $e \preceq y \preceq e \preceq e$. It is obvious.

3.2.1. $y \in [e, k]$.

Let $z \in [0, e]$. Since $y \preceq_{U^2} z$, $e \preceq y \preceq z \preceq e$. It is obvious.

3.2.1.1. $z \in [e, k]$.

Since $x \preceq_{U^2} y$ and $y \preceq_{U^2} z$, there exist $m_1, m_2 \in [e, k]$ such that $U^2(x, m_1) = y$ and $U^2(y, m_2) = z$. Then, it must be that

$$z = U^2(y, m_2) = U^2(U^2(x, m_1), m_2) = U^2(x, U^2(m_1, m_2)).$$

Since $U^2(m_1, m_2) \in [e, k]$, $x \preceq_{U^2} z$.

3.2.1.2. $z \in [e, k]$.

Since $y \preceq_{U^2} z$, $y \preceq z$. Also, there exist $m \in [e, k]$ such that $U^2(x, m) = y$. Since $x = U^2(x, e) \preceq U^2(x, m) = y \preceq z$, $x \preceq_{U^2} z$.

3.2.1.1. $y \notin [e, k]$.

Since $y \notin [e, k]$, it must be that $z \notin [e, k]$. The other hand, we have that $x \preceq y$.

3.2.2.1. $y$ and $z$ be in one of interval $[k, f]$ or $[f, 1]$ at the same time.

Let $y, z \in [k, f]$. There exist $n \in [k, f]$ such that $U^2(z, n) = y$ from $y \preceq_{U^2} z$. Since $x \preceq y = U^2(z, n) \preceq U^2(z, f) = z$, $x \preceq_{U^2} z$.

3.2.2. $y$ and $z$ don’t be in one of interval $[k, f]$ or $[f, 1]$ at the same time.

In this case, we have that $y \preceq z$. Since $x \preceq y$ and $y \preceq z$, $x \preceq z$. Thus, $x \preceq_{U^2} z$.

3.3. $x \in [k, f]$ or $x \in [f, 1]$.

Similar proof can be done for $x \in [k, f]$ and $x \in [f, 1]$ as done in $[0, e]$ and $[e, f]$ respectively.

3.4. $x \in [0, e] \cup [e, k] \cup [k, f] \cup [f, 1]$.

In this case, we have that $x \preceq y$ for all $y \in L$.

3.4.1. $y$ and $z$ be in one of interval $[0, e], [e, k], [k, f]$ or $[f, 1]$ at the same time.

Let $y, z \in [0, e]$. Since $y \preceq_{U^2} z$, there exist $\ell \in [0, e]$ such that $U^2(z, \ell) = y$. It is obtained that $x \preceq_{U^2} z$ since $x \preceq y = U^2(z, \ell) \preceq U^2(z, e) = z$.

Similar proof can be done for other cases as $[0, e]$.
3.4.2. \( y \) and \( z \) don’t be in one of interval \([0, e]\), \([e, k]\), \([k, f]\) or \([f, 1]\) at the same time or either \( y \) or \( z \) don’t be in \([0, e] \cup [e, k] \cup [k, f] \cup [f, 1]\) or neither \( y \) nor \( z \) don’t be in \([0, e] \cup [e, k] \cup [k, f] \cup [f, 1]\).

In this case, \( y \succeq U^2 z \) implies that \( y \leq z \). Since \( x \leq y \) and \( y \leq z \), it is obtained that \( x \leq z \). Thus, \( x \succeq U^2 z \). So the transitivity holds.

**Proposition 3.** Let \((L, \leq, 0, 1)\) be a bounded lattice and \(U^2 \in U_{k(e,f)}\). If \( x \preceq U^2 y \) for any \( x, y \in L \), then \( x \leq y \).

**Proof.** Let \( x \preceq U^2 y \) for \( x, y \in L \). If \( x, y \in [0, e]\{x, y \in [k, f]\}, \) then there exists an element \( \ell \leq e(n \in [k, f]) \) such that

\[
U^2(\ell, y) = x(U^2(n, y) = x).
\]

Since \( x = U^2(\ell, y) \leq U^2(e, y) = y(x = U^2(n, y) \leq U^2(f, y) = y) \), we have that \( x \leq y \). Let \( x, y \in [e,k]\{x, y \in [f, 1]\}. \) Then, there exists an element \( m \in [e,k]\{n \in [f, 1]\} \) such that

\[
U^2(m, x) = y(U^2(n, x) = y).
\]

Since \( x = U^2(e, x) \leq U(m, x) = y(x = U(f, x) \leq U(n, x) = y) \), we have that \( x \leq y \). Otherwise, since \( x \preceq U^2 y \), it is clear that \( x \leq y \).

**Remark.** The converse of Proposition 3 may not be satisfied. For example. Consider the lattice \((L = \{0, a, b, c, d, e, 1\}, \leq, 0, 1)\) such that \( 0 < a < b < c < d < e < 1 \) and define the function \(U^2 \in U_{b(a,c)}\) as follows.

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**Table 1:** The 2-uninorm \(U^2\) on \(L\).

It is clear that the function \(U^2\) is an 2-uninorm on \(L\). Although \( d \leq e, d \not\preceq U^2 e \) since there doesn’t exist an element \( m \in [f, 1]\) such that \( e = U^2(m, d) \). The order \( \preceq U^2 \) on \(L\) has its diagram as follows (see Figure 1).

![Fig. 1: \((L, \preceq U^2)\).](image)
Remark. Even if \((L, \leq, 0, 1)\) is a chain, the partially ordered set \((L, \leq_{U_2})\) may not be a chain. To show that consider the above mentioned lattice and 2-uninorm on it. It is easily seen that \((L, \leq_{U_2})\) even if \((L, \leq, 0, 1)\) is a chain.

**Proposition 4.** Let \((L, \leq, 0, 1)\) be a bounded lattice and \(U^2 \in U_{k(e,f)}\). Then, \((L, \leq_{U_2})\) is a bounded partially ordered set.

**Proof.** It is clear that \((L, \leq_{e})\) is a partially ordered set by Proposition 2. Let \(x \in [0, e]\). Since \(U^2(0,x) \leq U(0,e) = 0\), we have that \(U(0,x) = 0\). Thus, \(0 \leq_{U_2} x\). Let \(x \not\in [0, e]\). Then, it follows \(0 \leq_{U_2} x\) from \(0 \leq x\). So, for any \(x \in L\), \(0 \leq_{U_2} x\). That 1 is the greatest element with respect to \(\leq_{U_2}\) is shown in a similar way.

**Remark.** (i) Note that if \(U^2 \in U_{k(e,f)}\) is an 2-uninorm on bounded lattice \(L, U^2 \downarrow [0,k]^2 := U_1\) is a disjunctive uninorm on \([0,k]\) with identity element \(e\) and zero element \(k\) and \(U^2 \downarrow [k,1]^2 := U_2\) is a conjunctive uninorm on \([k,1]\) with identity element \(f\) and zero element \(k\).

(ii) Let \(U^2 \in U_{k(e,f)}\) is a 2-uninorm on bounded lattice \(L\).

(a) \(T_1^* = U^2 \downarrow [0,e]^2 : [0,e]^2 \rightarrow [0,e]\) and \(T_2^* = U^2 \downarrow [k,f]^2 : [k,f]^2 \rightarrow [k,f]\) are t-norms.

(b) \(S_1^* = U^2 \downarrow [e,k]^2 : [e,k]^2 \rightarrow [e,k]\) and \(S_2^* = U^2 \downarrow [f,1]^2 : [f,1]^2 \rightarrow [f,1]\) are t-conorms.

**Proposition 5.** \((L, \leq, 0, 1)\) be a bounded lattice and \(U^2 \in U_{k(e,f)}\) is a 2-uninorm on bounded lattice \(L\). Then, \(T_1^*, T_2^*, S_1^*\) and \(S_2^*\) are divisible if and only if \(\leq_{U_2} = \leq\).

**Proof.** (i) If \(a \leq_{U_2} b\) for any \(a, b \in L\), then \(a \leq b\). Conversely, let \(a \leq b\). Suppose that \(a, b \in [0,e]\). \(\leq_{T_1^*} = \leq\) since \(T_1^*\) divisible. Thus \(a \leq_{T_1^*} b\), whence there exist an element \(\ell \in [0,e]\) such that \(T_1^*(a,\ell) = a\). Since \(U^2(b,\ell) = U^2 \downarrow [0,e]\), \(T_1^*(b,\ell) = a\). Suppose that \(a, b \in [e,k]\), \(\leq_{S_1^*} = \leq\) since \(S_1^*\) divisible. Thus \(a \leq_{S_1^*} b\), whence there exist an element \(m \in [e,k]\) such that \(S_1^*(a,m) = b\). Since \(U^2(a,m) = U^2 \downarrow [e,k]\), \(S_1^*(a,m) = S_1^*(a,m) = b\). Suppose that \(a, b \in [k,f]\). \(\leq_{T_2^*} = \leq\) since \(T_2^*\) divisible. Thus \(a \leq_{T_2^*} b\), whence there exist an element \(n \in [k,f]\) such that \(T_2^*(b,n) = a\). Since \(U^2(b,n) = U^2 \downarrow [k,f]\), \(T_2^*(b,n) = T_2^*(b,n) = a\). Suppose that \(a, b \in [f,1]\). \(\leq_{S_2^*} = \leq\) since \(S_2^*\) divisible. Thus \(a \leq_{S_2^*} b\), whence there exist an element \(p \in [f,1]\) such that \(S_2^*(a,p) = b\). Since \(U^2(a,p) = U^2 \downarrow [f,1]\), \(S_2^*(a,p) = S_2^*(a,p) = b\). Otherwise \(a \leq b\) implies that \(a \leq_{U_2} b\).

(ii) Let \(\leq_{U_2} = \leq\). Suppose that \(a \leq b\) for \(a, b \in [0,e]\). Then, \(a \leq_{U_2} b\). Since \(a, b \in [0,e]\), there exist \(\ell \in [0,e]\) such that \(U^2(b,\ell) = a\). Since \(U^2(b,\ell) = U^2 \downarrow [0,e]\), \(T_1^*(b,\ell) = a\). This implies \(T_1^*\) divisible. Suppose that \(a \leq b\) for \(a, b \in [e,k]\). Then, \(a \leq_{U_2} b\). Since \(a, b \in [e,k]\), there exist \(m \in [e,k]\) such that \(U^2(a,m) = b\). Since \(U^2(a,m) = U^2 \downarrow [e,k]\), \(S_1^*(a,m) = S_1^*(a,m) = b\). Suppose that \(a \leq b\) for \(a, b \in [k,f]\). Then, \(a \leq_{U_2} b\). Since \(a, b \in [k,f]\), there exist \(n \in [k,f]\) such that \(U^2(b,n) = a\). Since \(U^2(b,n) = U^2 \downarrow [k,f]\), \(T_2^*(b,n) = T_2^*(b,n) = a\). This implies \(T_2^*\) divisible. Suppose that \(a \leq b\) for \(a, b \in [f,1]\). Then, \(a \leq_{U_2} b\). Since \(a, b \in [f,1]\), there exist \(p \in [f,1]\) such that \(U^2(a,p) = b\). Since \(U^2(a,p) = U^2 \downarrow [f,1]\), \(S_2^*(a,p) = S_2^*(a,p) = b\). This implies \(S_2^*\) divisible.

**Definition 12.** \(L, \leq, 0, 1\) be a bounded lattice and \(V\) a binary operator on \(L\) which is commutative. Then, \(\{e_1, e_2, \ldots, e_n\}_{z_1 \leq z_2 \leq \ldots \leq z_n} \) is called an \(n\)-neutral element of \(V\) if \(V(e_i, x) = x\) for all \(x \in [z_{i-1}, z_i]\) for \(0 = z_0 < z_1 < \ldots < z_n = 1\) and \(e_i \in [z_{i-1}, z_i], i = 1, 2, \ldots, n\).

**Definition 13.** A binary operator \(U^n\) on \(L\), is a \(n\)- uninorm if it is associative, monotone, non-decreasing in each variable and commutative and has a neutral element \(\{e_1, e_2, \ldots, e_n\}_{z_1 \leq z_2 \leq \ldots \leq z_n-1} \).

Similarly, the order given in (2) can be generalized for \(n\)-uninorms as follows.

**Proposition 6.** Let \(U^n\) be an \(n\)-uninorm on a bounded lattice \(L\) with an \(n\)-neutral element \(\{e_1, e_2, \ldots, e_n\}_{z_1 \leq z_2 \leq \ldots \leq z_n-1}, i = 1, 2, \ldots, n\). Then, the relation given in (3)

\[
x \leq^{U^n} y : \Leftrightarrow \begin{cases} 
3 \ell \in [z_{i-1}, e_i] \text{ such that } U(\ell, y) = x, & \text{when } x, y \in [z_{i-1}, e_i] \\
3 m \in [e_i, z_i] \text{ such that } U(m, x) = y, & \text{when } x, y \in [e_i, z_i] \\
x \leq y, & \text{otherwise},
\end{cases}
\]
is a partial order on bounded lattice \( L \).

**Proof.** The proof can be done as done in Proposition 2.

### 4 Conclusion

A partial order on a bounded lattice \( L \) from a 2-uninorm on \( L \) is given and discussed. So, we have extended the T-partial (S-partial) and V-order to a more general form. According to the underlying t-norm and t-conorm of a uninorm, we have characterized the order induced by the uninorm. We have studied some properties of the order induced by a uninorm. Moreover, we have generalized n-uninorms.

### Competing interests

The authors declare that they have no competing interests.

### Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

### References