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Some applications of generalized open sets via operations

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Abstract: This paper introduces the concept of an operation on τ_g. Using this operation, we define the concept of g-γ-open sets, and study some of their related notions. Also, we introduce the concept of $g\gamma$-generalized closed sets and then investigate some of its properties. Furthermore, we introduce and investigate g-γ-T_i spaces ($i \in \{0, \frac{1}{2}, 1, 2\}$) and g-(γ, β)-continuous functions by utilizing the operation γ on τ_g. Finally, some basic properties of functions with g-β-closed graphs have been obtained.

Keywords: g-γ-open sets, $g\gamma g$-closed sets, g-γ-T_i spaces ($i \in \{0, \frac{1}{2}, 1, 2\}$), g-(γ, β)-continuous functions, g-β-closed graphs.

1 Introduction

Generalized open sets play a very important role in general topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in general topology and real analysis concerns the variously modified forms of continuity, separation axioms compactness, connectedness etc. by utilizing generalized open sets.

Generalized closed (g-closed) sets in a topological space were introduced by Levine [5] in order to extend many of the important properties of closed sets to a larger family. For instance, it was shown that compactness, normality, and completeness in a uniform space are inherited by g-closed subsets.

Kasahara [6] defined the concept of an operation on τ and introduced the concept of α-closed graphs of functions. After the work of Kasahara, Jankovic [4] defined the concept of operation-closures of α and investigated function with strongly closed graph. Ogata [8] defined and studied the concept of operation-open sets (γ-open sets), and used it to investigate operation-separation axioms and operation-functions.

In recent years, many concepts of operation γ in a topological space (X, τ) have been developed. An, Cuong and Maki [1] developed an operation γ on the collection of all preopen subsets of (X, τ) to introduce the notion of pre-γ-open sets. Krishnan, Ganster and Balachandran [7] defined and investigated the concept of the mapping γ on the collection of all semiopen subsets of (X, τ), and introduced the notion of semi γ-open sets and studied some of their properties. Tahiliani [9] developed an operation γ on the collection of all β-open subsets of (X, τ) to describe the notion of β-γ-open sets and Carpintero, Rajesh and Rosas [2] developed an operation γ on the collection of all b-open subsets of (X, τ) to define the notion of b-γ-open sets.

The aim of this paper is to introduce the concept of an operation γ on τ_g and to define the notion of g-γ-open sets of (X, τ) by using the operation γ on τ_g. Also, some notions of g-γ-open sets with their relationships are studied. In Section

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4, we introduce the concept of $g\gamma$-generalized closed sets and then investigate some of its properties. In Section 5, g-γ-T_i spaces where $i \in \{0, \frac{1}{2}, 1, 2\}$ by utilizing the operation γ on τ_g are introduced and investigated. In the last two sections, some basic properties of g-(γ,β)-continuous functions with g-β-closed graphs have been obtained.

2 Preliminaries

Throughout this paper, the space (X,τ) (or simply X) always mean topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, the closure of A is denoted by $Cl(A)$. A subset A of a topological space (X,τ) is said to be generalized closed (briefly g-closed) [5] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an open set in X. The complement of a g-closed set of X is g-open. The family of all g-open subsets of a space (X,τ) is denoted by τ_g. In general, every closed set of a space X is g-closed. A space (X,τ) is $T_{\frac{1}{2}}$ [5] if every g-closed subset of X is closed.

Definition 1.[8] *An operation γ on the topology τ on X is a mapping $\gamma\colon \tau \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U. A nonempty subset A of a topological space (X,τ) with an operation γ on τ is said to be γ-open if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$. The complement of a γ-open subset of a space X as γ-closed. The family of all γ-open subsets of a space (X,τ) is denoted by τ_γ.*

Definition 2. *[4] A point $x \in X$ is in the γ-closure of a set $A \subseteq X$ if $\gamma(U) \cap A \neq \phi$ for each open set U containing x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $Cl_\gamma(A)$.*

Definition 3. *[8] A subset A of (X,τ) with an operation γ on τ is said to be γ-g-closed if $Cl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is γ-open in (X,τ).*

Definition 4. *[8] A topological space (X,τ) with an operation γ on τ is said to be*

1. γ-T_0 if for any two distinct points x,y in X, there exists an open set U such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.
2. γ-T_1 if for any two distinct points x,y in X, there exist two open sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.
3. γ-T_2 if for any two distinct points x,y in X, there exist two open sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \phi$.
4. γ-$T_{\frac{1}{2}}$ if every γ-g-closed set in X is γ-closed.

Theorem 1. *[5] If a topological space (X,τ) is $T_{\frac{1}{2}}$, then $\tau_g = \tau$.*

3 g-γ-Open Sets

Definition 5. *An operation γ on τ_g is a mapping $\gamma\colon \tau_g \to P(X)$ such that $U \subseteq \gamma(U)$ for every $U \in \tau_g$, where $P(X)$ is the power set of X and $\gamma(U)$ is the value of γ at U.*

From this definition, we can easy to find $\gamma(X) = X$ for any operation $\gamma\colon \tau_g \to P(X)$.

Definition 6. *Let (X,τ) be a topological space and $\gamma\colon \tau_g \to P(X)$ be an operation on τ_g. A nonempty set A of X is said to be g-γ-open if for each $x \in A$, there exists a g-open set U such that $x \in U$ and $\gamma(U) \subseteq A$. The complement of a g-γ-open set of X is g-γ-closed. Assume that the empty set ϕ is also g-γ-open set for any operation $\gamma\colon \tau_g \to P(X)$. The family of all g-γ-open subsets of a space (X,τ) is denoted by $\tau_{g\gamma}$.*

Theorem 2. *The union of any collection of g-γ-open sets in a topological space X is a g-γ-open.*

Proof. Let $x \in \bigcup_{\lambda \in \Lambda}\{A_\lambda\}$, where $\{A_\lambda\}_{\lambda \in \Lambda}$ be a class of g-γ-open sets in X. Then $x \in A_\lambda$ for some $\lambda \in \Lambda$. Since A_λ is g-γ-open set in X, then there exists a g-open set V such that $x \in V \subseteq \gamma(V) \subseteq A_\lambda \subseteq \bigcup_{\lambda \in \Lambda}\{A_\lambda\}$. Therefore, $\bigcup_{\lambda \in \Lambda}\{A_\lambda\}$ is g-γ-open set in X.

Example 1. The intersection of any two g-γ-open sets in (X,τ) is generally not a g-γ-open sets. To see this, let $X = \{a,b,c\}$ and $\tau = P(X) = \tau_g$. Let $\gamma\colon\ \tau_g \to P(X)$ be an operation on τ_g defined as follows:
For every $A \in \tau_g$

$$\gamma(A) = \begin{cases} A & \text{if } A \neq \{c\} \\ \{b,c\} & \text{if } A = \{c\} \end{cases}$$

Thus, $\tau_{g\gamma} = P(X)\backslash\{c\}$. Then $\{a,c\} \in \tau_{g\gamma}$ and $\{b,c\} \in \tau_{g\gamma}$, but $\{a,c\} \cap \{b,c\} = \{c\} \notin \tau_{g\gamma}$.

Remark. Since the union of two g-open sets is generally not a g-open set. So the concept of g-open set and g-γ-open set are independent (That is, $\tau_g \neq \tau_{g\gamma}$). It is shown by the following two examples.

Example 2. In 1, the set $\{c\}$ is g-open, but it is not g-γ-open.

Example 3. Let $X = \{a,b,c\}$ and $\tau = \{\phi, X, \{a\}\}$. Then $\tau_g = P(X)\backslash\{b,c\}$. Define an operation $\gamma\colon\ \tau_g \to P(X)$ by $\gamma(A) = A$ for all $A \in \tau_g$. Here, $\tau_{g\gamma} = P(X)$. Then the set $\{b,c\}$ is g-γ-open, but $\{b,c\}$ is not a g-open set.

Definition 7. *A topological space (X,τ) with an operation γ on τ_g is said to be g-γ-regular if for each $x \in X$ and for each g-open set U containing x, there exists a g-open set W such that $x \in W$ and $\gamma(W) \subseteq U$.*

Theorem 3. *Let (X,τ) be a topological space and $\gamma\colon\ \tau_g \to P(X)$ be an operation on τ_g. Then the following conditions are equivalent:*

1.$\tau_g \subseteq \tau_{g\gamma}$.
2.(X,τ) is a g-γ-regular space.
3.For every $x \in X$ and for every g-open set U of (X,τ) containing x, there exists a g-γ-open set W of (X,τ) containing x such that $W \subseteq U$.

*Proof.*1.$\Rightarrow$ (2) Let $x \in X$ and U be a g-open set in X such that $x \in U$. It follows from assumption that U is a g-γ-open set. This implies that there exists a g-open set W such that $x \in W$ and $\gamma(W) \subseteq U$. Therefore, the space (X,τ) is g-γ-regular.

2.$\Rightarrow$ (3) Let $x \in X$ and U be a g-open set in (X,τ) containing x. Then by (2), there is a g-open set W such that $x \in W \subseteq \gamma(W) \subseteq U$. Again, by using (2) for the set W, it is shown that W is g-γ-open. Hence W is a g-γ-open set containing x such that $W \subseteq U$.

3.$\Rightarrow$ (1) By applying the part (3) and 2, it follows that every g-open set of X is g-γ-open in X. That is, $\tau_g \subseteq \tau_{g\gamma}$.

Remark. Since every open set is g-open. Then by 6 and 1, every γ-open set is g-γ-open (this means that $\tau_\gamma \subseteq \tau_{g\gamma}$), but the converse is not true in general. For instance, in 3, we have $\tau_\gamma = \tau$. Therefore, the set $\{b\} \in \tau_{g\gamma}$, but the set $\{b\} \notin \tau_\gamma$.

Lemma 1. *If the space (X,τ) is $T_{\frac{1}{2}}$, then the concept of g-γ-open set and γ-open set coincide (That is $\tau_{g\gamma} = \tau_\gamma$).*

Proof. Follows from their definitions and 1.

Definition 8. *Let (X,τ) be any topological space. An operation γ on τ_g is said to be*

1.g-open if for each $x \in X$ and for every g-open set U containing x, there exists a g-γ-open set W containing x such that $W \subseteq \gamma(U)$.
2.g-regular if for each $x \in X$ and for every pair of g-open sets U_1 and U_2 such that both containing x, there exists a g-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$.

Proposition 1. *Let a mapping γ be g-regular operation on τ_g. If the subsets A and B are g-γ-open in a topological space (X,τ), then $A \cap B$ is also g-γ-open set in (X,τ).*

Proof. Suppose $x \in A \cap B$ for any g-γ-open subsets A and B in (X,τ) both containing x. Then there exist g-open sets U_1 and U_2 such that $x \in U_1 \subseteq A$ and $x \in U_2 \subseteq B$. Since γ is a g-regular operation on τ_g, then there exists a g-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2) \subseteq A \cap B$. Therefore, $A \cap B$ is g-γ-open set in (X,τ).

Remark. By applying 1, it is easy to show that $\tau_{g\gamma}$ forms a topology on X for any g-regular operation γ on τ_g.

Definition 9. *The point $x \in X$ is in the g-closure$_\gamma$ of a set A if $\gamma(U) \cap A \neq \phi$ for each g-open set U containing x. The set of all g-closure$_\gamma$ points of A is called g-closure$_\gamma$ of A and is denoted by $gCl_\gamma(A)$.*

Definition 10. *Let A be any subset of a topological space (X,τ) and γ be an operation on τ_g. The g-γ-closure of A is defined as the intersection of all g-γ-closed sets of X containing A and it is denoted by $g_\gamma Cl(A)$. That is,*
$g_\gamma Cl(A) = \bigcap\{F : A \subseteq F, X \backslash F \in \tau_{g\gamma}\}$.

Theorem 4. *Let A be any subset of a topological space (X,τ) and γ be an operation on τ_g. Then $x \in g_\gamma Cl(A)$ if and only if $A \cap U \neq \phi$ for every g-γ-open set U of X containing x.*

Proof. Let $x \in g_\gamma Cl(A)$ and let $A \cap U = \phi$ for some g-γ-open set U of X containing x. Then $A \subseteq X \backslash U$ and $X \backslash U$ is g-γ-closed set in X. So $g_\gamma Cl(A) \subseteq X \backslash U$. Thus, $x \in X \backslash U$. This is a contradiction. Hence $A \cap U \neq \phi$ for every g-γ-open set U of X containing x.

Conversely, suppose that $x \notin g_\gamma Cl(A)$. So there exists a g-γ-closed set F such that $A \subseteq F$ and $x \notin F$. Then $X \backslash F$ is a g-γ-open set such that $x \in X \backslash F$ and $A \cap (X \backslash F) = \phi$. Contradiction of hypothesis. Therefore, $x \in g_\gamma Cl(A)$.

Lemma 2. *The following statements are true for any subsets A and B of a topological space (X,τ) with an operation γ on τ_g.*

1.$g_\gamma Cl(A)$ is g-γ-closed set in X and $gCl_\gamma(A)$ is g-closed set in X.
2.$A \subseteq gCl_\gamma(A) \subseteq g_\gamma Cl(A)$.
3.$g_\gamma Cl(\phi) = gCl_\gamma(\phi) = \phi$ and $g_\gamma Cl(X) = gCl_\gamma(X) = X$.
 (a)A is g-γ-closed if and only if $g_\gamma Cl(A) = A$ and,
 (b)A is g-γ-closed if and only if $gCl_\gamma(A) = A$.
4.If $A \subseteq B$, then $g_\gamma Cl(A) \subseteq g_\gamma Cl(B)$ and $gCl_\gamma(A) \subseteq gCl_\gamma(B)$.
 (a)$g_\gamma Cl(A \cap B) \subseteq g_\gamma Cl(A) \cap g_\gamma Cl(B)$ and,
 (b)$gCl_\gamma(A \cap B) \subseteq gCl_\gamma(A) \cap gCl_\gamma(B)$.
 (c)$g_\gamma Cl(A) \cup g_\gamma Cl(B) \subseteq g_\gamma Cl(A \cup B)$ and,
 (d)$gCl_\gamma(A) \cup gCl_\gamma(B) \subseteq gCl_\gamma(A \cup B)$.
5.$g_\gamma Cl(g_\gamma Cl(A)) = g_\gamma Cl(A)$.

Proof. Straightforward.

Theorem 5. *For any subsets A, B of a topological space (X,τ). If γ is a g-regular operation on τ_g, then*

1.$g_\gamma Cl(A) \cup g_\gamma Cl(B) = g_\gamma Cl(A \cup B)$.

2.$gCl_\gamma(A) \cup gCl_\gamma(B) = gCl_\gamma(A \cup B)$.

*Proof.*1.It is enough to proof that $g_\gamma Cl(A \cup B) \subseteq g_\gamma Cl(A) \cup g_\gamma Cl(B)$ since the other part follows directly from 2 (7). Let $x \notin g_\gamma Cl(A) \cup g_\gamma Cl(B)$. Then there exist two g-γ-open sets U and V containing x such that $A \cap U = \phi$ and $B \cap V = \phi$. Since γ is a g-regular operation on τ_g, then by 1, $U \cap V$ is g-γ-open in X such that

$$(U \cap V) \cap (A \cup B) = \phi.$$

Therefore, we have $x \notin g_\gamma Cl(A \cup B)$ and hence

$$g_\gamma Cl(A \cup B) \subseteq g_\gamma Cl(A) \cup g_\gamma Cl(B).$$

2.Let $x \notin gCl_\gamma(A) \cup gCl_\gamma(B)$. Then there exist g-open sets U_1 and U_2 such that $x \in U_1$, $x \in U_2$, $A \cap \gamma(U_1) = \phi$ and $A \cap \gamma(U_2) = \phi$. Since γ is a g-regular operation on τ_g, then there exists a g-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$. Thus, we have

$$(A \cup B) \cap \gamma(W) \subseteq (A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)).$$

This implies that $(A \cup B) \cap \gamma(W) = \phi$ since $(A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)) = \phi$. This means that $x \notin gCl_\gamma(A \cup B)$ and hence $gCl_\gamma(A \cup B) \subseteq gCl_\gamma(A) \cup gCl_\gamma(B)$. Using 2 (7), we have the equality.

Theorem 6. *Let A be any subset of a topological space (X, τ). If γ is a g-open operation on τ_g, then $gCl_\gamma(A) = g_\gamma Cl(A)$, $gCl_\gamma(gCl_\gamma(A)) = gCl_\gamma(A)$ and $gCl_\gamma(A)$ is g-γ-closed set in X.*

Proof. First we need to show that $g_\gamma Cl(A) \subseteq gCl_\gamma(A)$ since by 2 (2), we have $gCl_\gamma(A) \subseteq g_\gamma Cl(A)$. Now let $x \notin gCl_\gamma(A)$, then there exists a g-open set U containing x such that $A \cap \gamma(U) = \phi$. Since γ is a g-open on τ_g, then there exists a g-γ-open set W containing x such that $W \subseteq \gamma(U)$. So $A \cap W = \phi$ and hence by 4, $x \notin g_\gamma Cl(A)$. Therefore, $g_\gamma Cl(A) \subseteq gCl_\gamma(A)$. Hence $gCl_\gamma(A) = g_\gamma Cl(A)$. Moreover, using the above result and by 2 (8), we get $gCl_\gamma(gCl_\gamma(A)) = gCl_\gamma(A)$ and by 2 (4b), we obtain $gCl_\gamma(A)$ is g-γ-closed set in X.

Theorem 7. *Let A be any subset of a topological space (X, τ) and γ be an operation on τ_g. Then the following statements are equivalent.*

1. A is g-γ-open set.
2. $gCl_\gamma(X \backslash A) = X \backslash A$.
3. $g_\gamma Cl(X \backslash A) = X \backslash A$.
4. $X \backslash A$ is g-γ-closed set.

Proof. Clear.

Definition 11. *A subset N of a topological space (X, τ) is called a g-γ-neighbourhood of a point $x \in X$, if there exists a g-γ-open set U in X such that $x \in U \subseteq N$.*

Lemma 3. *Let $U \subseteq (X, \tau)$ be a g-γ-open if and only if it is a g-γ-neighbourhood of each of its points.*

Proof. Let U be any g-γ-open set in (X, τ). Then by 11, it is clear that U is a g-γ-neighbourhood of each of its points, since for every $x \in U$, $x \in U \subseteq U$ and $U \in \tau_{g\gamma}$.

Conversely, suppose U is a g-γ-neighbourhood of each of its points. Then for each $x \in U$, there exists a g-γ-open set V_x containing x such that $V_x \subseteq U$. Then $U = \bigcup_{x \in U} V_x$. Since each V_x is g-γ-open. It follows from 2 that U is g-γ-open set in X.

Definition 12. *Let A be any subset of a topological space (X,τ) and γ be an operation on τ_g. The g-γ-interior of A is defined as the union of all g-γ-open sets of X contained in A and it is denoted by $g_\gamma Int(A)$. That is,*

$$g_\gamma Int(A) = \bigcup\{U : U \in \tau_{g\gamma} \text{ and } U \subseteq A\}.$$

Some important properties of g-γ-interior operator will be given in 4.

Lemma 4. *Let A and B be subset of a topological space (X,τ) and γ be an operation on τ_g. Then the following conditions hold.*

1. $g_\gamma Int(A)$ *is g-γ-open set in X and* $g_\gamma Int(A) \subseteq A$.
2. $g_\gamma Int(\phi) = \phi$ *and* $g_\gamma Int(X) = X$.
3. *A is g-γ-open if and only if* $g_\gamma Int(A) = A$.
4. *If $A \subseteq B$, then* $g_\gamma Int(A) \subseteq g_\gamma Int(B)$.
5. $g_\gamma Int(A \cap B) \subseteq g_\gamma Int(A) \cap g_\gamma Int(B)$.
6. $g_\gamma Int(A) \cup g_\gamma Int(B) \subseteq g_\gamma Int(A \cup B)$.
7. $g_\gamma Int(g_\gamma Int(A)) = g_\gamma Int(A)$.
8. $g_\gamma Int(X \backslash A) = X \backslash g_\gamma Cl(A)$.

Proof. Straightforward.

Theorem 8. *If γ is a g-regular operation on τ_g, then for any subsets A, B of a space X, we have*

$$g_\gamma Int(A) \cap g_\gamma Int(B) = g_\gamma Int(A \cap B).$$

Proof. Follows directly from 5 (1) and using 4 (8).

Lemma 5. *Let (X,τ) be a topological space and γ be a g-regular operation on τ_g. Then $g_\gamma Cl(A) \cap U \subseteq g_\gamma Cl(A \cap U)$ holds for every g-γ-open set U and every subset A of X.*

Proof. Suppose that $x \in g_\gamma Cl(A) \cap U$ for every g-γ-open set U, then $x \in g_\gamma Cl(A)$ and $x \in U$. Let V be any g-γ-open set of X containing x. Since γ is g-regular on τ_g. So by 1, $U \cap V$ is g-γ-open set containing x. Since $x \in g_\gamma Cl(A)$, then by 4, we have $A \cap (U \cap V) \neq \phi$. This means that $(A \cap U) \cap V \neq \phi$. Therefore, again by 4, we obtain that $x \in g_\gamma Cl(A \cap U)$. Thus, $g_\gamma Cl(A) \cap U \subseteq g_\gamma Cl(A \cap U)$.

The proof of the following lemma is similar to 5 and using 4 (8).

Lemma 6. *Let (X,τ) be a topological space and γ be a g-regular operation on τ_g. Then $g_\gamma Int(A \cup F) \subseteq g_\gamma Int(A) \cup F$ holds for every g-γ-closed set F and every subset A of X.*

4 $g\gamma g$-closed sets

Definition 13. *A subset A of a topological space (X,τ) with an operation γ on τ_g is said to be $g\gamma$-generalized closed (briefly $g\gamma g$-closed) if $gCl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is a g-γ-open set in X.*

Lemma 7. *Let (X,τ) be a topological space and γ be an operation on τ_g. A set A in (X,τ) is $g\gamma g$-closed if and only if $A \cap g_\gamma Cl(\{x\}) \neq \phi$ for every $x \in gCl_\gamma(A)$.*

Proof. Suppose A is $g\gamma g$-closed set in X and suppose (if possible) that there exists an element $x \in gCl_\gamma(A)$ such that $A \cap g_\gamma Cl(\{x\}) = \phi$. This follows that $A \subseteq X \backslash g_\gamma Cl(\{x\})$. Since $g_\gamma Cl(\{x\})$ is g-γ-closed implies $X \backslash g_\gamma Cl(\{x\})$ is g-γ-open and A is $g\gamma g$-closed set in X. Then, we have that $gCl_{\gamma}(A) \subseteq X \backslash g_{\gamma}Cl(\{x\})$. This means that $x \notin gCl_{\gamma}(A)$. This is a contradiction. Hence $A \cap g_{\gamma}Cl(\{x\}) \neq \phi$.

Conversely, let $U \in \tau_{g\gamma}$ such that $A \subseteq U$. To show that $gCl_{\gamma}(A) \subseteq U$. Let $x \in gCl_{\gamma}(A)$. Then by hypothesis, $A \cap g_{\gamma}Cl(\{x\}) \neq \phi$. So there exists an element $y \in A \cap g_{\gamma}Cl(\{x\})$. Thus $y \in A \subseteq U$ and $y \in g_{\gamma}Cl(\{x\})$. By 4, $\{x\} \cap U \neq \phi$. Hence $x \in U$ and so $gCl_{\gamma}(A) \subseteq U$. Therefore, A is $g\gamma g$-closed set in (X,τ).

Theorem 9. *Let A be a subset of topological space (X,τ) and γ be an operation on τ_g. If A is $g\gamma g$-closed, then $gCl_{\gamma}(A) \backslash A$ does not contain any non-empty g-γ-closed set.*

Proof. Let F be a non-empty g-γ-closed set in X such that $F \subseteq gCl_{\gamma}(A) \backslash A$. Then $F \subseteq X \backslash A$ implies $A \subseteq X \backslash F$. Since $X \backslash F$ is g-γ-open set and A is $g\gamma g$-closed set, then $gCl_{\gamma}(A) \subseteq X \backslash F$. That is $F \subseteq X \backslash gCl_{\gamma}(A)$. Hence $F \subseteq X \backslash gCl_{\gamma}(A) \cap gCl_{\gamma}(A) \backslash A \subseteq X \backslash gCl_{\gamma}(A) \cap gCl_{\gamma}(A) = \phi$. This shows that $F = \phi$. This is contradiction. Therefore, $F \nsubseteq gCl_{\gamma}(A) \backslash A$.

Theorem 10. *If $\gamma\colon \tau_g \to P(X)$ is a g-open operation, then the converse of the 9 is true.*

Proof. Let U be a g-γ-open set in (X,τ) such that $A \subseteq U$. Since $\gamma\colon \tau_g \to P(X)$ is a g-open operation, then by 6, $gCl_{\gamma}(A)$ is g-γ-closed set in X. Thus, using 2, we have $gCl_{\gamma}(A) \cap X \backslash U$ is a g-γ-closed set in (X,τ). Since $X \backslash U \subseteq X \backslash A$, $gCl_{\gamma}(A) \cap X \backslash U \subseteq gCl_{\gamma}(A) \backslash A$. Using the assumption of the converse of the 9, $gCl_{\gamma}(A) \subseteq U$. Therefore, A is $g\gamma g$-closed set in (X,τ).

Corollary 1. *Let A be a $g\gamma g$-closed subset of topological space (X,τ) and let γ be an operation on τ_g. Then A is g-γ-closed if and only if $gCl_{\gamma}(A) \backslash A$ is g-γ-closed set.*

Proof. Let A be a g-γ-closed set in (X,τ). Then by 2 (4b), $gCl_{\gamma}(A) = A$ and hence $gCl_{\gamma}(A) \backslash A = \phi$ which is g-γ-closed set.

Conversely, suppose $gCl_{\gamma}(A) \backslash A$ is g-γ-closed and A is $g\gamma g$-closed. Then by 9, $gCl_{\gamma}(A) \backslash A$ does not contain any non-empty g-γ-closed set and since $gCl_{\gamma}(A) \backslash A$ is g-γ-closed subset of itself, then $gCl_{\gamma}(A) \backslash A = \phi$ implies $gCl_{\gamma}(A) \cap X \backslash A = \phi$. Hence $gCl_{\gamma}(A) = A$. This follows from 2 (4b) that A is g-γ-closed set in (X,τ).

Theorem 11. *Let (X,τ) be a topological space and γ be an operation on τ_g. If a subset A of X is $g\gamma g$-closed and g-γ-open, then A is g-γ-closed.*

Proof. Since A is $g\gamma g$-closed and g-γ-open set in X, then $gCl_{\gamma}(A) \subseteq A$ and hence by 2 (4b), A is g-γ-closed.

Theorem 12. *In any topological space (X,τ) with an operation γ on τ_g. For an element $x \in X$, the set $X \backslash \{x\}$ is $g\gamma g$-closed or g-γ-open.*

Proof. Suppose that $X \backslash \{x\}$ is not g-γ-open. Then X is the only g-γ-open set containing $X \backslash \{x\}$. This implies that $gCl_{\gamma}(X \backslash \{x\}) \subseteq X$. Thus $X \backslash \{x\}$ is a $g\gamma g$-closed set in X.

Corollary 2. *In any topological space (X,τ) with an operation γ on τ_g. For an element $x \in X$, either the set $\{x\}$ is g-γ-closed or the set $X \backslash \{x\}$ is $g\gamma g$-closed.*

Proof. Suppose $\{x\}$ is not g-γ-closed, then $X \backslash \{x\}$ is not g-γ-open. Hence by 12, $X \backslash \{x\}$ is $g\gamma g$-closed set in X.

Definition 14. *Let A be any subset of a topological space (X,τ) and γ be an operation on τ_g. Then the $\tau_{g\gamma}$-kernel of A is denoted by $\tau_{g\gamma}$-ker(A) and is defined as follows.*

$$\tau_{g\gamma} - ker(A) = \cap\{U : A \subseteq U and U \in \tau_{g\gamma}\}.$$

In other words, $\tau_{g\gamma}$-ker(A) is the intersection of all g-γ-open sets of (X,τ) containing A.

Theorem 13. *Let $A \subseteq (X,\tau)$ and γ be an operation on τ_g. Then A is $g\gamma g$-closed if and only if $gCl_\gamma(A) \subseteq \tau_{g\gamma}$-ker(A).*

Proof. Suppose that A is $g\gamma g$-closed. Then $gCl_\gamma(A) \subseteq U$, whenever $A \subseteq U$ and U is g-γ-open. Let $x \in gCl_\gamma(A)$. Then by 7, $A \cap g_\gamma Cl(\{x\}) \neq \phi$. So there exists a point z in X such that $z \in A \cap g_\gamma Cl(\{x\})$ implies that $z \in A \subseteq U$ and $z \in g_\gamma Cl(\{x\})$. By 4, $\{x\} \cap U \neq \phi$. Hence we show that $x \in \tau_{g\gamma}$-ker(A). Therefore, $gCl_\gamma(A) \subseteq \tau_{g\gamma}$-ker($A$).

Conversely, let $gCl_\gamma(A) \subseteq \tau_{g\gamma}$-ker($A$). Let U be any g-γ-open set containing A. Let x be a point in X such that $x \in gCl_\gamma(A)$. Then $x \in \tau_{g\gamma}$-ker(A). Namely, we have $x \in U$, because $A \subseteq U$ and $U \in \tau_{g\gamma}\}$. That is $gCl_\gamma(A) \subseteq \tau_{g\gamma}$-ker($A$)$\subseteq U$. Therefore, A is $g\gamma g$-closed set in X.

5 g-γ-T_i Spaces for $i \in \{0, \frac{1}{2}, 1, 2\}$

In this section, we introduce some types of g-γ- separation axioms called g-γ-T_i for $i \in \{0, \frac{1}{2}, 1, 2\}$. Some basic properties of these spaces are investigated.

Definition 15. *A topological space (X,τ) with an operation γ on τ_g is said to be g-γ-T_0 if for any two distinct points x, y in X, there exists a g-open set U such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.*

Definition 16. *A topological space (X,τ) with an operation γ on τ_g is said to be g-γ-T_1 if for any two distinct points x, y in X, there exist two g-open sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.*

Definition 17. *A topological space (X,τ) with an operation γ on τ_g is said to be g-γ-T_2 if for any two distinct points x, y in X, there exist two g-open sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \phi$.*

Definition 18. *A topological space (X,τ) with an operation γ on τ_g is said to be g-γ-$T_{\frac{1}{2}}$ if every $g\gamma g$-closed set in X is g-γ-closed set.*

Theorem 14. *For any topological space (X,τ) with an operation γ on τ_g. Then (X,τ) is g-γ-$T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is g-γ-closed or g-γ-open.*

Proof. Let X be a g-γ-$T_{\frac{1}{2}}$ space and let $\{x\}$ is not g-γ-closed set in (X,τ). By 2, $X\backslash\{x\}$ is $g\gamma g$-closed. Since (X,τ) is g-γ-$T_{\frac{1}{2}}$, then $X\backslash\{x\}$ is g-γ-closed set which means that $\{x\}$ is g-γ-open set in X.

Conversely, let F be any $g\gamma g$-closed set in the space (X,τ). We have to show that F is g-γ-closed (that is $gCl_\gamma(F) = F$ (by 2 (4b))). It is sufficient to show that $gCl_\gamma(F) \subseteq F$. Let $x \in gCl_\gamma(F)$. By hypothesis $\{x\}$ is g-γ-closed or g-γ-open for each $x \in X$. So we have two cases.

Case 1. If $\{x\}$ is g-γ-closed set. Suppose $x \notin F$, then $x \in gCl_\gamma(F)\backslash F$ contains a non-empty g-γ-closed set $\{x\}$. A contradiction since F is $g\gamma g$-closed set and according to the 9. Hence $x \in F$. This follows that $gCl_\gamma(F) \subseteq F$ and hence $gCl_\gamma(F) = F$. This means from by 2 (4b) that F is g-γ-closed set in (X,τ). Thus (X,τ) is g-γ-$T_{\frac{1}{2}}$ space.

Case 1. If $\{x\}$ is g-γ-open set. Then by 4, $F \cap \{x\} \neq \phi$ which implies that $x \in F$. So $gCl_\gamma(F) \subseteq F$. Thus by 2 (4b), F is g-γ-closed. Therefore, (X,τ) is g-γ-$T_{\frac{1}{2}}$ space.

Theorem 15. *Let γ be a g-open operation on τ_g. Then (X,τ) is a g-γ-T_0 space if and only if $gCl_\gamma(\{x\}) \neq gCl_\gamma(\{y\})$, for every pair x, y of X with $x \neq y$.*

Proof. **Necessary Part.** Let x, y be any two distinct points of a g-γ-T_0 space (X,τ). Then by definition, we assume that there exists a g-γ-open set U such that $x \in U$ and $y \notin \gamma(U)$. Since γ is a g-open operation on τ_g, then there exists a g-γ-open set W such that $x \in W$ and $W \subseteq \gamma(U)$. Hence $y \in X\backslash\gamma(U) \subseteq X\backslash W$. Since $X\backslash W$ is a g-γ-closed set in (X,τ). Then we obtain that $gCl_\gamma(\{y\}) \subseteq X\backslash W$ and therefore $gCl_\gamma(\{x\}) \neq gCl_\gamma(\{y\})$.

Sufficient Part. Suppose for any $x,y \in X$ with $x \neq y$, we have $gCl_\gamma(\{x\}) \neq gCl_\gamma(\{y\})$. Now, we assume that there exists $z \in X$ such that $z \in gCl_\gamma(\{x\})$, but $z \notin gCl_\gamma(\{y\})$. If $x \in gCl_\gamma(\{y\})$, then $\{x\} \subseteq gCl_\gamma(\{y\})$, which implies that $gCl_\gamma(\{x\}) \subseteq gCl_\gamma(\{y\})$ (by 2 (5)). This implies that $z \in gCl_\gamma(\{y\})$. This contradiction shows that $x \notin gCl_\gamma(\{y\})$. This means that by 9, there exists a g-open set U such that $x \in U$ and $\gamma(U) \cap \{y\} = \phi$. Thus, we have that $x \in U$ and $y \notin \gamma(U)$. It gives that the space (X,τ) is g-γ-T_0.

Theorem 16. *The space (X,τ) is g-γ-T_1 if and only if for every point $x \in X$, $\{x\}$ is a g-γ-closed set.*

Proof. **Necessary Part.** Let x be a point of a g-γ-T_1 space (X,τ). Then for any point $y \in X$ such that $x \neq y$, there exists a g-open set V_y such that $y \in V_y$ but $x \notin \gamma(V_y)$. Thus, $y \in \gamma(V_y) \subseteq X\backslash\{x\}$. This implies that $X\backslash\{x\} = \cup\{\gamma(V_y) : y \in X\backslash\{x\}\}$. It is shown that $X\backslash\{x\}$ is g-γ-open set in (X,τ). Hence $\{x\}$ is g-γ-closed set in (X,τ).

Sufficient Part. Let $x,y \in X$ such that $x \neq y$. By hypothesis, we get $X\backslash\{y\}$ and $X\backslash\{x\}$ are g-γ-open sets such that $x \in X\backslash\{y\}$ and $y \in X\backslash\{x\}$. Therefore, there exist g-open sets U and V such that $x \in U$, $y \in V$, $\gamma(U) \subseteq X\backslash\{y\}$ and $\gamma(V) \subseteq X\backslash\{x\}$. So, $y \notin \gamma(U)$ and $x \notin \gamma(V)$. This implies that (X,τ) is g-γ-T_1.

Theorem 17. *For any topological space (X,τ) and any operation γ on τ_g, the following properties hold.*

1. Every g-γ-T_2 space is g-γ-T_1.
2. Every g-γ-T_1 space is g-γ-$T_{\frac{1}{2}}$.
3. Every g-γ-$T_{\frac{1}{2}}$ space is g-γ-T_0.

Proof. The proofs are obvious by their definitions.

Remark. By 17, 3 and [8], we obtain the following diagram of implications. Moreover, the following **??** below show that the reverse implications are not true in general.

$$g-\gamma-T_2 \to g-\gamma-T_1 \to g-\gamma-T_{\frac{1}{2}} \to g-\gamma-T_0$$
$$\gamma-T_2 \to \gamma-T_1 \to \gamma-T_{\frac{1}{2}} \to \gamma-T_0$$

Example 4. Let $X = \{a,b,c\}$ and $\tau = \{\phi,X\}$. Then $\tau_g = P(X)$. Define an operation γ: $\tau_g \to P(X)$ by $\gamma(A) = A$ for all $A \in \tau_g$. Here, $\tau_{g\gamma} = P(X)$ and $\tau = \tau_\gamma$. Then the space (X,τ) is g-γ-T_i ($i \in \{0,\frac{1}{2},1,2\}$), but not γ-T_i ($i \in \{0,\frac{1}{2},1,2\}$).

Example 5. Let $X = \{a,b,c\}$ and $\tau = \{\phi,X,\{a\},\{a,b\}\}$. Then $\tau_g = \{\phi,X,\{a\},\{b\},\{a,b\}\}$. Let γ: $\tau_g \to P(X)$ be an operation on τ_g defined as follows. For every set $A \in \tau_g$

$$\gamma(A) = \begin{cases} A & \text{if } b \in A \\ Cl(A) & \text{if } b \notin A \end{cases}$$

Thus, $\tau_{g\gamma} = \{\phi,X,\{b\},\{a,b\}\}$. Then the space (X,τ) is g-γ-T_0, but it is not g-γ-$T_{\frac{1}{2}}$. Since $\{b,c\}$ is $g\gamma g$-closed set in (X,τ), but $\{b,c\}$ is not g-γ-closed set in (X,τ).

Example 6. Let $X = \{a,b,c\}$ and $\tau = \{\phi,X,\{a\},\{b\},\{a,b\},\{a,c\}\}$. Then $\tau_g = \tau$. Let γ: $\tau_g \to P(X)$ be an operation on τ_g defined as follows:

For every set $A \in \tau_g$

$$\gamma(A) = \begin{cases} A, & \text{if } a \in A \\ Cl(A), & \text{if } a \notin A \end{cases}$$

Thus, $\tau_{g\gamma} = \tau$. Therefore, the space (X,τ) is g-γ-$T_{\frac{1}{2}}$, but it is not g-γ-T_1.

Example 7. Suppose $X = \{a,b,c\}$ and τ be the discrete topology on X. Define an operation γ on τ_g as follows. For every $A \in \tau_g$

$$\gamma(A) = \begin{cases} A, & \text{if } A = \{a,b\} \text{ or } \{a,c\} \text{ or } \{b,c\} \\ X, & \text{otherwise} \end{cases}$$

Then (X,τ) is g-γ-T_1 space, but (X,τ) is not g-γ-T_2.

Lemma 8. *Let (X,τ) be a $T_{\frac{1}{2}}$ space. Then (X,τ) is g-γ-T_i if and only if it is γ-T_i, where $i \in \{0,1,2\}$.*

6 g-(γ,β)-Continuous Functions

Throughout Section 6 and Section 7, let (X,τ) and (Y,σ) be two topological spaces and let $\gamma\colon \tau_g \to P(X)$ and $\beta\colon \sigma_g \to P(Y)$ be operations on τ_g and σ_g respectively. In this section, we introduce a new class of functions called g-(γ,β)-continuous. Some characterizations and properties of this function are investigated.

Definition 19. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be g-(γ,β)-continuous if for each $x \in X$ and each g-open set V containing $f(x)$, there exists a g-open set U containing x such that $f(\gamma(U)) \subseteq \beta(V)$.*

Theorem 18. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a g-(γ,β)-continuous function, then,*

1. $f(gCl_\gamma(A)) \subseteq gCl_\beta(f(A))$, for every $A \subseteq (X,\tau)$.
2. $f^{-1}(F)$ is g-γ-closed set in (X,τ), for every g-β-closed set F of (Y,σ).

*Proof.*1. Let $y \in f(gCl_\gamma(A))$ and V be any g-open set containing y. Then by hypothesis, there exists $x \in X$ and g-open set U containing x such that $f(x) = y$ and $f(\gamma(U)) \subseteq \beta(V)$. Since $x \in gCl_\gamma(A)$, we have $\gamma(U) \cap A \neq \phi$. Hence $\phi \neq f(\gamma(U) \cap A) \subseteq f(\gamma(U)) \cap f(A) \subseteq \beta(V) \cap f(A)$. This implies that $y \in gCl_\beta(f(A))$. Therefore, $f(gCl_\gamma(A)) \subseteq gCl_\beta(f(A))$.
2. Let F be any g-β-closed set of (Y,σ). By using (1), we have $f(gCl_\gamma(f^{-1}(F))) \subseteq gCl_\beta(F) = F$. Therefore, $gCl_\gamma(f^{-1}(F)) = f^{-1}(F)$. Hence $f^{-1}(F)$ is g-γ-closed set in (X,τ).

Theorem 19. *In 18, the properties of g-(γ,β)-continuity of f, (1) and (2) are equivalent to each other if either the space (Y,σ) is g-β-regular or the operation β is g-open.*

Proof. It follows from the proof of 18 that we know the following implications: "g-(γ,β)-continuity of f" $\Rightarrow$ (1) $\Rightarrow$ (2). Thus, when the space (Y,σ) is g-β-regular, we prove the implication: (2) $\Rightarrow$ g-(γ,β)-continuity of f. Let $x \in X$ and let $V \in \sigma_g$ such that $f(x) \in V$. Since (Y,σ) is a g-β-regular space, then by 3, $V \in \sigma_{g\beta}$. By using (2) of 18, $f^{-1}(V) \in \tau_{g\gamma}$ such that $x \in f^{-1}(V)$. So there exists a g-open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(V)$. This implies that $f(\gamma(U)) \subseteq V \subseteq \beta(V)$. Therefore, f is g-(γ,β)-continuous.

Now, when β is a g-open operation, we show the implication: (2) $\Rightarrow$ g-(γ,β)-continuity of f. Let $x \in X$ and let $V \in \sigma_g$ such that $f(x) \in V$. Since β is a g-open operation, then there exists $W \in \sigma_{g\beta}$ such that $f(x) \in W$ and $W \subseteq \beta(V)$. By using (2) of 18, $f^{-1}(W) \in \tau_{g\gamma}$ such that $x \in f^{-1}(W)$. So there exists a g-open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(W) \subseteq f^{-1}(\beta(V))$. This implies that $f(\gamma(U)) \subseteq \beta(V)$. Hence f is g-(γ,β)-continuous.

Definition 20. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be*

1. g-(γ,β)-closed if the image of each g-γ-closed set of X is g-β-closed in Y.
2. g-β-closed if the image of each g-closed set of X is g-β-closed in Y.

Theorem 20. *Suppose that a function $f: (X,\tau) \to (Y,\sigma)$ is both g-(γ,β)-continuous and g-β-closed,then,*

1. For every $g\gamma g$-closed set A of (X,τ), the image $f(A)$ is $g\beta g$-closed in (Y,σ).
2. If (X,τ) is $T_{\frac{1}{2}}$, then the inverse set $f^{-1}(B)$ is $g\gamma g$-closed in (X,τ), for every $g\beta g$-closed set B of (Y,σ).

*Proof.*1. Let G be any g-β-open set in (Y,σ) such that $f(A) \subseteq G$. Since f is g-(γ,β)-continuous function, then by using 18 (2), $f^{-1}(G)$ is g-γ-open set in (X,τ). Since A is $g\gamma g$-closed and $A \subseteq f^{-1}(G)$, we have $gCl_{\gamma}(A) \subseteq f^{-1}(G)$, and hence $f(gCl_{\gamma}(A)) \subseteq G$. Thus, by 2 (1), $gCl_{\gamma}(A)$ is g-closed set and since f is g-β-closed, then $f(gCl_{\gamma}(A))$ is g-β-closed set in Y. Therefore, $gCl_{\beta}(f(A)) \subseteq gCl_{\beta}(f(gCl_{\gamma}(A))) = f(gCl_{\gamma}(A)) \subseteq G$. This implies that $f(A)$ is $g\beta g$-closed in (Y,σ).
2. Let H be any g-γ-open set of a $T_{\frac{1}{2}}$ space (X,τ) such that $f^{-1}(B) \subseteq H$. Let $C = gCl_{\gamma}(f^{-1}(B)) \cap (X \backslash H)$, then by 7, $C = gCl_{\gamma}(f^{-1}(B)) \cap gCl_{\gamma}(X \backslash H)$ and hence by 2 (1) and 1, C is g-closed set in (X,τ). Since f is g-β-closed function. Then $f(C)$ is g-β-closed in (Y,σ). Since f is g-(γ,β)-continuous function, then by using 18 (1), we have $f(C) = f(gCl_{\gamma}(f^{-1}(B))) \cap f(X \backslash H) \subseteq gCl_{\beta}(B) \cap f(X \backslash H) \subseteq gCl_{\beta}(B) \cap (Y \backslash B) = gCl_{\beta}(B) \backslash B$. This implies from 9 that $f(C) = \phi$, and hence $C = \phi$. So $gCl_{\gamma}(f^{-1}(B)) \subseteq H$. Therefore, $f^{-1}(B)$ is $g\gamma g$-closed in (X,τ).

Theorem 21. *Let $f: (X,\tau) \to (Y,\sigma)$ be an injective, g-(γ,β)-continuous and g-β-closed function. If (Y,σ) is g-β-$T_{\frac{1}{2}}$, then (X,τ) is g-γ-$T_{\frac{1}{2}}$.*

Proof. Let G be any $g\gamma g$-closed set of (X,τ). Since f is g-(γ,β)-continuous and g-β-closed function. Then by 20 (1), $f(G)$ is $g\beta g$-closed in (Y,σ). Since (Y,σ) is g-β-$T_{\frac{1}{2}}$, then $f(G)$ is g-β-closed in Y. Again, since f is g-(γ,β)-continuous, then by 18 (2), $f^{-1}(f(G))$ is g-γ-closed in X. Hence G is g-γ-closed in X since f is injective. Therefore, (X,τ) is a g-γ-$T_{\frac{1}{2}}$ space.

Theorem 22. *Let a function $f: (X,\tau) \to (Y,\sigma)$ be surjective, g-(γ,β)-continuous and g-β-closed. If (X,τ) is g-γ-$T_{\frac{1}{2}}$, then (Y,σ) is g-β-$T_{\frac{1}{2}}$.*

Proof. Let H be a $g\beta g$-closed set of (Y,σ). Since f is g-(γ,β)-continuous and g-β-closed function. Then by 20 (2), $f^{-1}(H)$ is $g\gamma g$-closed in (X,τ). Since (X,τ) is g-γ-$T_{\frac{1}{2}}$, then we have, $f^{-1}(H)$ is g-γ-closed set in X. Again, since f is g-β-closed function, then $f(f^{-1}(H))$ is g-β-closed in Y. Therefore, H is g-β-closed in Y since f is surjective. Hence (Y,σ) is g-β-$T_{\frac{1}{2}}$ space.

Theorem 23. *If a function $f: (X,\tau) \to (Y,\sigma)$ is injective g-(γ,β)-continuous and the space (Y,σ) is g-β-T_2, then the space (X,τ) is g-γ-T_2.*

Proof. Let x_1 and x_2 be any distinct points of a space (X,τ). Since f is an injective function and (Y,σ) is g-β-T_2. Then there exist two g-open sets U_1 and U_2 in Y such that $f(x_1) \in U_1$, $f(x_2) \in U_2$ and $\beta(U_1) \cap \beta(U_2) = \phi$. Since f is g-(γ,β)-continuous, there exist g-open sets V_1 and V_2 in X such that $x_1 \in V_1$, $x_2 \in V_2$, $f(\gamma(V_1)) \subseteq \beta(U_1)$ and $f(\gamma(V_2)) \subseteq \beta(U_2)$. Therefore $\beta(U_1) \cap \beta(U_2) = \phi$. Hence (X,τ) is g-γ-T_2.

Theorem 24. *If a function $f: (X,\tau) \to (Y,\sigma)$ is injective g-(γ,β)-continuous and the space (Y,σ) is g-β-T_i, then the space (X,τ) is g-γ-T_i for $i \in \{0,1\}$.*

Proof. The proof is similar to 23.

Definition 21. *A function $f: (X,\tau) \to (Y,\sigma)$ is said to be g-(γ,β)-homeomorphism if f is bijective, g-(γ,β)-continuous and f^{-1} is g-(β,γ)-continuous.*

Theorem 25. *Assume that a function $f: (X,\tau) \to (Y,\sigma)$ is g-(γ,β)-homeomorphism. If (X,τ) is g-γ-$T_{\frac{1}{2}}$, then (Y,σ) is g-β-$T_{\frac{1}{2}}$.*

Proof. Let $\{y\}$ be any singleton set of (Y,σ). Then there exists an element x of X such that $y = f(x)$. So by hypothesis and 14, we have $\{x\}$ is g-γ-closed or g-γ-open set in X. By using 18, $\{y\}$ is g-β-closed or g-β-open set. Hence the space by 14, (Y,σ) is g-β-$T_{\frac{1}{2}}$.

7 Functions with $g-\beta$-closed graphs

For a function $f\colon (X,\tau)\to(Y,\sigma)$, the subset $\{(x,f(x)) : x\in X\}$ of the product space $(X\times Y,\tau\times\sigma)$ is called the graph of f and is denoted by $G(f)$ [3]. In this section, we further investigate general operator approaches of closed graphs of functions. Let $\lambda\colon (\tau\times\sigma)_g\to P(X\times Y)$ be an operation on $(\tau\times\sigma)_g$.

Definition 22. *The graph $G(f)$ of $f\colon (X,\tau)\to(Y,\sigma)$ is called g-β-closed if for each $(x,y)\in(X\times Y)\backslash G(f)$, there exist g-open sets $U\subseteq X$ and $V\subseteq Y$ containing x and y, respectively, such that $(U\times\beta(V))\cap G(f)=\phi$.*

The proof of the following lemma follows directly from the above definition.

Lemma 9. *A function $f\colon (X,\tau)\to(Y,\sigma)$ has g-β-closed graph if and only if for each $(x,y)\in(X\times Y)\backslash G(f)$, there exist $U\in\tau_g$ containing x and $V\in\sigma_g$ containing y such that $f(U)\cap\beta(V)=\phi$.*

Definition 23. *An operation $\lambda\colon (\tau\times\sigma)_g\to P(X\times Y)$ is said to be g-associated with γ and β if $\lambda(U\times V)=\gamma(U)\times\beta(V)$ holds for each $U\in\tau_g$ and $V\in\sigma_g$.*

Definition 24. *The operation $\lambda\colon (\tau\times\sigma)_g\to P(X\times Y)$ is said to be g-regular with respect to γ and β if for each $(x,y)\in X\times Y$ and each g-open set W containing (x,y), there exist g-open sets U in X and V in Y such that $x\in U$, $y\in V$ and $\gamma(U)\times\beta(V)\subseteq\lambda(W)$.*

Theorem 26. *Let $\lambda\colon (\tau\times\tau)_g\to P(X\times X)$ be a g-associated operation with γ and γ. If $f\colon (X,\tau)\to(Y,\sigma)$ is a g-(γ,β)-continuous function and (Y,σ) is a g-β-T_2 space, then the set $A=\{(x,y)\in X\times X : f(x)=f(y)\}$ is a g-λ-closed set of $(X\times X,\tau\times\tau)$.*

Proof. We want to prove that $gCl_\lambda(A)\subseteq A$. Let $(x,y)\in(X\times X)\backslash A$. Since (Y,σ) is g-β-T_2. Then there exist two g-open sets U and V in (Y,σ) such that $f(x)\in U$, $f(y)\in V$ and $\beta(U)\cap\beta(V)=\phi$. Moreover, for U and V there exist g-open sets R and S in (X,τ) such that $x\in R$, $y\in S$ and $f(\gamma(R))\subseteq\beta(U)$ and $f(\gamma(S))\subseteq\beta(V)$ since f is g-(γ,β)-continuous. Therefore we have $(x,y)\in\gamma(R)\times\gamma(S)=\lambda(R\times S)\cap A=\phi$ because $R\times S\in(\tau\times\tau)_g$. This shows that $(x,y)\notin gCl_\lambda(A)$.

Corollary 3. *Suppose $\lambda\colon (\tau\times\tau)_g\to P(X\times X)$ is g-associated operation with γ and γ, and it is g-regular with γ and γ. A space (X,τ) is g-γ-T_2 if and only if the diagonal set $\triangle=\{(x,x) : x\in X\}$ is g-λ-closed of $(X\times X,\tau\times\tau)$.*

Theorem 27. *Let $\lambda\colon (\tau\times\sigma)_g\to P(X\times Y)$ be a g-associated operation with γ and β. If $f\colon (X,\tau)\to(Y,\sigma)$ is g-(γ,β)-continuous and (Y,σ) is g-β-T_2, then the graph of f, $G(f)=\{(x,f(x))\in X\times Y\}$ is a g-λ-closed set of $(X\times Y,\tau\times\sigma)$.*

Proof. The proof is similar to 26.

Definition 25. *Let (X,τ) be a topological space and γ be an operation on τ_g. A subset S of X is said to be g-γ-compact if for every g-open cover $\{U_i, i\in\mathbb{N}\}$ of S, there exists a finite subfamily $\{U_1,U_2,\ldots,U_n\}$ such that $S\subseteq\gamma(U_1)\cup\gamma(U_2)\cup\ldots\cup\gamma(U_n)$.*

Theorem 28. *Suppose that γ is g-regular and $\lambda\colon (\tau\times\sigma)_g\to P(X\times Y)$ is g-regular with respect to γ and β. Let $f\colon (X,\tau)\to(Y,\sigma)$ be a function whose graph $G(f)$ is g-λ-closed in $(X\times Y,\tau\times\sigma)$. If a subset S is g-$\beta$-compact in (Y,σ), then $f^{-1}(S)$ is g-γ-closed in (X,τ).*

Proof. Suppose that $f^{-1}(S)$ is not g-γ-closed then there exist a point x such that $x\in gCl_\gamma(f^{-1}(S))$ and $x\notin f^{-1}(S)$. Since $(x,s)\notin G(f)$ and each $s\in S$ and $gCl_\lambda(G(f))\subseteq G(f)$, there exists a g-open set W of $(X\times Y,\tau\times\sigma)$ such that $(x,s)\in W$ and $\beta(W)\cap G(f)=\phi$. By g-regularity of λ, for each $s\in S$ we can take two g-open sets $U(s)$ and $V(s)$ in (Y,σ) such that $x\in U(s)$, $s\in V(s)$ and $\gamma(U(s))\times\beta(V(s))\subseteq\lambda(W)$. Then we have $f(\gamma(U(s)))\cap\beta(V(s))=\phi$. Since $\{V(s) : s\in S\}$ is g-open cover of S, then by g-γ-compactness there exists a finite number $s_1,s_2,\ldots,s_n\in S$ such that $S\subseteq\beta(V(s_1))\cup\beta(V(s_2))\cup\ldots\cup\beta(V(s_n))$. By the g-regularity of γ, there exist a g-open set U such that $x\in U$, $\gamma(U)\subseteq\gamma(U(s_1))\cap\gamma(U(s_2))\cap\ldots\cap\gamma(U(s_n))$. Therefore, we have $\gamma(U)\cap f^{-1}(S)\subseteq U(s_i)\cap f^{-1}(\beta(V(s_i)))=\phi$. This shows that $x\notin gCl_\gamma(f^{-1}(S))$. This is a contradiction. Therefore, $f^{-1}(S)$ is g-γ-closed.

Theorem 29. *Suppose that the following condition hold.*

1. $\gamma\colon \tau_g \to P(X)$ *is g-open,*
2. $\beta : \sigma_g \to P(Y)$ *is g-regular, and*
3. $\lambda : (\tau \times \sigma)_g \to P(X \times Y)$ *is associated with* γ *and* β*, and* λ *is g-regular with respect to* γ *and* β*.*

Let $f\colon (X,\tau) \to (Y,\sigma)$ *be a function whose graph* $G(f)$ *is g-*λ*-closed in* $(X \times Y, \tau \times \sigma)$*. If every cover of A by g-*γ*-open sets of* (X,τ) *has finite sub cover, then* $f(A)$ *is g-*β*-closed in* (Y,σ)*.*

Proof. Similar to 28.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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