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Some more results on $i-$convergence of filters

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Abstract: In this paper, we have proved some more results on $I-$convergence of filters. We have proved the equivalence of $I-$convergence and ordinary convergence of filters as well as the equivalence of $I-$convergence of nets and filters.

Keywords: $I-$convergence of filters, equivalence of $I-$convergence,ordinary convergence of filters.

1 Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by H. Fast [4] and I. J. Schoenberg [21]. Any convergent sequence is statistically convergent but the converse is not true [18]. Moreover, a statistically convergent sequence need not even be bounded [18]. Let $\mathbb{N}$ denotes the set of natural numbers. If $K \subset \mathbb{N}$, then K_n will denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ stands for the cardinality of K_n. The natural density of K is defined by

$$d(K) = \lim_n \frac{|K_n|}{n},$$

if the limit exists [5,17].

The concept of $I-$convergence of real sequences [7,8] is a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers. The notion of ideal convergence for single sequences was first defined and studied by Kostyrko et. al. [7]. Mursaleen et. al. [13] defined and studied the notion of ideal convergence in random $2-$normed spaces and construct some interesting examples. Several works on $I-$convergence and statistical convergence have been done in [1,3,6,7,8,9,12,13,14,15,16,20].

The idea of $I-$convergence of real sequences coincides with the idea of ordinary convergence if I is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if I is the ideal of subsets of $\mathbb{N}$ of natural density zero [10].

The idea of $I-$convergence has been extended from real number space to metric space [7] and to a normed linear space [19] in recent works. Later B. K. Lahiri and P. Das [10] extended the idea of $I-$convergence to an arbitrary topological space and observed that the basic properties are preserved in a topological space. In [11], they also introduced the idea of $I-$convergence of nets in a topological space and examined how far it affects the basic properties. [6] introduced the idea of $I-$convergence of filters in a topological space X and studied its various properties. [6] proved that basic properties of convergence of filters in a topological space X also hold in case of $I-$convergence of filters. We start with the following definitions.

Definition 1. *Let X be a non-empty set. Then a family $\mathscr{F} \subset 2^X$ is called a* **filter** *on X if*

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(i) $\emptyset \notin \mathscr{F}$,
(ii) *$A,B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$ and*
(iii) *$A \in \mathscr{F}, B \supset A$ implies $B \in \mathscr{F}$.*

Definition 2. *Let X be a non-empty set. Then a family $I \subset 2^X$ is called an* **ideal** *of X if*

(i) $\emptyset \in I$,
(ii) *$A,B \in I$ implies $A \cup B \in I$ and*
(iii) *$A \in I, B \subset A$ implies $B \in I$.*

Definition 3. *Let X be a non-empty set. Then a filter $\mathscr{F}$ on X is said to be* **non-trivial** *if $\mathscr{F} \neq \{X\}$.*

Definition 4. *Let X be a non-empty set. Then an ideal I of X is said to be* **non-trivial** *if $I \neq \{\emptyset\}$ and $X \notin I$.*

Note (i) $\mathscr{F} = \mathscr{F}(I) = \{A \subset X : X \setminus A \in I\}$ is a filter on X, called the **filter associated with the ideal** I.
(ii) $I = I(\mathscr{F}) = \{A \subset X : X \setminus A \in \mathscr{F}\}$ is an ideal of X, called the **ideal associated with the filter** $\mathscr{F}$.
(iii) A non-trivial ideal I of X is called **admissible** if I contains all the singleton subsets of X.
Several examples of non-trivial admissible ideals have been considered in [7].

Throughout this paper, $X = (X,\tau)$ will stand for a topological space and $I = I(\mathscr{F})$ will be the ideal of X associated with the filter $\mathscr{F}$ on X.

Before proving some more results on $I-$convergence of filters, we give a brief discussion on $I-$convergence of filters as given by [6].

Definition 5. *A filter $\mathscr{F}$ on X is said to be $I-$***convergent** *to $x_0 \in X$ if for each nbd U of x_0, $\{y \in X : y \notin U\} \in I$. In this case, x_0 is called an $I-$***limit of** *$\mathscr{F}$ and is written as $I - \lim \mathscr{F} = x_0$.*

Theorem 1. *A filter $\mathscr{F}$ on X is $I-$convergent to x_0 if and only if for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$.*

Proposition 1. *If X is Hausdorff, then any $I-$convergent filter $\mathscr{F}$ on X has a unique $I-$limit.*

Notation In case more than one filter is involved, we use the notation $I(\mathscr{F})$ to denote the ideal associated with the corresponding filter $\mathscr{F}$.

Proposition 2. *Let $E \subset X$ and $\mathscr{F}$ be a filter on E which is $I-$convergent to $x_0 \in X$, where $I = I(\mathscr{F})$ is an admissible ideal of E. Then x_0 is a limit point of E. Conversely, if x_0 is a limit point of E, then there is a filter on $E \setminus \{x_0\}$ which is $I-$convergent to x_0, for some admissible ideal I of E.*

Proposition 3. *Let X and Y be two topological spaces and $f : X \to Y$ be a map. Let $\mathscr{F}$ be a filter on X. Then $f : X \to Y$ is continuous at $x_0 \in X$ if and only if $I_X - \lim \mathscr{F} = x_0$ in X implies $I_Y - \lim f(\mathscr{F}) = f(x_0)$, where $I_X = I_X(\mathscr{F})$, $f(\mathscr{F})$ is a filter on Y generated by the base $\{f(F) : F \in \mathscr{F}\}$ and $I_Y = I_Y(f(\mathscr{F}))$.*

1.1 Characterization of closure

Proposition 4. *Let $E \subset X$. Then $x_0 \in \overline{E}$ if and only if there is a filter $\mathscr{F}$ on X such that $E \in \mathscr{F}$ and $I - \lim \mathscr{F} = x_0$.*

Proposition 5. *Let $\mathscr{F}$ be a filter on X such that $I - \lim \mathscr{F} = x_0$. Then every filter $\mathscr{F}'$ finer than $\mathscr{F}$ also $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Remark. Let $\mathscr{F}$ be a filter on X and $\mathscr{F}'$ be another filter on X finer than $\mathscr{F}$. Then $I(\mathscr{F}') - \lim \mathscr{F}' = x_0$ need not imply that $I(\mathscr{F}) - \lim \mathscr{F} = x_0$.

Proposition 6. *Let $\mathscr{F}$ be a filter on X such that $I-\lim\mathscr{F}=x_0$. Then every filter $\mathscr{F}'$ on X coarser than $\mathscr{F}$ also $I-$converges to x_0, where $I=I(\mathscr{F})$.*

Note The above proposition need not be true if we replace $I(\mathscr{F})-\lim\mathscr{F}'$ by $I(\mathscr{F}')-\lim\mathscr{F}'$.

Proposition 7.*Let $\mathscr{F}$ be a filter on X and $\mathscr{G}$ be any other filter on X finer than $\mathscr{F}$. Then $I(\mathscr{F})-\lim\mathscr{G}=x_0$ implies $I(\mathscr{G})-\lim\mathscr{G}=x_0$. But not conversely.*

Proposition 8.*Let τ_1 and τ_2 be two topologies on X such that τ_1 is coarser than τ_2. Let $\mathscr{F}$ be a filter on X such that $I-\lim\mathscr{F}=x_0$ w.r.t τ_2. Then $I-\lim\mathscr{F}=x_0$ w.r.t τ_1. But the converse need not be true.*

Proposition 9.*Let $\mathscr{M}$ be a collection of all those filters $\mathscr{G}$ on a space X which $I(\mathscr{G})-$converges to the same point $x_0\in X$. Then the intersection $\mathscr{F}$ of all the filters in $\mathscr{M}$ $I(\mathscr{F})-$converges to x_0.*

Proposition 10.*If every $I-$convergent filter $\mathscr{F}$ on X has a unique $I-$limit, then the space X is Hausdorff.*

Theorem 2.*A filter $\mathscr{F}$ I_X-converges to x in $X=\prod_{\alpha\in\Lambda}X_\alpha$ if and only if $p_\alpha(\mathscr{F})$ $I_{X_\alpha}-$converges to $p_\alpha(x),\forall\ \alpha$, where $I_X=I_X(\mathscr{F})$ and $I_{X_\alpha}=I_{X_\alpha}(p_\alpha(\mathscr{F}))$.*

This paper is an extension of the work done on $I-$convergence of filters in [6] and is inspired from [2,22].

2 Equivalence of $I-$convergence and convergence of a filter $\mathscr{F}$

We recall the following.
Let $\mathscr{F}$ be a filter on X and let $\mathscr{D}$ be a set that is bijective with the filter $\mathscr{F}$. We shall call $\mathscr{D}$ an index set for $\mathscr{F}$ and denote the bijective correspondance by $\mathscr{F}=\{F_d : d\in\mathscr{D}\}$.

Note It is easy to show that $\mathscr{D}$ becomes a poset with the partial order defined by

$$c\leq d \ \ if\ and\ only\ if\ F_c\supset F_d.$$

In this case, we speak of an indexed filter.

Definition 6. *Let $\mathscr{F}$ be an indexed filter on X with index set $\mathscr{D}$. Any net $\lambda:\mathscr{D}\to X$ with $\lambda(d)\in F_d$ is called a* **derived net** *of $\mathscr{F}$.*

Definition 7. *Let λ be a net in X with directed set $\mathscr{D}$. Then $\mathscr{F}=\{F\subset X : \lambda$ is eventually in $F\}$ is called a* **derived filter** *of λ. By λ eventually in F, we mean that some tail of λ is contained in F. By tail of λ, we mean the set $\Lambda_d=\{\lambda(c) : c\geq d$ in $\mathscr{D}\}$.*

Definition 8. *A net $\lambda:\mathscr{D}\to X$ in X is said to be* **convergent** *to $x_0\in X$ if for each nbd U of x_0, there is some $d\in\mathscr{D}$ such that $c\geq d$ in $\mathscr{D}$ implies that $\lambda(c)\in U$. In other words, some tail $\Lambda_d=\{\lambda(c) : c\geq d$ in $\mathscr{D}\}\subset U$.*

Theorem 3. *A filter $\mathscr{F}$ on X $I-$converges to $x_0\in X$ if and only if every derived net λ of $\mathscr{F}$ converges to x_0.*

Proof. Suppose $I-lim\mathscr{F}=x_0$. This means that for each nbd U of x_0, $\{V\in\mathscr{P}(X) : U\cap V=\emptyset\}\subset I$. Let us index $\mathscr{F}$ with an index set $\mathscr{D}$ so that $\mathscr{F}=\{F_s : s\in\mathscr{D}\}$. Let us give some direction to $\mathscr{D}$ so that $c\geq d$ in $\mathscr{D}$ if and only if $F_c\subset F_d$. Let λ be a derived net of $\mathscr{F}$ so obtained. We have to show that the net λ converges to x_0. For this, let U be a nbd of x_0. Since $U\cap(X\setminus U)=\emptyset$, we find that $X\setminus U\in I$. This implies that $U\in\mathscr{F}$. Now $U\in\mathscr{F}$ implies $U=F_d$, for some $d\in\mathscr{D}$. Now if $c\geq d$, then $F_c\subset F_d$ and so $\lambda(c)\in F_c\subset F_d=U$. Thus there is some tail $\Lambda_d=\{\lambda(c) : c\geq d\ in\ \mathscr{D}\}$ of λ such that $\Lambda_d\subset U$. That is, λ is eventually in U. Thus $\lambda\to x_0$.

Conversely, suppose every derived net λ converges to x_0. We shall show that $I-\lim\mathscr{F}=x_0$. For this, let U be a nbd of x_0. We claim that $\{V\in\mathscr{P}(X):U\cap V=\emptyset\}\subset I$. Since $\lambda\to x_0$, tail $\Lambda_d=\{\lambda(c):c\geq d\ in\ \mathscr{D}\}\subset U$. Since λ is a derived net, there exists $F_c\in\mathscr{F}$ for $c\in\mathscr{D}$ such that $\lambda(c)\in F_c$. Thus $\Lambda_d\in\mathscr{F}$. Now, let $V\in\mathscr{P}(X)$ such that $U\cap V=\emptyset$. Then $V\subset X\setminus U$ implies $V\subset X\setminus\Lambda_d$ $(\because\Lambda_d\subset U)$ and so $V\in I$ $(\because\Lambda_d\in\mathscr{F})$. Therefore, $\{V\in\mathscr{P}(X):U\cap V=\emptyset\}\subset I$. Hence $I-\lim\mathscr{F}=x_0$. This completes the proof.

We recall [22] that a filter $\mathscr{F}$ on a topological space X is said to **converge** to x_0(written as $\mathscr{F}\to x_0$) if $\mathscr{F}$ is finer than the nbd filter at x_0 (i.e., $\mathscr{U}_{x_0}\subset\mathscr{F}$). Using Theorem $2\cdot 4$, we can prove the equivalence of $I-$convergence and convergence of a filter $\mathscr{F}$.

Theorem 4. *A filter $\mathscr{F}$ on X $I-$converges to $x_0\in X$ if and only if $\mathscr{F}$ converges to x_0.* **Proof** *We know that a filter $\mathscr{F}$ converges to x_0 in a topological space X if and only if every derived net λ does [22]. Using Theorem $2\cdot 4$, we get the required result.*

Theorem 5. *A net $\lambda:\mathscr{D}\to X$ converges to $x_0\in X$ if and only if the derived filter $\mathscr{F}$ of λ $I-$ converges to x_0.* **Proof** *Suppose a net $\lambda:\mathscr{D}\to X$ converges to $x_0\in X$. This means that for each nbd U of x_0, some tail $\Lambda_d=\{\lambda(c):c\geq d\ in\ \mathscr{D}\}\subset U$. Since $\mathscr{F}$ is a derived filter, each nbd $U\in\mathscr{F}$ (by definition of derived filter and the given condition). We have to show that $I-\lim\mathscr{F}=x_0$. For this, let U be a nbd of x_0. We claim that $\{V\in\mathscr{P}(X):U\cap V=\emptyset\}\subset I$. So, let $V\in\mathscr{P}(X)$ such that $U\cap V=\emptyset$. Now $U\cap V=\emptyset$ implies that $V\subset X\setminus U$. Since $U\in\mathscr{F},X\setminus U\in I$. Further, I is an ideal of X and $V\subset X\setminus U$ implies that $V\in I$. Therefore, $\{V\in\mathscr{P}(X):U\cap V=\emptyset\}\subset I$. Conversely, suppose $\mathscr{F}$ is a derived filter of λ such that $I-\lim\mathscr{F}=x_0$. This means that for each nbd U of x_0, $\{V\in\mathscr{P}(X):U\cap V=\emptyset\}\subset I\cdots(*)$. We shall show that the net $\lambda:\mathscr{D}\to X$ converges to x_0. Suppose λ does not converge to x_0. This means that there is some nbd U of x_0 such that λ is not eventually in U. That is, $\Lambda_d=\{\lambda(c):c\geq d\ in\ \mathscr{D}\}\nsubseteq U$, for any tail Λ_d. Since $\mathscr{F}$ is a derived filter, $U\notin\mathscr{F}$. From $(*)$, $U\cap(X\setminus U)=\emptyset$ implies that $X\setminus U\in I$. This further implies that $U\in\mathscr{F}$, which is not true. Therefore, our supposition is wrong. Hence $\lambda\to x_0$.*

3 Equivalence of $I-$convergence of filters and nets

We first define the $I-$convergence of nets in X.

Definition 9. *Let I be a non-trivial ideal of subsets of X. Let $\lambda:\mathscr{D}\to X$ be a net in X, where $\mathscr{D}$ is a directed set. Then λ is said to be $I-$***convergent** *to x_0 in X if for each nbd U of x_0, $\{\lambda(c)\in X:\lambda(c)\notin U\}\in I$.*

Theorem 6. *A filter $\mathscr{F}$ on X $I-$converges to $x_0\in X$ if and only if every derived net λ of $\mathscr{F}$ $I-$converges to x_0, where $I=I(\mathscr{F})$.*

Proof. Suppose $I-\lim\mathscr{F}=x_0\cdots(*)$. Let us index $\mathscr{F}$ with an index set $\mathscr{D}$ so that $\mathscr{F}=\{F_d:d\in\mathscr{D}\}$. Let us give some direction to $\mathscr{D}$ so that $c\geq d$ in $\mathscr{D}$ if and only if $F_c\subset F_d$. Let $\lambda:\mathscr{D}\to X$ be the derived net of $\mathscr{F}$ so obtained. This means that $\lambda(c)\in F_c$, for some $c\in\mathscr{D}$. We have to show that $I-\lim\lambda=x_0$. For this, let U be a nbd of x_0. We claim that $\{\lambda(c)\in X:\lambda(c)\notin U\}\in I$. So, let $\lambda(c)\in X$ such that $\lambda(c)\notin U$. Then by the given condition $(*)$, $\{\lambda(c)\}\in I$. Hence $\{\lambda(c)\in X:\lambda(c)\notin U\}\in I$.

Conversely, suppose every derived net $\lambda:\mathscr{D}\to X$ of $\mathscr{F}$ $I-$converges to $x_0\cdots(**)$. We have to show that $I-\lim\mathscr{F}=x_0$. For this, let U be a nbd of x_0. We claim that $\{y\in X:y\notin U\}\in I$. So, let $y\in X$ such that $y\notin U$. We need to show that $\{y\}\in I$. If $y\neq\lambda(c)$, for any $c\in\mathscr{D}$, then clearly $y\notin F_c$, for any $c\in\mathscr{D}$ and so $y\in X\setminus F_c$, for some $c\in\mathscr{D}$. This implies that $\{y\}\in I$. If $y=\lambda(c)$, for some $c\in\mathscr{D}$, then $y\notin U$ implies that $\lambda(c)\notin U$. By the given condition $(**)$, $\{\lambda(c)\}\in I$. This implies that $\{y\}\in I$.

Lemma 1. *A filter $\mathscr{F}$ on X converges to x_0 in X if and only if every derived net λ of $\mathscr{F}$ $I-$converges to x_0.* **Proof** *It follows from Theorems* $2\cdot 5$ *and* $3\cdot 2$.

Theorem 7. *Let $\lambda : \mathscr{D} \to X$ be a net in X and $\mathscr{F}$ be a derived filter of λ. Then λ $I-$converges to x_0 in X if and only if the derived filter $\mathscr{F}$ $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Proof. Let $\lambda : \mathscr{D} \to X$ be a net in X and $\mathscr{F}$ be a derived filter of λ. Suppose $I-\lim\lambda = x_0$. Then for each nbd U of x_0, $\{\lambda(c) \in X : \lambda(c) \notin U\} \in I \cdots (*)$. We have to show that $I-\lim\mathscr{F} = x_0$. For this, let U be a nbd of x_0. We claim that $\{y \in X : y \notin U\} \in I$. So, let $y \in X$ such that $y \notin U$. If $y = \lambda(c)$, for some $c \in \mathscr{D}$, then clearly by the given condition $\{y\} \in I$. If $y \neq \lambda(c)$, for any $c \in \mathscr{D}$, then we proceed as follows. $y \neq \lambda(c)$, for any $c \in \mathscr{D}$ implies that $y \notin \Lambda_d$, for any tail Λ_d of λ. Thus $\Lambda_d \subset U$. Since $\mathscr{F}$ is a derived filter, by definition $U \in \mathscr{F}$. This implies that $X \setminus U \in I$. Now $y \notin U$ implies that $y \in X \setminus U$ and so $\{y\} \in I$. Therefore, $\{y \in X : y \notin U\} \in I$. Hence $I-\lim\mathscr{F} = x_0$.

Conversely, suppose that $I-\lim\mathscr{F} = x_0$. Then for each nbd U of x_0, $\{y \in X : y \notin U\} \in I \cdots (**)$. We have to show that $I-\lim\lambda = x_0$. For this, let U be a nbd of x_0. We claim that $\{\lambda(c) \in X : \lambda(c) \notin U\} \in I$. So, let $\lambda(c) \in X$ such that $\lambda(c) \notin U$. Clearly, by the given condition $(**)$, $\{\lambda(c)\} \in I$. Therefore, $I-\lim\lambda = x_0$.

We have the following definition of $I-$convergence of nets in X as given by [11].

Definition 10. *Let $\mathscr{D}$ be a directed set. Let I be a non-trivial ideal of subsets of $\mathscr{D}$. A net $\lambda : \mathscr{D} \to X$ is said to be $I-$***convergent** *to x_0 in X if for each nbd U of x_0, $\{c \in \mathscr{D} : \lambda(c) \notin U\} \in I$.*

With the help of an example, we shall show that $I(\mathscr{F})-\lim\mathscr{F} = x_0$ need not imply that $I(\lambda)-\lim\lambda = x_0$, where $I(\mathscr{F})$ is the ideal associated with the filter $\mathscr{F}$ and $I(\lambda)$ is the non-trivial ideal of subsets of the directed set $\mathscr{D}$ of λ.

Example 1. Let $X = \{1,2,3\}$ with $\tau = \{\emptyset,\{1\},X\}$. Here, $\mathscr{U}_1 = \{\{1\},\{1,2\},\{1,3\},X\}, \mathscr{U}_2 = \{X\}$ and $\mathscr{U}_3 = \{X\}$. Let $\mathscr{F} = \{\{1\},\{1,2\},\{1,3\},X\}$. Then $I(\mathscr{F}) = \{\emptyset,\{2\},\{3\},\{2,3\}\}$. We can easily see that $1,2$ and 3 are $I(\mathscr{F})-$ limits of $\mathscr{F}$. Now let $\mathscr{D} = \{a,b,c,d\}$. Then there is a one-to-one correspondance $\phi : \mathscr{D} \to \mathscr{F}$ given by $\phi(a) = F_a = X$, $\phi(b) = F_b = \{1,3\}$, $\phi(c) = F_c = \{1,2\}$ and $\phi(d) = F_d = \{1\}$. Let $\lambda : \mathscr{D} \to X$ be the directed net so obtained such that $\lambda(i) \in F_i$, for $i = a,b,c,d$. Suppose $\lambda = \{1,2,1,1\}$. Let $I(\lambda) = \{\emptyset,\{a\},\{c\},\{a,c\}\}$. We can see that for $x = 1$ and $U = \{1\}$, $\{t \in \mathscr{D} : \lambda(t) \notin \{1\}\} = \{b\} \notin I(\lambda)$. Therefore, $I(\lambda)-\lim\lambda \neq 1$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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