New Trends in Mathematical Sciences

http://dx.doi.org/10.20852/ntmsci.2017.131

An inverse nodal problem for differential pencils with complex spectral parameter dependent boundary conditions

Tuba Gulsen, Emrah Yilmaz, Hikmet Koyunbakan

Department of Mathematics, Faculty of Science, Firat University, Elazig, Turkey

Received: 24 October 2016, Accepted: 19 December 2016
Published online: 7 January 2017.

Abstract: In this study, we are concerned with an inverse nodal problem for second order differential pencil on a finite interval with complex spectral parameter dependent boundary conditions by using nodal points. We give some reconstruction formulas for potential functions p and q as a limit.

Keywords: Inverse nodal problem, differential pencil, Eigenvalues.

1 Introduction

Theory of inverse problems constitutes a vast field of mathematics. There are some approaches related to this subject; one of them is to study inverse eigenvalue problem (see [1], [2], [3], [4], [5], [6], [7]) and the other one is to study inverse nodal problem. The inverse nodal problem was introduced as a new type of spectral data which is so called nodal points-zeros of eigenfunctions by McLaughlin in 1988 [8]. She proved that the knowledge of a dense subset of nodal points of the eigenfunction can alone determine the potential function of Sturm-Liouville equation up to a constant. Independently, Shen studied the relations between the nodal points and the density function of string equation in 1988 [9]. Later, many authors have studied inverse nodal problem for different operators (see [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]).

In this study, we consider boundary value problem $L = L(p,q,h_0,h_1,H_0,H_1)$ of the form

$$l_\lambda y(x,\lambda) = y''(x) + \left[\lambda^2 - 2\lambda p(x) - q(x)\right] y(x) = 0, \;\; 0 < x < \pi, \tag{1}$$

with the boundary conditions

$$y'(0) + (i\lambda h_1 + h_0)\, y(0) = 0,$$

$$y'(\pi) + (i\lambda H_1 + H_0)\, y(\pi) = 0, \tag{2}$$

where $i^2 = -1$, λ is a spectral parameter; $p \in W_1^2[0,\pi], q \in L_1[0,\pi]$ are complex-valued functions; $h_0, h_1, H_0, H_1 \in \mathbb{C}, h_1 \neq \pm 1, H_1 \neq \pm 1$ [21]. Equation (1) is also known as diffusion equation in literature.

Let us emphasize some historical and physical improvement of differential pencils. Jaulent and Jean [22] stated the actual background of diffusion operators and discussed the inverse problem for the diffusion equation. Also, Gasymov

* Corresponding author e-mail: emrah231983@gmail.com

and Guseinov studied the spectral theory of this operator [23]. The problem of describing the interactions between colliding particles is of fundemental interesting in physics. In many cases, a description can be carried out through a well known theoretical model. In particular, one is interested in collisions of two spinless particles, and it is supposed that the $s-$ wave binding energies and $s-$wave scattering matrix are exactly known from collision experiments. $s-$wave Schrödinger equation with a radial static potential V can be written as

$$y'' + [E - V(x)]y = 0, \ x \geq 0, \tag{3}$$

where the potential function depends on energy in some way and has the following form of energy dependence

$$V(x,E) = U(x) + 2\sqrt{E}Q(x). \tag{4}$$

$U(x)$ and $Q(x)$ are complex-valued functions. (3) reduces to the Klein-Gordon $s-$wave equation with the static potential $Q(x)$, for a particle of zero mass and the energy $\sqrt{E}$ with the additional condition $U(x) = -Q^2(x)$ [22]. The Klein Gordon equation is considered one of the most important mathematical models in quantum field theory. The equation appears in relativistic physics and is used to describe dispersive wave phenomena in general. The Klein-Gordon equation arise in physics in linear and nonlinear forms [24].

Differential equations with a nonlinear dependence on the spectral parameter frequently appear in mathematics as well as in applications. Some aspects of this type problems were studied by many authors (see [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38]). Such problems play an important role in mathematics and have many applications in natural sciences and engineering.

Let λ_n be the $n-$th eigenvalue, D_0^{π} be a circle with radius π, centered 0 and x_j^n be the nodal points of the $n-$th eigenfunction where $j = 1,2,...,n-1$ for the problem (1), (2). The numbers x_j^n, $j = 1,2,...,n-1$ are the nodal points of the eigenfunction $y(x,\lambda_n)$, then $y(x_j^n,\lambda_n) = 0$. $n-th$ eigenfunction of the problem (1.1),(1.2) has exactly $n-$nodes inside the circle D_0^{π}. We assume that l_j^n is the distance of two consecutive nodal points x_j^n and x_{j+1}^n, $l_j^n = x_{j+1}^n - x_j^n$. Let us define the function $j_n(x)$ to be the largest index j such that $0 \leq x_j^n \leq x$ for $n > 0$, and the function $j_n(x)$ to be the largest index j such that $0 \leq x \leq x_j^n$ for $n < 0$. Thus, $j = j_n(x)$ iff $x \in [x_j^n, x_{j+1}^n)$ for $n > 0$ and $x \in [x_{j+1}^n, x_j^n)$ for $n < 0$ on D_0^{π}.

In this study, inverse nodal problem for differential pencil L on a finite interval is studied. We have reconstructed the potential functions p and q from the nodal points of eigenfunctions as complex, provided p,q are smooth enough.

2 Main results

In this section, we will try to obtain some asymptotic results for nodal parameters and some reconstruction formulas for potentials p and q which has been obtained as solution of an inverse nodal problem.

Lemma 1. *[21] Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions to $l_\lambda y(x,\lambda) = 0$ with the initial conditions*

$$\left(\varphi(0,\lambda), \varphi'(0,\lambda)\right) = (0,1) = \left(\psi'(0,\lambda), \psi(0,\lambda)\right).$$

Then, the following representations hold:

$$\begin{aligned}\varphi(x,\lambda) = &\frac{\sin[\lambda x - \alpha(x)]}{\lambda} - b_1(x)\frac{\cos[\lambda x - \alpha(x)]}{\lambda^2} + a_1(x)\frac{\sin[\lambda x - \alpha(x)]}{\lambda^2} \\ &+ b_2(x)\frac{\cos[\lambda x - \alpha(x)]}{\lambda^3} + a_2(x)\frac{\sin[\lambda x - \alpha(x)]}{\lambda^3} + o\left(\frac{e^{\tau x}}{\lambda^3}\right),\end{aligned}$$

and

$$\psi(x,\lambda)=\cos\left[\lambda x-\alpha(x)\right]-c_1(x)\frac{\cos\left[\lambda x-\alpha(x)\right]}{\lambda}+b_1(x)\frac{\sin\left[\lambda x-\alpha(x)\right]}{\lambda}$$
$$+d_2(x)\frac{\cos\left[\lambda x-\alpha(x)\right]}{\lambda^2}+d_1(x)\frac{\sin\left[\lambda x-\alpha(x)\right]}{\lambda^2}+o\left(\frac{e^{\tau x}}{\lambda^2}\right),$$

where

$$\alpha(x)=\int_0^x p(t)dt, b_1(x)=\frac{1}{2}\int_0^x\left[p^2(t)+q(t)\right]dt,\ a_1(x)=\frac{1}{2}\left[p(x)+p(0)\right],$$
$$b_2(x)=\frac{1}{4}\left[p'(x)-p'(0)\right]-\frac{1}{2}b_1(x)\left[p(x)+p(0)\right]-\frac{1}{2}\int_0^x p(t)\left[p^2(t)+q(t)\right]dt,$$
$$a_2(x)=\frac{1}{8}\left[5p^2(x)+5p^2(0)+2p(0)p(x)\right]+\frac{q(x)+q(0)}{4}-\frac{1}{2}b_1^2(x),$$
$$c_1(x)=\frac{1}{2}\left[p(0)-p(x)\right],$$
$$d_2(x)=\frac{1}{8}\left[5p^2(x)-2p(0)p(x)-3p^2(0)+2q(x)-2q(0)\right]-\frac{1}{2}b_1^2(x),$$
$$d_1(x)=-\frac{1}{4}\left[p'(x)+p'(0)\right]+\frac{1}{2}b_1(x)\left[p(x)-p(0)\right]+\frac{1}{2}\int_0^x p(t)\left[p^2(t)+q(t)\right]dt,$$

and $\tau=|Im\lambda|$.

Lemma 2. *[21] Let $\lambda_n, n\in A=\{\pm 0,\pm 1,\pm 2,...\}$, be the eigenvalues of the pencil L. Then, the sequence $\{\lambda_n : n\in A\}$ satisfies the following asymptotic expression as $|n|\to\infty$*

$$\lambda_n=n+\frac{1}{2\pi i}\log\frac{(h_1+1)(H_1-1)}{(h_1-1)(H_1+1)}+\frac{k_5}{n+\frac{1}{2\pi i}\log\frac{(h_1+1)(H_1-1)}{(h_1-1)(H_1+1)}}+\frac{k_6+k_7}{\left[n+\frac{1}{2\pi i}\log\frac{(h_1+1)(H_1-1)}{(h_1-1)(H_1+1)}\right]^2}+O\left(\frac{1}{n^3}\right).$$

For the convenience, Yang sets

$$a_1=a_1(\pi), b_1=b_1(\pi),\ c_i=c_i(\pi)(i=1,2,3), d_i=d_i(\pi)(i=1,2),\ e_i=e_i(\pi)(i=1,2)$$
$$k_1=a_1+a_1h_1H_1+i\left(H_1b_1-h_1b_1-h_0H_1-h_1H_0\right),$$
$$k_2=b_1-h_0+H_0-b_1h_1H_1-ic_1(h_1+H_1),$$
$$k_3=e_2-(h_0-H_0)b_1+a_2h_1H_1-h_0H_0+i(H_1d_1-h_1c_3-a_1h_0H_1-a_1h_1H_0),$$
$$k_4=e_1-h_0c_1-H_0c_1+b_2h_1H_1+i(H_1d_2-h_1c_2+b_1h_0H_1+b_1h_1H_0),$$
$$k_5=-\frac{1}{\pi}\frac{ik_1(h_1-H_1)+k_2(h_1H_1-1)}{(h_1H_1-1)^2-(h_1-H_1)^2},$$
$$k_6=-\frac{1}{\pi}\frac{ik_3(h_1-H_1)+k_4(h_1H_1-1)}{(h_1H_1-1)^2-(h_1-H_1)^2},$$
$$k_7=\frac{\left[k_1(h_1-H_1)i+k_2(h_1H_1-1)\right]\left[k_1(h_1-H_1)-k_2(h_1H_1-1)i\right]}{\pi\left[(h_1H_1-1)^2-(h_1-H_1)^2\right]},$$

where

$$c_2(x)=-\frac{1}{8}\left[3p^2(x)+2p(0)p(x)-5p^2(0)+2q(x)-2q(0)\right]-\frac{1}{2}b_1^2(x),$$

$$c_3(x) = -\frac{1}{4}\left[p'(x)+p'(0)\right] - \frac{1}{2}b_1(x)\left[p(x)-p(0)\right] + \frac{1}{2}\int_0^x p(t)\left[p^2(t)+q(t)\right]dt,$$

$$e_1(x) = \frac{1}{4}\left[p'(x)-p'(0)\right] - \frac{1}{2}b_1(x)\left[p(x)+p(0)\right] + \frac{1}{2}\int_0^x p(t)\left[p^2(t)+q(t)\right]dt$$

and

$$e_2(x) = \frac{1}{8}\left[3p^2(x)-2p(0)p(x)+3p^2(0)+2q(x)+2q(0)\right] + \frac{1}{2}b_1^2(x).$$

Theorem 1. *Suppose that $p \in W_1^2[0,\pi]$ and $q \in L_1[0,\pi]$, then*

$$x_j^n = \frac{\pi j}{n} + \frac{j\log A}{2n^2 i} + \frac{1}{n}\int_0^{x_j^n} p(t)dt + \frac{\log A}{2n^2 i\pi}\int_0^{x_j^n} p(t)dt + \frac{1}{2n^2}\int_0^{x_j^n}\left[p^2(t)+q(t)\right]dt + O\left(\frac{1}{n^3}\right), \tag{5}$$

and

$$l_j^n = \frac{\pi}{n} + \frac{\log A}{2n^2 i} + \frac{1}{n}\int_{x_j^n}^{x_{j+1}^n} p(t)dt + \frac{\log A}{2n^2 i\pi}\int_{x_j^n}^{x_{j+1}^n} p(t)dt + \frac{1}{2n^2}\int_{x_j^n}^{x_{j+1}^n}\left[p^2(t)+q(t)\right]dt + O\left(\frac{1}{n^3}\right), \tag{6}$$

where $A = \dfrac{(h_1+1)(H_1-1)}{(h_1-1)(H_1+1)}$ *and* $i^2=-1$.

Proof. We will prove only for $\varphi(x,\lambda)$. The other case for $\psi(x,\lambda)$ can be proved similarly. If $\varphi(x,\lambda)$ equals to zero, then as long as $\cos[\lambda x - \alpha(x)]$ is not close to 0,

$$0 = \frac{\tan[\lambda x-\alpha(x)]}{\lambda} - \frac{b_1(x)}{\lambda^2} + a_1(x)\frac{\tan[\lambda x-\alpha(x)]}{\lambda^2} + \frac{b_2(x)}{\lambda^3} + a_2(x)\frac{\tan[\lambda x-\alpha(x)]}{\lambda^3} + O\left(\frac{1}{\lambda^3}\right).$$

Hence

$$\tan[\lambda x-\alpha(x)]\left[1+O\left(\frac{1}{\lambda}\right)\right] = \frac{b_1(x)}{\lambda} - \frac{b_2(x)}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right).$$

Now take $\lambda=\lambda_n$ and $x=x_j^n$. Hence by Taylor's expansion for the tangent function, we get

$$\lambda_n x_j^n - \alpha(x) = \pi j + \frac{b_1(x)}{\lambda_n} - \frac{b_2(x)}{\lambda_n^2} + O\left(\frac{1}{\lambda_n^2}\right),$$

or

$$x_j^n = \frac{\alpha(x)}{\lambda_n} + \frac{\pi j}{\lambda_n} + \frac{b_1(x)}{\lambda_n^2} - \frac{b_2(x)}{\lambda_n^3} + O\left(\frac{1}{\lambda_n^3}\right).$$

And, by using estimates of $b_1(x)$, $b_2(x)$ and $\alpha(x)$, it yields

$$x_j^n = \frac{\pi j}{\lambda_n} + \frac{1}{\lambda_n}\int_0^{x_j^n} p(t)dt + \frac{1}{2\lambda_n^2}\int_0^{x_j^n}\left[p^2(t)+q(t)\right]dt - \frac{b_2(x_j^n)}{\lambda_n^3} + O\left(\frac{1}{\lambda_n^3}\right). \tag{7}$$

Furthermore, by using following asymptotic formulas in (7)

$$\frac{1}{\lambda_n} = \frac{1}{n} + \frac{\log A}{2n^2\pi i} + \frac{k_5}{n^2\left(n+\frac{1}{2\pi i}\log A\right)} + O\left(\frac{1}{n^4}\right),$$
$$\frac{1}{\lambda_n^2} = \frac{1}{n^2} - \frac{(\log A)^2}{4n^4\pi^2} + \frac{k_5^2}{n^4\left(n+\frac{1}{2\pi i}\log A\right)} + \frac{\log A}{n^3\pi i} + \frac{2k_5}{n^3\left(n+\frac{1}{2\pi i}\log A\right)} + \frac{(\log A)k_5}{n^4\pi i\left(n+\frac{1}{2\pi i}\log A\right)} + O\left(\frac{1}{n^5}\right),$$

and

$$\frac{1}{\lambda_n^3}=\frac{1}{n^3}-\frac{3(\log A)^2}{4n^5\pi^2}+\frac{\log A}{n^4\pi i}+\frac{2k_5}{n^4\left(n+\frac{1}{2\pi i}\log A\right)}+\frac{\log A}{2n^4\pi i}+\frac{k_5}{n^4\left(n+\frac{1}{2\pi i}\log A\right)}+O\left(\frac{1}{n^6}\right),$$

we conclude that

$$x_j^n=\frac{\pi j}{n}+\frac{j\log A}{2n^2 i}+\frac{1}{n}\int_0^{x_j^n}p(t)dt+\frac{\log A}{2n^2 i\pi}\int_0^{x_j^n}p(t)dt+\frac{1}{2n^2}\int_0^{x_j^n}\left[p^2(t)+q(t)\right]dt+O\left(\frac{1}{n^3}\right).$$

Equality (5) gives the asymptotic expansion for nodal lengths l_j^n as

$$l_j^n=\frac{\pi}{n}+\frac{\log A}{2n^2 i}+\frac{1}{n}\int_{x_j^n}^{x_{j+1}^n}p(t)dt+\frac{\log A}{2n^2 i\pi}\int_{x_j^n}^{x_{j+1}^n}p(t)dt+\frac{1}{2n^2}\int_{x_j^n}^{x_{j+1}^n}\left[p^2(t)+q(t)\right]dt+O\left(\frac{1}{n^3}\right).$$

Lemma 3. *[12] Suppose that $q\in L_1[0,\pi]$. Then, for almost every $x\in[0,\pi]$ with $j=j_n(x)$*

$$\lim_{n\to\infty}\lambda_n\int_{x_j^n}^{x_{j+1}^n}q(t)dt=q(x).$$

Theorem 2. *Let $q\in L_1[0,\pi]$, then*

$$q(x)=\lim_{n\to\infty}\left[2n^2\left(nl_j^n-\pi\right)-\frac{n}{i}\log A-p(x)\left(2n+\frac{1}{\pi i}\right)-p^2(x)\right]$$

where

$$p(x)=\lim_{n\to\infty}n\left(nl_j^n-\frac{\log A}{2ni}-\pi\right).$$

and $i^2=-1$.

Proof. Considering (6) in the form

$$l_j^n=\frac{\pi}{n}+\frac{\log A}{2n^2 i}+\frac{1}{n}\int_{x_j^n}^{x_{j+1}^n}p(t)dt+O\left(\frac{1}{n^2}\right),$$

so that

$$n\pi\left(\frac{n}{\pi}l_j^n-\frac{\log A}{2n\pi i}-1\right)=n\int_{x_j^n}^{x_{j+1}^n}p(t)dt+O\left(\frac{1}{n}\right).$$

By Lemma 3, for almost every $x\in[0,\pi]$, we obtain

$$p(x)=\lim_{n\to\infty}n\left(nl_j^n-\frac{\log A}{2ni}-\pi\right).$$

Now, we will get a reconstruction formula for the potential function q. Let consider (6) in the form

$$l_j^n = \frac{\pi}{n} + \frac{\log A}{2n^2 i} + \frac{1}{n}\int_{x_j^n}^{x_{j+1}^n} p(t)dt + \frac{\log A}{2n^2 i\pi}\int_{x_j^n}^{x_{j+1}^n} p(t)dt + \frac{1}{2n^2}\int_{x_j^n}^{x_{j+1}^n}\left[p^2(t)+q(t)\right]dt + O\left(\frac{1}{n^3}\right).$$

After some algebraic computations, we can get

$$n\pi\left(\frac{n}{\pi}l_j^n - 1\right) = \frac{\log A}{2i} + n\int_{x_j^n}^{x_{j+1}^n} p(t)dt + \frac{\log A}{2i\pi}\int_{x_j^n}^{x_{j+1}^n} p(t)dt + \frac{1}{2}\int_{x_j^n}^{x_{j+1}^n}\left[p^2(t)+q(t)\right]dt + O\left(\frac{1}{n}\right),$$

and

$$2n^2\left(nl_j^n - \pi\right) = \frac{n}{i}\log A + 2np(x) + \frac{p(x)}{\pi i} + n\int_{x_j^n}^{x_{j+1}^n}\left[p^2(t)+q(t)\right]dt + O(1),$$

for sufficiently large n. Hence, by using Lemma 3, it yields

$$q(x) = \lim_{n\to\infty}\left[2n^2\left(nl_j^n - \pi\right) - \frac{n}{i}\log A - p(x)\left(2n + \frac{1}{\pi i}\right) - p^2(x)\right].$$

This completes the proof.

3 Conclusion

In this study, we give some results about inverse nodal problem for differential pencils with complex spectral parameter in boundary conditions. Asymptotic formulas of nodal parameters and potential functions for this problem are obtained. We think that it will offer a different perspective to spectral theory.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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