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A new technique of Laplace Padé reduced differential transform method for (1+3) dimensional wave equations

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Abstract: The aim of this paper is to give a good strategy for solving some linear and non-linear partial differential equations in mechanics, physics, engineering and various other technical fields by Modified Reduced Differential Transform Method. In this article we use the method named with Laplace-Padé Reduced Differential Transform Method. This method is obtained by combining Laplace-Padé resummation method, which is a useful technique to find exact solutions, and the Reduced Differential Transform Method. We apply the method to the wave equations and give some examples to see its effectiveness and usefulness. The results and the findings showed that this method leads us to exact solutions with a few iterations or the approximate solutions with small errors.

Keywords: Laplace-Padé Reduced Differential Transform Method (LPRDTM), Modified Reduced Differential Transform Method (MRDTM), Reduced Differential Transform Method (RDTM), Partial differential equations (PDEs), wave equation.

1 Introduction

As widely known, the importance of research linear and non-linear PDEs have a large number of essential application studies in different branches of engineering and science such as fluid physics, plasma physics, non-linear fiberoptics, fluidmechanics, thermodynamic mechanic, thermodynamic, heat transfer, oceanography and atmospheric science[1–3]. Many researchers have paid attention to the solutions of linear and non-linear PDEs by various methods, such as, the differential transform method (DTM)[4–7], RDTM [8–15], the variational iteration method (VIM)[16,17] the homotopy analysis method (HAM)[18–20] and the Adomian decomposition method[21] among others.

Consider the following general form of (1+3) dimensional wave Equation,

$$u_{tt}+a(x,y,z,t)u_{xx}+b(x,y,z,t)u_{yy}+c(x,y,z,t)u_{zz}=e(x,y,z,t) \tag{1}$$

subject to initial condition

$$\begin{aligned} u(x,y,z,0) &= f(x,y,z) \\ u_t(x,y,z,0) &= g(x,y,z). \end{aligned} \tag{2}$$

We apply LPRDTM (combining Laplace-Padé resummation method and RDTM) to solve wave equation of the form (1).

This study is organized as follows. In Section 2, we briefly describe LPRDTM. Two numerical examples are introduced in Section 3 for demonstrating the complete study. Conclusion is given in the last section.

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2 Laplace-Padé reduced differential transform method (LPRDTM)

2.1 Reduced Differential Transform Method (RDTM)

We will briefly introduce RDTM [8] for wave equation in this section. We consider the wave equation in the operator form

$$L_{tt}\left(u(x,y,z,t)\right)+L\left(u(x,y,z,t)\right)=e(x,y,z,t) \tag{3}$$

with initial conditions

$$\begin{aligned} u(x,y,z,0)&=f(x,y,z)\\ u_t(x,y,z,0)&=g(x,y,z) \end{aligned} \tag{4}$$

where $L_{tt}\left(u(x,y,z,t)\right)=u_{tt}$ and $L\left(u(x,y,z,t)\right)=a(x,y,z,t)u_{xx}+b(x,y,z,t)u_{yy}+c(x,y,z,t)u_{zz}$ are linear operators.

Definition 1. *If $u\left(x,y,z,t\right)$ is differentiated continuously with respect to time t and analyticfunction and spaces x, y and z in the domain of interest, then the spectrum function*

$$U_k(x,y,z)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u\left(x,y,z,t\right)\right]_{t=0} \tag{5}$$

is the reduced transformed function of $u\left(x,y,z,t\right)$. In this study, the lowercase $u\left(x,y,z,t\right)$ represent the original function while the uppercase $U_k\left(x,y,z\right)$ stand for the transformed function. The differential inverse transform of $U_k\left(x,y,z\right)$ is defined as.

$$u\left(x,y,z,t\right)=\sum_{k=0}^{\infty}U_k\left(x,y,z\right)\,t^k \tag{6}$$

Combining equation (5) and (6). it can be obtained that

$$u\left(x,y,z,t\right)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u\left(x,y,z,t\right)\right]_{t=0}t^k. \tag{7}$$

From the above definition, one can find that the concept of the reduced (1+3) dimensional differential transform is derived from the power series expansion. The following theorem of the fundamental operators of RDTM is given below (for details see[8–10].

Theorem 1. *Assume that the reduced differential transform functions of* $u\left(x,y,z,t\right), v\left(x,y,z,t\right)$ and $w\left(x,y,z,t\right)$ *are* $U_k(x,y,z), V_k(x,y,z)$ and $W_k(x,y,z)$ *respectively. Then,*

(i) *If $w\left(x,y,z,t\right)=u\left(x,y,z,t\right)\pm\alpha v\left(x,y,z,t\right)$, then $W_k(x,y,z)=U_k(x,y,z)\pm\alpha V_k(x,y,z)$($\alpha$ is a constant),*

(ii) *If $w\left(x,y,z,t\right)=u\left(x,y,z,t\right)v\left(x,y,z,t\right)$,then $W_k(x,y,z)=\sum\limits_{r=0}^{k}V_r(x,y,z)U_{k-r}(x,y,z)=\sum\limits_{r=0}^{k}U_r(x,y,z)V_{k-r}(x,y,z)$,*

(iii) *If $w\left(x,y,z,t\right)=\frac{\partial^r}{\partial t^r}u\left(x,y,z,t\right)$, then $W_k(x,y,z)=(k+1)\ldots(k+r)U_{k+r}(x,y,z)=\frac{(k+r)!}{k!}U_{k+r}(x,y,z)$,*

(iv) *If $w\left(x,y,z,t\right)=x^my^nz^qt^p$, then $W_k(x,y,z)=\begin{cases}x^my^nz^q, k=p\\ 0, k\neq p\end{cases}$,*

(v) *If $w\left(x,y,z,t\right)=\frac{\partial}{\partial x^m\partial y^n\partial z^q}u\left(x,y,z,t\right)$, then $W_k(x,y,z)=\frac{\partial}{\partial x^m\partial y^n\partial z^q}U_k(x,y,z)$.*

Assume that the reduced differential transform functions of $a\left(x,y,z,t\right), b\left(x,y,z,t\right), c\left(x,y,z,t\right)$ and $e\left(x,y,z,t\right)$ *are* $A_k(x,y,z), B_k(x,y,z), C_k(x,y,z)$ and $E_k(x,y,z)$ *respectively. Then, According to Theorem 1 and the RDTM, from (3) we can construct the following iteration formula.*

$$\begin{aligned}(k+1)(k+2)U_{k+2}(x,y,z)=E_k(x,y,z)-\sum_{r=0}^{k}A_r(x,y,z)\frac{\partial}{\partial x^2}U_{k-r}(x,y,z)\\ -\sum_{r=0}^{k}B_r(x,y,z)\frac{\partial}{\partial y^2}U_{k-r}(x,y,z)-\sum_{r=0}^{k}C_r(x,y,z)\frac{\partial}{\partial z^2}U_{k-r}(x,y,z)\end{aligned} \tag{8}$$

From initial condition (3), we write

$$\begin{aligned} U_0(x,y,z) &= f(x,y,z) \\ U_1(x,y,z) &= g(x,y,z) \end{aligned} \tag{9}$$

Substituting (9) into (8) and by a straight forward iterative calculations, we get the following $U_k(x,y,z)$ values. Then the inverse transformation of the set of values $\{U_k(x,y,z)\}_{k=0}^{n}$ gives approximation solution as,

$$\tilde{u}_n(x,y,z,t) = \sum_{k=0}^{n} U_k(x,y,z)t^k \tag{10}$$

where n is order of approximation solution. Therefore, the exact solution of problem is given by

$$u(x,y,z,t) = \lim_{n\to\infty} \tilde{u}_n(x,y,z,t). \tag{11}$$

2.2 Padé Approximant

Let $u(x,y,z,t)$ be an analytical function with Maclaurin's expansion

$$u(x,y,z,t) = \sum_{k=0}^{n} u_n t^k \tag{12}$$

Then the Padé approximant to $u(x,y,z,t)$ of order $[K,L]$ which we denote by $[K/L]_u(x,y,z,t)$ is defined by [22,23],

$$\left[\frac{K}{L}\right]_u (x,y,z,t) = \frac{p_0 + p_1 t + \cdots + p_L t^K}{q_0 + q_1 t + \cdots + q_M t^L}, \tag{13}$$

where $q_0 = 1$, and the numerator and denominator polynomials have no common factors. The numerator and denominator polynomials are constructed as follows

$$u(x,y,z,t) - \left[\frac{K}{L}\right]_u (x,y,z,t) = O\left(t^{K+L+1}\right) \tag{14}$$

From (14), it is obtained

$$u(x,y,z,t)\sum_{n=0}^{L} q_n t^n - \sum_{n=0}^{K} p_n t^n = O\left(t^{K+L+1}\right). \tag{15}$$

From (15), the following algebraic linear systems are obtained as

$$\begin{aligned} u_K q_1 + \cdots + u_{K-L+1} q_M &= -u_{K+1}, \\ u_{K+1} q_1 + \cdots + u_{K-L+2} q_M &= -u_{K+2}, \\ &\vdots \\ u_{K+L-1} q_1 + \cdots + u_K q_L &= -u_{K+L}, \end{aligned} \tag{16}$$

$$\begin{aligned} p_0 &= u_0 \\ p_1 &= u_1 + u_0 q_1 \\ &\vdots \\ p_K &= u_K + u_{K-1} q_1 + \cdots + u_0 q_K. \end{aligned} \tag{17}$$

From (16), we calculate first all the coefficients $q_n, 1 \leq n \leq L$, Then, we determine the coefficients $p_n, 0 \leq n \leq K$, from (17).

2.3 Laplace-Padé reduced differential transform method (LPRDTM)

Some numerical methods give power series solutions. However, this kind of solutions have narrow domains of convergence. Hence, Laplace-Padé [23–26] resummation method is used to extend the domain of convergence of the solutions or to obtain exact solutions. We describe the LPRDTM which is combination of RDTM and Laplace-Padé resummation method as follows.

(i) Firstly, let's get series as in (2.8) by using RDTM up to a certain iteration for the initial value problem.
(ii) Secondly, we apply Laplace transformation to the obtained series (10).
(iii) Next, we write $1/t$instead of s in the obtained equation.
(iv) After that, we transform the series obtained from (iii) into a meromorphic function by creating its Padé approximant of order $[K/L]$ where$K$ and L are randomly chosen, but they should be smaller than the order of the series. In this step, the Padé approximant enlarges the domain of the obtained series solution to get better convergence and accuracy.
(v) Then, we write $1/s$instead of t in the obtained equation.
(vi) Finally, we get the approximate or exact solution by using the inverse Laplace s transformation.

3 Numerical consideration

We consider two examples applied to LPRDTM.

Example 1. Considering the following wave equation,

$$u_{tt} = \frac{1}{6}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right) \tag{18}$$

subject to initial conditions:

$$\begin{aligned} u(x,y,z,0) &= x^2y^2z^2, \\ u_t(x,y,z,0) &= -x^2y^2z^2. \end{aligned} \tag{19}$$

Now we apply the steps *(i)-(vi)* to our example. If we apply RDTM and Theorem 2.1 in (3.1) it gives us

$$U_{k+2}(x,y,z) = \frac{1}{6(k+1)(k+2)}\left(x^2\frac{\partial^2}{\partial x^2}U_k(x,y,z) + y^2\frac{\partial^2}{\partial y^2}U_k(x,y,z) + z^2\frac{\partial^2}{\partial z^2}U_k(x,y,z)\right) \tag{20}$$

where $U_k(x,y,z)$'s are the transformed functions. By the initial conditions (19) we write

$$\begin{aligned} U_0(x,y,z) &= x^2y^2z^2 \\ U_1(x,y,z) &= -x^2y^2z^2. \end{aligned} \tag{21}$$

By substituting (21) into (20) respectively, we obtain

$$U_2(x,y,z) = \frac{x^2y^2z^2}{2} \tag{22}$$

$$U_3(x,y,z) = -\frac{x^2y^2z^2}{6}, \tag{23}$$

$$U_4(x,y,z) = \frac{x^2y^2z^2}{24}. \tag{24}$$

By (10)

$$\tilde{u}_4(x,y,z,t) = \sum_{k=0}^{4} U_k(x,y,z)t^k. \tag{25}$$

Substituting (21), (22), (23) and (24) in (25), we get

$$\tilde{u}_4(x,y,z,t) = x^2y^2z^2 - x^2y^2z^2t + \frac{x^2y^2z^2}{2}t^2 - \frac{x^2y^2z^2}{6}t^3 + \frac{x^2y^2z^2}{24}t^4. \tag{26}$$

If we apply Laplace transform to $\tilde{u}_4(x,y,z,t)$, it gives

$$L\left[\tilde{u}_4(x,y,z,t)\right] = \frac{x^2y^2z^2(s^4 - s^3 + s^2 - s + 1)}{s^5}. \tag{27}$$

We write $1/t$ instead of s in (27) then

$$L\left[\tilde{u}_4(x,y,z,t)\right] = x^2y^2z^2(1 - t + t^2 - t^3 + t^4)t. \tag{28}$$

All $\left[K/L\right]$ -Padé approximants of (28) with $L \geq 1$, $K \geq 1$ and $K + L \leq 4$ give

$$\left[\frac{K}{L}\right]_{\tilde{u}_4}(x,y,z,t) = \frac{x^2y^2z^2t}{1+t}. \tag{29}$$

Now by changing $1/s$ into t in (3.12), we obtain $[K/L]_{\tilde{u}_4}$ in terms of s as follow,

$$\left[\frac{L}{M}\right]_{\tilde{u}_4}(x,y,z,t) = \frac{x^2y^2z^2}{1+s}. \tag{30}$$

Finally, when we apply the inverse Laplace transform to (3.13), it gives us an approximate solution. In fact, the approximate solution corresponds the exact solution.

$$\tilde{u}_4(x,y,z,t) = u(x,y,z,t) = x^2y^2z^2e^{-t}. \tag{31}$$

Example 2. Now we consider the following another wave equation.

$$u_{tt} = au_{xx} + bu_{yy} + cu_{zz} \tag{32}$$

where $a, b and c$ are constant. Subject to initial conditions.

$$\begin{aligned} u(x,y,z,0) &= a_1x + a_2x^2 + b_1y + b_2y^2 + c_1z + c_2z^2, \\ u_t(x,y,z,0) &= a'_1x + a'_2x^2 + b'_1y + b'_2y^2 + c'_1z + c'_2z^2. \end{aligned} \tag{33}$$

Now we apply the steps *(i)-(vi)* to our example. If we apply RDTM and Theorem 1 in (32) it gives us

$$U_{k+2}(x,y,z) = \frac{1}{(k+1)(k+2)}\left(a\frac{\partial^2}{\partial x^2}U_k(x,y,z) + b\frac{\partial^2}{\partial y^2}U_k(x,y,z) + c\frac{\partial^2}{\partial z^2}U_k(x,y,z)\right) \tag{34}$$

where $U_k(x,y,z)$'s are the transformed functions. By the initial conditions (33) we write

$$\begin{aligned} U_0(x,y,z) &= a_1x + a_2x^2 + b_1y + b_2y^2 + c_1z + c_2z^2 \\ U_1(x,y,z) &= a'_1x + a'_2x^2 + b'_1y + b'_2y^2 + c'_1z + c'_2z^2. \end{aligned} \tag{35}$$

Now, substituting (35) into (34) respectively, we obtain

$$U_2(x,y,z) = aa_2 + bb_2 + cc_2 \tag{36}$$

$$U_3(x,y,z) = \frac{1}{3}\left(aa'_2 + bb'_2 + cc'_2\right), \tag{37}$$

$$U_4(x,y,z) = 0. \tag{38}$$

By (10)

$$\tilde{u}_4(x,y,z,t) = \sum_{k=0}^{4} U_k(x,y,z)t^k. \tag{39}$$

Substituting (35), (36), (37) and (38) in (39), we get

$$\begin{aligned} \tilde{u}_4(x,y,z,t) &= a_1x + a_2x^2 + b_1y + b_2y^2 + c_1z + c_2z^2 + \left(a'_1x + a'_2x^2 + b'_1y + b'_2y^2 + c'_1z + c'_2z^2.\right)t \\ &+ (aa_2 + bb_2 + cc_2)\,t^2 + \frac{1}{3}\left(aa'_2 + bb'_2 + cc'_2\right)t^3. \end{aligned} \tag{40}$$

If we apply Laplace transform to $\tilde{u}_4(x,y,z,t)$, it gives

$$\begin{aligned} L\left[\tilde{u}_4(x,y,z,t)\right] &= \frac{a_1x + a_2x^2 + b_1y + b_2y^2 + c_1z + c_2z^2}{s} + \frac{a'_1x + a'_2x^2 + b'_1y + b'_2y^2 + c'_1z + c'_2z^2}{s^2} \\ &+ \frac{2aa_2 + 2bb_2 + 2cc_2}{s^3} + \frac{2aa'_2 + 2bb'_2 + 2cc'_2}{s^4}. \end{aligned} \tag{41}$$

We write $1/t$ instead of s in (27) then

$$L\left[\tilde{u}_4(x,y,z,t)\right] = \begin{pmatrix} a_1x + a_2x^2 + b_1y + b_2y^2 + c_1z + c_2z^2 + (a'_1x + a'_2x^2 + b'_1y + b'_2y^2 + c'_1z + c'_2z^2)t \\ +(2aa_2 + 2bb_2 + 2cc_2)t^2 + (2aa'_2 + 2bb'_2 + 2cc'_2)t^3 \end{pmatrix} t. \tag{42}$$

All $[K/L]$ t-Padé approximants of (42) with $L \geq 1$, $K \geq 1$ and $K + L \leq 4$ give

$$\left[\frac{K}{L}\right]_{\tilde{u}_4}(x,y,z,t) = \frac{\begin{pmatrix} 2a_1b_1xy + 2a_1b_2xy^2 + 2a_1c_1xz + 2a_1c_2xz^2 + 2a_2b_1x^2y + 2a_2b_2x^2y^2 \\ +2a_2c_1x^2z + 2a_2c_2x^2z^2 + 2b_1c_1yz + 2b_1c_2yz^2 + 2b_2c_1y^2z + 2b_2c_2y^2z^2 + {a_1}^2x^2 \\ +{a_2}^2x^4 + {b_1}^2y^2 + {b_2}^2y^4 + {c_1}^2z^2 + {c_2}^2z^4 + 2a_1a_2x^3 + 2b_1b_2y^3 + 2c_1c_2z^3 \end{pmatrix} t}{a_1x + a_2x^2 + b_1y + b_2y^2 + c_1z + c_2z^2 - (a'_1x + a'_2x^2 + b'_1y + b'_2y^2 + c'_1z + c'_2z^2)\,t}. \tag{43}$$

Now by changing $1/s$ into t in (3.12), we obtain $[K/L]_{\tilde{u}_4}$ in terms of s as follow:

$$\left[\frac{K}{L}\right]_{\tilde{u}_4}(x,y,z,t) = \frac{\begin{pmatrix} 2a_1b_1xy + 2a_1b_2xy^2 + 2a_1c_1xz + 2a_1c_2xz^2 + 2a_2b_1x^2y + 2a_2b_2x^2y^2 \\ +2a_2c_1x^2z + 2a_2c_2x^2z^2 + 2b_1c_1yz + 2b_1c_2yz^2 + 2b_2c_1y^2z + 2b_2c_2y^2z^2 + {a_1}^2x^2 \\ +{a_2}^2x^4 + {b_1}^2y^2 + {b_2}^2y^4 + {c_1}^2z^2 + {c_2}^2z^4 + 2a_1a_2x^3 + 2b_1b_2y^3 + 2c_1c_2z^3 \end{pmatrix}}{(a_1x + a_2x^2 + b_1y + b_2y^2 + c_1z + c_2z^2)\,s - (a'_1x + a'_2x^2 + b'_1y + b'_2y^2 + c'_1z + c'_2z^2)}. \tag{44}$$

Finally, when we apply the inverse Laplace transform to (30), it gives us an approximate solution. In fact, the approximate solution corresponds the exact solution.

$$\tilde{u}_4(x,y,z,t) = u(x,y,z,t) = \left(a_1x + a_2x^2 + b_1y + b_2y^2 + c_1z + c_2z^2\right) e^{\left(\frac{a'_1x+a'_2x^2+b'_1y+b'_2y^2+c'_1z+c'_2z^2}{a_1x+a_2x^2+b_1y+b_2y^2+c_1z+c_2z^2}\right)t}. \tag{45}$$

4 Conclusion

In this study, LPRDTM, which is combining Laplace-Padé resummation method as a useful technique to find exact solutions and the RDTM, has been successfully applied to different types of (1+3) dimensional wave equations. Our obtained results show that LPRDTM gives exact solutions of the equations by using only two iterations. Hence, LPRDTM is useful to ease CPU load and it helps us to have higher accuracy, efficiency and perfect harmony for solutions. Additionally, we point out that LPRDTM is very powerful and easy applicable mathematical tool for PDEs.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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