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Some properties of orders generated by uninorm and 2-uninorm

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Abstract: In this paper, the order definition obtained from uninorm has been reorganized and some features have been examined in this way. Order-weakest uninorm and order-strongest uninorm was determined. Using the notions of order-weakest uninorm and order-strongest uninorm, order-weakest 2-uninorm and order-strongest 2-uninorm was also determined. A way to obtain partially ordered relation via orders obtained from uninorms on subinterval of bounded lattice is given. The relation between the order obtained 2-uninorm and this new construction method is investigated.

Keywords: Uninorm, 2-uninorm, bounded lattice, partial order.

1 Introduction

Uninorms can be seen as a more general class of t-norms and t-conorms. Since t-norms and t-conorms have been studied extensively, uninorms have been also studied extensively since they defined by Yager and Rybalov [17]. In addition to this, it can be said that they have extra interest because they have a lot of application areas [9,18]. Althought they were first described on unit reel interval, they were also defined and studied by researchers on bounded lattice [4,6,12,16]. How important is it that uninorms are a generalization of t-norms and t-conorms, 2- uninorms are also important for researchers to define and study on them [1,2,5].

Partially ordered relation obtained from logical operators has been investigated by researcher deeply [13,14,15]. In [13], a partial order defined by means of t-norms bounded lattices has been introduced. This partial order $\preceq_T$ is called a T-partial order on L. In addition, there have been some initiatives to define the order obtained by uninorms [11]. But, it was first defined in [7]. Again in [7], the order obtained by 2-uninorms is introduced on chain but without proof. Finally, the order obtained from 2-uninorms on bounded lattice is given with proof, the some properties of the order are examined [8].

In this study, the order definitions $\preceq_U$ and $\preceq_{U^2}$ have been reorganized. By this way, order-weakest uninorms and order-strongest uninorms were determined. In addition, it is showed that order-weakest uninorms and order-strongest uninorms may not be the only one. This new form of $\preceq_{U^2}$ also made it possible to obtain a new order definition from two orders obtained from two uninorms defined on subintervals of bounded lattice. The paper is organized as follows: I shortly recall some basic notions and results in Section 2. In Section 3, firstly, the order notion of $\preceq_U$ was reconsidered. In this way, order-weakest and order-strongest uninorm were studied. The example was given to show they dont need to be one. In same section, it was studied on $\preceq_{U^2}$ similarly. Using this new definition, a method was given to obtain partially ordered relation from two partially ordered relations on subintervals of bounded lattice.

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2 Notations, definitions and a review of previous results

Definition 1. [12] *Let* $(L, \leq, 0, 1)$ *be a bounded lattice. An operation* $U : L^2 \to L$ *is called a uninorm on* L*, if it is commutative, associative, increasing with respect to the both variables and has a neutral element* $e \in L$*.*

In this study, the notation $\mathscr{U}(e)$ *will be used for the set of all uninorms on* L *with neutral element* $e \in L$*.*

Definition 2. [13] *An operation* T *(*S*) on a bounded lattice* L *is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element* 1 *(*0*.)*

Example 1. Let $(L, \leq, 0, 1)$ be a bounded lattice. Smallest t-norm T_W and greatest t-norm $T_\wedge$ on bounded lattice L are given respectively as follows.

$$T_W(x,y) = \begin{cases} y \text{ , if } x = 1 \\ x \text{ , if } y = 1 \\ 0 \text{ , otherwise} \end{cases}$$

$$T_\wedge(x,y) = x \wedge y.$$

Smallest t-conorm $S\vee$ and greatest t-norm S_W on bounded lattice L are given respectively as follows.

$$S_\vee(x,y) = x \vee y$$

$$S_W(x,y) = \begin{cases} y \text{ , if } x = 0 \\ x \text{ , if } y = 0 \\ 1 \text{ , otherwise.} \end{cases}$$

Definition 3. [13,14] *A t-norm* T *(or a t-conorm* S*) on a bounded lattice* L *is divisible if the following condition holds. For all* $x, y \in L$ *with* $x \leq y$ *there is* $z \in L$ *such that* $x = T(y,z)$ *(or* $y = S(x,z)$*).*

Definition 4. [13] *Let* L *be a bounded lattice,* T *a t-norm on* L*. The order defined by*

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L$$

is called a $T-$ *partial order (triangular order) for t-norm* T*.*

Similarly, the notion $S-$ *partial order* can be defined as follows.

Definition 5. *Let* L *be a bounded lattice,* S *be a t-conorm on* L*. The order defined by is called a* $S-$ *partial order for t-conorm* S*.*

$$x \preceq_S y :\Leftrightarrow S(\ell, x) = y \text{ for some } \ell \in L$$

is called a $S-$ *partial order for t-conorm* S*.*

Note that many properties satisfied for $T-$ *partial order* are also satisfied for $S-$ *partial order*.

Definition 6. [7] *Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* $U \in \mathscr{U}(e)$*. Define the following relation,for* $x, y \in L$*, as*

$$x \preceq_U y :\Leftrightarrow \begin{cases} \text{if } \ x,y \in [0,e] \ \text{ and there exist } \ k \in [0,e] \ \text{ such that } \ U(k,y) = x \ \text{ or,} \\ \text{if } \ x,y \in [e,1] \ \text{ and there exist } \ \ell \in [e,1] \ \text{ such that } \ U(x,\ell) = y \ \text{ or,} \\ \text{if } \ (x,y) \in L^* \ \text{ and } \ x \leq y, \end{cases} \tag{1}$$

where $I_e = \{x \in L \mid x \| e\}$ *and* $L^* = [0,e] \times [e,1] \cup [0,e] \times I_e \cup [e,1] \times [0,e] \cup [e,1] \times I_e \cup I_e \times [0,e] \cup I_e \times [e,1] \cup I_e \times I_e$*.*

Here, note that the notation $x||y$ *denotes that* x *and* y *are incomparable.*

Proposition 1. [7] *The relation* $\preceq_U$ *defined in (1) is a partial order on* L.

Definition 7. [5] *Let* $(L,\leq,0,1)$ *be a bounded lattice. An operator* $F : L^2 \to L$ *is called* 2*-uninorm if it is commutative, associative, increasing with respect to both variables and fulfilling*

$$\forall x \leq k\; F(e,x) = x \text{ and } \forall x \geq k\; F(f,x) = x,$$

where $e,k,f \in L$ *with* $0 \leq e \leq k \leq f \leq 1$.

By $U_{k(e,f)}$ *we denote the class of all* 2*-uninorms on bounded lattice* L.

Definition 8. [8] *Let* $U^2 \in U_{k(e,f)}$. *Define the following relation. For every* $x,y \in L$,

$$x \preceq_{U^2} y :\Leftrightarrow \begin{cases} \exists \ell \leq e \quad \text{such that} \quad U^2(\ell,y) = x, & \text{when} \quad x,y \in [0,e] \quad \text{or}, \\ \exists m \in [e,k] \quad \text{such that} \quad U^2(x,m) = y, & \text{when} \quad x,y \in [e,k] \quad \text{or}, \\ \exists n \in [k,f] \quad \text{such that} \quad U^2(y,n) = x, & \text{when} \quad x,y \in [k,f] \quad \text{or}, \\ \exists p \in [f,1] \quad \text{such that} \quad U^2(x,p) = y, & \text{when} \quad x,y \in [f,1] \quad \text{or}, \\ x \leq y, & \text{otherwise}. \end{cases} \tag{2}$$

Proposition 2. [8] *The relation* $\preceq_{U^2}$ *defined in (2) is a partial order on bounded lattice* L.

3 Order-weakest and order strongest uninorms and 2-uninorms

In this section, the partially ordered relations obtained from uninorm and 2-uninorm on bounded lattice L has been reorganized. By this way, order-weakest and order-strongest uninorms are determined. In addition, considering the relation between uninorms and 2-uninorms, order-weakest 2-uninorm and order-strongest 2-uninorms are also determined. Also, it is showed that order weakest uninorms or 2-uninorms and order-strongest uninorms or 2-uninorms dont need to be one. Further, the partially ordered relation obtained two partially ordered relations obtained from two uninorms on subintervals of bounded lattice is given.

Proposition 3. [12] *Let* $(L,\leq,0,1)$ *be a bounded lattice, and* U *a uninorm with a neutral element* $e \in L$. *Then*

(i) $T^* = U \downarrow [0,e]^2 : [0,e]^2 \to [0,e]$ *is a t-norm on* $[0,e]$.
(ii) $S^* = U \downarrow [e,1]^2 : [e,1]^2 \to [e,1]$ *is a t-conorm on* $[e,1]$.

Considering Proposition 3, the definiton of $\preceq_U$ can be reorganized as follow.

Definition 9. *Let* $(L,\leq,0,1)$ *be a bounded lattice and* $U \in \mathscr{U}(e)$ *such that* $U \downarrow [0,e]^2 = T^*$ *and* $U \downarrow [e,1]^2 = S^*$. *(1) can be represented as following,for* $x,y \in L$, *as*

$$x \preceq_U y :\Leftrightarrow \begin{cases} \text{if} \quad x,y \in [0,e] \quad \text{and} \quad x \preceq_{T^*} y \quad \text{or}, \\ \text{if} \quad x,y \in [e,1] \quad \text{and} \quad x \preceq_{S^*} y \quad \text{or}, \\ \text{if} \quad (x,y) \in L^* \quad \text{and} \quad x \leq y, \end{cases} \tag{3}$$

where $L^* = [0,e] \times [e,1] \cup [0,e] \times I_e \cup [e,1] \times [0,e] \cup [e,1] \times I_e \cup I_e \times [0,e] \cup I_e \times [e,1] \cup I_e \times I_e$.

Remark. [3] Let T be a t-norm ans S be a t-conorm on bounded lattice L and consider the t-norms T_W and $T_\wedge$ and t-conorms S_W and $S_\vee$.

T_W is the order-weakest and $T_\wedge$ is order-strongest t-norm on L,i.e.,

$$\preceq_{T_W} \subseteq \preceq_T \subseteq \preceq_{T_\wedge}.$$

Similarly, $S_{\vee}$ is the order-weakest and S_W is order-strongest t-conorm on L,i.e.,

$$\preceq_{S_W} \subseteq \preceq_S \subseteq \preceq_{S_\vee} .$$

Proposition 4. *Let* $(L,\leq,0,1)$ *be a bounded lattice and* $U_W \in \mathscr{U}(e)$ *such that* $U_W \downarrow [0,e]^2 = T_W$ *and* $U_W \downarrow [e,1]^2 = S_W$. *Then* $\preceq_{U_W} \subseteq \preceq_U$ *for all* $U \in \mathscr{U}(e)$.

Proof. $U \in \mathscr{U}(e)$ be an arbitrary uninorm such that $U \downarrow [0,e]^2 = T^*$ and $U \downarrow [e,1]^2 = S^*$. Let $(x,y) \in \preceq_{U_W}$.

(i) $x,y \in [0,e]$. Then, it is obtained that $(x,y) \in \preceq_{T_W}$. Since $\preceq_{T_W} \subseteq \preceq_T$ for any t-norm on $[0,e]$, $\preceq_{T_W} \subseteq \preceq_{T^*}$. Therefore, $(x,y) \in \preceq_U$.

(ii) $x,y \in [e,1]$. Then, it is obtained that $(x,y) \in \preceq_{S_W}$. Since $\preceq_{S_W} \subseteq \preceq_S$ for any t-conorm on $[e,1]$, $\preceq_{S_W} \subseteq \preceq_{S^*}$. Therefore, $(x,y) \in \preceq_U$.

(iii) For other cases, $(x,y) \in \preceq_{U_W}$ implies that $(x,y) \in \leq$. Therefore, $(x,y) \in \preceq_U$.

Thus, it is obtained that $\preceq_{U_W} \subseteq \preceq_U$.

Proposition 5. *Let* $(L,\leq,0,1)$ *be a bounded lattice and* $U_{\wedge\vee} \in \mathscr{U}(e)$ *such that* $U_{\wedge\vee} \downarrow [0,e]^2 = T_\wedge$ *and* $U_{\wedge\vee} \downarrow [e,1]^2 = S_\vee$. *Then* $\preceq_U \subseteq \preceq_{U_{\wedge\vee}}$ *for all* $U \in \mathscr{U}(e)$.

Proof. The proof can be done similar proof of Proposition 4.

Corollary 1. *Let* $(L,\leq,0,1)$ *be a bounded lattice and* $U \in \mathscr{U}(e)$ *be an arbitrary uninorm on L. Then,* U_W *is the order-weakest and* $U_{\wedge\vee}$ *is order-strongest uninorms on L,i.e.,*

$$\preceq_{U_W} \subseteq \preceq_U \subseteq \preceq_{U_{\wedge\vee}}.$$

Remark. U_W is the order-weakest and $U_{\wedge\vee}$ is order-strongest uninorm on L mentioned in Corollary 1 are not the necessarily the ones. Let show that following example:

Example 2. Consider the lattice $(L = \{0,a,b,c,d,e,1\},\leq,0,1)$ whose lattice diagram is displayed in Figure 1.

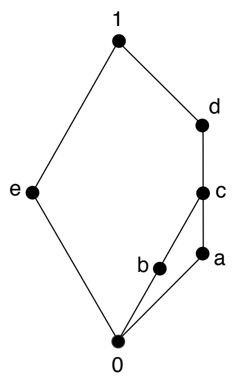

Fig. 1: $(L,\leq)$

Let define the following U_1 and U_2 uninorms with neutral element c given in Table 1 and Table 2 respectively:

U_1	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	0	0	a	a	0	a
b	0	0	0	b	b	0	b
c	0	a	b	c	d	e	1
d	0	a	b	d	1	e	1
e	0	0	0	e	e	0	e
1	0	a	b	1	1	e	1

Table 1: U_1 Uninormu.

U_2	0	a	b	c	1	c	1
0	0	0	0	0	0	0	0
a	0	0	0	a	a	0	a
b	0	0	0	b	b	0	b
c	0	a	b	c	d	e	1
d	0	a	b	d	1	e	1
e	0	0	0	e	e	e	e
1	0	a	b	1	1	e	1

Table 2: U_2 Uninormu.

One can easily check that U_1 and U_2 satisfies the conditions of Proposition 4, thus U_1 and U_2 can be seen as U_W but $U_1 \neq U_2$.

Let define the following U_3 and U_4 uninorms with neutral element c given in Table 3 and Table 4 respectively.

U_3	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	0	a	a	0	a
b	0	0	b	b	b	0	b
c	0	a	b	c	d	e	1
d	0	a	b	d	d	e	1
e	0	0	0	e	e	0	e
1	0	a	b	1	1	e	1

Table 3: U_3 Uninormu.

U_4	0	a	b	c	1	c	1
0	0	0	0	0	0	0	0
a	0	a	0	a	a	0	a
b	0	0	b	b	b	0	b
c	0	a	b	c	d	e	1
d	0	a	b	d	1	e	1
e	0	0	0	e	e	e	e
1	0	a	b	1	1	e	1

Table 4: U_4 Uninormu.

One can easily check that U_3 and U_4 satisfies the conditions of Proposition 5, thus U_3 and U_4 can be seen as $U_{\wedge\vee}$ but $U_3 \neq U_4$.

Let $U^2 \in \mathcal{U}_{k(e,f)}$. It is well known that $U^2 \downarrow [0,k]^2$ is an uninorm on $[0,k]$ with neutral element e and $U^2 \downarrow [k,1]^2$ is an uninorm on $[k,1]$ with neutral element f. Let we call $U^2 \downarrow [0,k]^2$ as ${U^2}_1$ and $U^2 \downarrow [k,1]^2$ as ${U^2}_2$.

Similar to the reorganization of the order $\preceq_U$, one can reorganize $\preceq_{U^2}$ as follow.

Definition 10. *Let $U^2 \in \mathcal{U}_{k(e,f)}$ such that $U^2 \downarrow [0,k]^2 = {U^2}_1$ and $U^2 \downarrow [k,1]^2 = {U^2}_2$. Define the following relation: For every $x,y \in L$,*

$$x \preceq_{U^2} y :\Leftrightarrow \begin{cases} \text{if} \quad x,y \in [0,k] \quad \text{and} \quad x \preceq_{{U^2}_1} y \quad \text{or}, \\ \text{if} \quad x,y \in [k,1] \quad \text{and} \quad x \preceq_{{U^2}_2} y \quad \text{or}, \\ x \leq y, \qquad \text{otherwise}. \end{cases} \tag{4}$$

Proposition 6. *Let $(L,\leq,0,1)$ be a bounded lattice and $U^{2W} \in \mathcal{U}_{k(e,f)}$ be an 2-uninorm on L such that ${U^{2W}}_1 = U_W$ on $[0,k]^2$ and ${U^{2W}}_2 = U_W$ on $[k,1]^2$. Then, $\preceq_{U^{2W}} \subseteq \preceq_{F^2}$ for all $F^2 \in \mathcal{U}_{k(e,f)}$.*

Proof. $F^2 \in \mathcal{U}_{k(e,f)}$ arbitrary 2-uninorm. Let $(x,y) \in \preceq_{U^{2W}}$.

(i) $x,y \in [0,k]$. Then, $(x,y) \in \preceq_{U^{2W}}$ implies that $(x,y) \in \preceq_{{U^{2W}}_1} = \preceq_{U_W}$. Since $\preceq_{U_W} \subseteq \preceq_U$ for any uninorm U on $[0,k]$, $\preceq_{U^{2W}} \subseteq \preceq_{{F^2}_1}$. Therefore, $(x,y) \in \preceq_{F^2}$.

(ii) $x,y \in [k,1]$. Then, $(x,y) \in \preceq_{U^{2W}}$ implies that $(x,y) \in \preceq_{{U^{2W}}_2} = \preceq_{U_W}$. Since $\preceq_{U_W} \subseteq \preceq_U$ for any uninorm U on $[k,1]$, $\preceq_{U_W} \subseteq \preceq_{{F^2}_2}$. Therefore, $(x,y) \in \preceq_{F^2}$.

(iii) For other cases, $(x,y) \in \preceq_{U^{2W}}$ implies that $(x,y) \in \leq$. Therefore, $(x,y) \in \preceq_{F^2}$.

Thus, it is obtained that $\preceq_{U^{2W}} \subseteq \preceq_{F^2}$.

Proposition 7. *Let $(L,\leq,0,1)$ be a bounded lattice and $U^{2S} \in U_{k(e,f)}$ be an 2-uninorm on L such that ${U^{2S}}_1 = U_{\wedge\vee}$ on $[0,k]^2$ and ${U^{2S}}_2 = U_{\wedge\vee}$ on $[k,1]^2$. Then, $\preceq_{G^2} \subseteq \preceq_{U^{2S}}$ for all $G^2 \in U_{k(e,f)}$.*

Proof. The proof can be done similar proof of Proposition 6.

Corollary 2. *Let $(L,\leq,0,1)$ be a bounded lattice and $U^2 \in U_{k(e,f)}$ be an arbitrary 2-uninorm on L. Then, U^{2W} is the order-weakest and U^{2S} is order-strongest 2-uninorms on L,i.e.,*

$$\preceq_{U^{2W}} \subseteq \preceq_{U^2} \subseteq \preceq_{U^{2S}} .$$

Remark. The order-weakest 2-uninorm U^{2W} and order-strongest 2-uninorm U^{2S} on L mentioned in Corollary 2 are not the necessarily the ones. This argue is clearly obtained that Remark 3, Proposition 6 and Proposition 7.

Remark. The relation (4) can be seen as a way to obtain order from two uninorms defined on subintervals $[0,k]$ and $[k,1]$ of L. Check following proposition.

Proposition 8. *Let $(L,\leq,0,1)$ be a bounded lattice, U_1 uninorm on subinterval $[0,k]$ of L with neutral element e and U_2 uninorm on subinterval $[k,1]$ of L with neutral element f. Following relation is partially ordered relation on L.*

For every $x,y \in L$,

$$x \preceq_{U_1U_2} y :\Leftrightarrow \begin{cases} if \quad x,y \in [0,k] \quad and \quad x \preceq_{U_1} y \quad or, \\ if \quad x,y \in [k,1] \quad and \quad x \preceq_{U_2} y \quad or, \\ x \leq y, \qquad otherwise. \end{cases} \tag{5}$$

Corollary 3.(5) *In Proposition 8 coincides with $\preceq_{U^2}$ if 2-uninorm $U^2 \in U_{k(e,f)}$ provide that ${U^2}_1 = U_1$ and ${U^2}_2 = U_2$,i.e., $\preceq_{U_1U_2} = \preceq_{U^2}$.*

Proposition 9. *Let L be a lattice and $U \in \mathscr{U}(e)$ such that $k \in L \setminus \{0,1\}$ is comparable with all elements of L. Then, $([0,k],\preceq_{U_1})$ and $([k,1],\preceq_{U_2})$ are lattices if and only if $(L,\preceq_{U_1U_2})$ is a lattice.*

Proof. Suppose that $([0,k],\preceq_{U_1})$ and $([k,1],\preceq_{U_2})$ are lattices.

(i) Let $x,y \in [0,k]$ be arbitrary. Since $([0,k],\preceq_{U_1})$ is a lattice, $x \vee_{U_1} y$ and $x \wedge_{U_1} y$ exist. Let call $x \vee_{U_1} y = a \in [0,k]$ and $x \wedge_{U_1} y = b \in [0,k]$. Since $x \vee_{U_1} y = a$, $x \preceq_{U_1} a$ and $y \preceq_{U_1} a$. Thus, it is obtained that $x \preceq_{U_1U_2} a$ and $y \preceq_{U_1U_2} a$, that is,$a \in \overline{\{x,y\}}_{U_1U_2}$.

Let $t \in \overline{\{x,y\}}_{U_1U_2}$ be arbitrary. Then, $x \preceq_{U_1U_2} t$ and $y \preceq_{U_1U_2} t$.

Since k is comparable with the elements of L, either $t \leq k$ or $k \leq t$.

Suppose that $t \leq k$. Then $x \preceq_{U_1} t$ and $y \preceq_{U_1} t$, that is, we have that $t \in \overline{\{x,y\}}_{U_1}$. Since $x \vee_{U_1} y = a$, $a \preceq_{U_1} t$. Then, it is obtained that $a \preceq_{U_1U_2} t$ since $a,t \in [0,k]$. So, $x \vee_{U_1U_2} y = a$. Similarly, it can be shown that $x \wedge_{U_1U_2} y = b$.

(ii) Let $x,y \in [k,1]$. Since $([k,1],\preceq_{U_2})$ is a lattice, $x \vee_{U_2} y$ and $x \wedge_{U_2} y$ exist. Let call $x \vee_{U_2} y = a^* \in [k,1]$ and $x \wedge_{U_2} y = b^* \in [k,1]$. Similarly, it is obtained that $x \vee_{U_1U_2} y = a^*$ and $x \wedge_{U_1U_2} y = b^*$.

(iii) Let $x \leq k$ and $k \leq y$. Then it is clear that $x \vee_{U_1U_2} y = y$ and $x \wedge_{U_1U_2} y = x$.

(iv) Let $k \leq x$ and $y \leq k$. Then it is clear that $x \vee_{U_1U_2} y = x$ and $x \wedge_{U_1U_2} y = y$.

Therefore, $(L,\preceq_{U_1U_2})$ is a lattice if $([0,k],\preceq_{U_1})$ and $([k,1],\preceq_{U_2})$ are lattices.

Conversely, if $(L,\preceq_{U_1U_2})$ is a lattice, it is clear that $([0,k],\preceq_{U_1})$ and $([k,1],\preceq_{U_2})$ are lattices.

Remark. If we drop the condition given in Proposition 9, that is, if k is not comparable with all elements of L, then the claim need not be satisfied. Check the following example.

Example 3. Consider the lattice $(L,\leq,0,1)$ given its lattices diagram as follows.

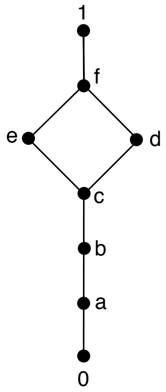

Fig. 2: $(L,\leq)$

Take the following uninorm U_1 on $[0,d]$ with neutral element a and its lattice diagram are as follows.

U_1	0	a	b	c	d
0	0	0	b	c	d
a	0	a	b	c	d
b	b	b	d	d	d
c	c	c	d	d	d
d	d	d	d	d	d

Table 5: The uninorm U_1.

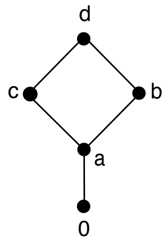

Fig. 3: $([0,d],\preceq_{U_1})$.

Also, uninorm U_2 on $[d,1]$ with neutral element f and its lattice diagram are as follows.

U_2	d	f	1
d	d	d	d
f	d	f	1
1	d	1	1

Table 6: The uninorm U_2.

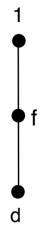

Fig. 4: $([d,1],\preceq_{U_2})$.

Finally, the order $\preceq_{U_1U_2}$ is depicted as follows.

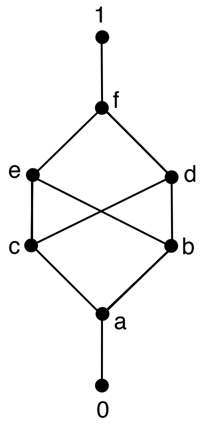

Fig. 5: $(L, \preceq_{U_1U_2})$

As it is easily seen in the figures, although $([0,d], \preceq_{U_1})$ and $([d,1], \preceq_{U_2})$ are lattices, $(L, \preceq_{U_1U_2})$ is not.

Corollary 4. *Let $(L, \leq, 0, 1)$ be a bounded lattice, U_1 uninorm on subinterval $[0,k]$ of L with neutral element e such that $U_1 \downarrow [0,e]^2$ divisible t-norm, $U_1 \downarrow [e,k]^2$ divisible t-conorm and U_2 uninorm on subinterval $[k,1]$ of L with neutral element f such that $U_2 \downarrow [k,f]^2$ divisible t-norm, $U_2 \downarrow [f,1]^2$ divisible t-conorm. Then, $\preceq_{U_1U_2} = \leq$.*

4 Conclusion

The order definition of $\preceq_U$ has been reorganized. By this way, order-weakest uninorms and order-strongest uninorms are determined. In addition, order-weakest uninorms and order-strongest uninorms may not be the only one. Similarly, the order definition of $\preceq_{U^2}$ has been reorganized through the underlying uninorms. This new form also made it possible to obtain a new order definition from two orders obtained from two uninorms defined on subintervals of bounded lattice.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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