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On the numerical solution of differential-algebraic equations (DAEs) by Laguerre Polynomials approximation

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Abstract: In this study, we investigate numerical solution of differential-algebraic equations (DAEs) using the Laguerre polynomials approximation. Two different problems are solved using the Laguerre polynomials approximation and the solutions are compared with the exact solutions. Firstly, we calculate the power series of a given equation system and then transform it into Laguerre polynomials approximation form, which gives an arbitrary order for solving the DAE numerically. Moreover, a Maple algorithm is developed for numerical solution of differential-algebraic equations (DAEs) with Laguerre polynomials approximation. In Maple Programming, we sketch graphs of obtained solutions, and are made tables to compare the solutions.

Keywords: Differential-Algebraic Equations (DAEs), Power Series, Leguerre Polynomials Approximation.

1 Introduction

Some numerical methods have been developed, Runge-Kutta, one-leg, implicit Runge-Kutta, Rosenbrock, one step and extrapolation, Padé approximation, Chebyshev approximation, Adomian decomposition, least squares approximation, etc. [1-18]. The purpose of this paper is to consider the numerical solution of differential-algebraic equations (DAEs) by using Laguerre polynomials approximation. Differential-algebraic equations (DAEs) can be used to describe the evolution of many interesting and important systems. Differential-algebraic equations (DAEs) are a set of differential equations with additional algebraic constraints in the form:

$$F(x,y(x),y'(x)) = 0 \tag{1}$$

with singular $F_{y'}$, where F and y are of the same dimension. In the following we denote partial derivatives by subscripts, so that $F_{y'} = \partial F/\partial y'$. The equation (1) is also called a fully *implicit DAE system*. We are here especially interested in *semi-explicit systems*, differential equations with algebraic constraints of the form

$$\begin{aligned} y'(x) &= f(x,y(x),z(x)) \\ 0 &= g(x,y(x),z(x)) \end{aligned} \tag{2}$$

where y represents the differential variables and z represents the algebraic variables [1-18]. The numerical methods devised for DAEs take into account the structure of the underlying DAE. We will calculate power series of the given differential-algebraic equations (DAEs) system then transform it into Laguerre polynomials approximation form, which give an arbitrary order for solving differential-algebraic equation numerically.

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2 The method

A differential-algebraic equation has the form

$$F(x,y,y') = 0 \tag{3}$$

with initial values

$$y(x_0) = y_0, y'(x_0) = y_1$$

where $F \in \mathbb{R}$ and $y \in \mathbb{R}^n$ are both vector functions for which we assumed sufficient differentiability and the initial values to be consistent, i.e.

$$F(x_0,y_0,y_0') = 0. \tag{4}$$

The solutions of the equation (3) can be assumed that

$$y = y_0 + y_1 x + e x^2 \tag{5}$$

where e is a vector function which is the same size as y_0 and y_1. Substitute the equation (5) into the equation (3) and convert the elementary functions in the equation (3) into series in $x = 0$ and neglect higher order term, we have the linear equation of e in the form

$$Ae = B \tag{6}$$

where A and B are constant matrices. Solving the equation (6); the coefficients of x^2 in the equation (5) can be determined. Repeating above procedure for higher order terms, we can get the arbitrary order power series of the solutions for the equation (3) [13,14,18,27]. The Power series given by above method can be transformed into Laguerre polynomials approximation and we have numerical solution of differential-algebraic equation in the equation (3).

2.1 Power series of solution for DAEs

We define another type of power series in the form

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + (f_n + p_1 e_1 + \cdots + p_m e_m) x^n \tag{7}$$

where $p_1, p_2, \ldots, p_m$ are constants. $e_1, e_2, \ldots, e_m$ are bases of vector e, m is the size of vector e. y is a vector with m elements in the equation (5). Every element can be represented by the power series in the equation (7)

$$y_i = y_{i,0} + y_{i,1} x + y_{i,2} x^2 + \cdots + e_i x^n, \tag{8}$$

where y_i is the ith element of y. Substituting the equation (8) into the equation (3), we can get the following:

$$f_i = (f_{i,n} + p_{i,1} e_1 + \cdots + p_{i,m} e_m) x^{n-j} + O(x^{n-j+1}), \tag{9}$$

where f_i is the ith element of $f(y,y',x)$ in the equation (3) and j is 0 if $f(y,y',x)$ have y', 1 if does not have. From the equation (9) and the equation (6), we can determine the linear equation in the equation (6) as follows:

$$A_{i,j} = P_{i,j}, \tag{10}$$

$$B_i = -f_{i,n}. \tag{11}$$

Solving this linear equation, we have $e_i (i = 1, \ldots, m)$. Substituting e_i into the equation (8), we have $y_i (i = 1, \ldots, m)$ which are polynomials of degree n. Repeating this procedure from the equation (10) and the equation (11) we can get the arbitrary order power series of the solution for DAEs in the equation (3). If we repeat the above procedure, we have numerical solution of DAEs in the equation (3) [13, 14, 18, 27].

Remark. When the initial value problem is

$$F(x_0, y_0, y_0') = 0, y(x_0) = y_0. \tag{12}$$

The solution of equation (12) can be assumed that

$$y = y_0 + ex \tag{13}$$

and repeating above procedure, we can get the solutions of equation (12) [13, 14, 18, 27].

3 Laguerre polynomials approximation

Laguerre polynomials are defined as solutions of Laguerre's differential equation:

$$xy'' + (1 - x)y' + ny = 0. \tag{14}$$

Solutions corresponding to the non-negative integer n can be expressed using **Rodrigues' formula**.

$$\begin{aligned} L_0(x) &= 1 \\ L_n(x) &= \frac{e^x}{n!}\frac{d^n}{dx^n}[(e^{-x}x^n)], n = 1, 2, \dots \end{aligned} \tag{15}$$

Thus,

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= -x + 1 \\ L_2(x) &= \frac{1}{2}(x^2 - 4x + 2) \\ L_3(x) &= \frac{1}{6}(-x^3 + 9x^2 - 18x + 6) \\ L_4(x) &= \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24) \\ L_5(x) &= \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120) \\ &\vdots \end{aligned} \tag{16}$$

The inverse relations are as follows:

$$\begin{aligned} 1 &= L_0(x) \\ x &= L_0(x) - L_1(x) \\ x^2 &= 2[L_0(x) - 2L_1(x) + L_2(x)] \\ x^3 &= 6[L_0(x) - 3L_1(x) + 3L_2(x) - L_3(x)] \\ x^4 &= 24[L_0(x) - 4L_1(x) + 6L_2(x) - 4L_3(x) + L_4(x)] \\ x^5 &= 120[L_0(x) - 5L_1(x) + 10L_2(x) - 10L_3(x) + 5L_4(x) - L_5(x)] \\ &\vdots \end{aligned} \tag{17}$$

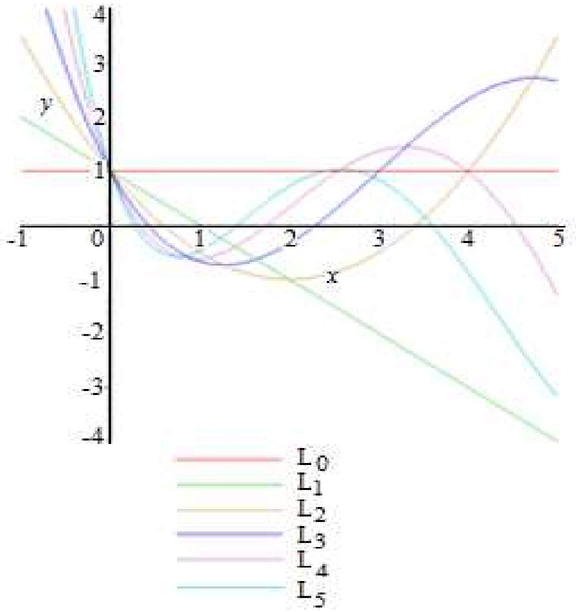

Fig. 1: Laguerre polynomials of degrees 0, 1, 2, 3, 4, 5.

3.1 The properties of Laguerre polynomials

Generating function The generating function of a Laguerre Polynomial is

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} L_n(x)t^n. \quad (18)$$

Orthogonality Laguerre Polynomials $L_n(x), (n = 0,1,2,3,\ldots)$, form a *complete orthogonal set* on the interval $0 < x < \infty$ with respect to the weighting function e^{-x}. It can be shown that

$$\int_0^{\infty} e^{-x} L_m(x)L_n(x)dx = \begin{cases} 0, m \neq n \\ 1, m = n. \end{cases} \quad (19)$$

Recurrence relation A Laguerre Polynomial at one point can be expressed in terms of neighboring Laguerre Polynomials at the same point [19,20,21,22,23,24,25].

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= -x+1 \\ (n+1)L_{n+1}(x) &= (2n+1-x)L_n(x) - nL_{n-1}(x), \quad n = 2,\ldots \end{aligned} \quad (20)$$

4 Test problems

In this section, two differential-algebraic equations are considered and these problems are solved by Laguerre Polynomials Approximation.

Example 1. Consider the following linear DAE of three variable

$$\begin{aligned} y_1'(x) &= \left(\alpha - \frac{1}{2-x}\right) y_1(x) + (2-x)\alpha y_3(x) + \left(\frac{3-x}{2-x}\right) e^x \\ y_2'(x) &= \left(\frac{1-\alpha}{x-2}\right) y_1(x) - y_2(x) + (\alpha-1) y_3(x) + 2e^x \\ 0 &= (x+2) y_1(x) + (x^2-4) y_2(x) - (x^2+x-2) e^x \end{aligned} \tag{21}$$

where α is a positive parameter (take $\alpha = 10$). For the initial condition

$$y_1(-1) = y_2(-1) = e^{-1}, y_3(-1) = -\frac{e^{-1}}{3} \tag{22}$$

the exact solution is given as

$$y_1(x) = y_2(x) = e^x, y_3(x) = \frac{e^x}{x-2} \quad \text{for } x \neq 2 \tag{23}$$

[16]. If the method is applied to the equation (21) we have

$$\begin{aligned} y_1(x) &= 0.9999998888 + 0.9999988749x + 0.4999948754x^2 + 0.1666527932x^3 + 0.04164190895x^4 \\ &+ 0.008302834609x^5 + 0.001362516449x^6 + 0.0001824798815x^7 + 0.00001824798815x^8 \\ &+ 0.000001013777120x^9, \\ y_2(x) &= 0.9999998888 + 0.9999988749x + 0.4999948754x^2 + 0.1666527932x^3 + 0.04164190895x^4 \\ &+ 0.008302834609x^5 + 0.001362516449x^6 + 0.0001824798815x^7 + 0.00001824798815x^8 \\ &+ 0.000001013777120x^9, \\ y_3(x) &= -0.4999374461 - 0.7493431788x - 0.6218566090x^2 - 0.3867549200x^3 - 0.2010737971x^4 \\ &- 0.08893877518x^5 - 0.03202602857x^6 - 0.008604472938x^7 - 0.001498942480x^8 - 0.0001249963548x^9. \end{aligned} \tag{24}$$

Then, if we use above algorithm in section 3.2, we obtain

$$\begin{aligned} y_1^*(x) &= 0.9999998627 + 0.999998942x + 0.499994717x^2 + 0.166652808x^3 + 0.041641893x^4 + 0.0083028344x^5 \\ &+ 0.00136251645x^6 + 0.000182479882x^7 + 0.00001824798818x^8 + 0.000001013777120x^9, \\ y_2^*(x) &= 0.9999998627 + 0.999998942x + 0.499994717x^2 + 0.166652808x^3 + 0.041641893x^4 + 0.0083028344x^5 \\ &+ 0.00136251645x^6 + 0.000182479882x^7 + 0.00001824798818x^8 + 0.000001013777120x^9, \\ y_3^*(x) &= -0.49993967 - 0.7493481x - 0.6218529x^2 - 0.3867592x^3 - 0.2010725x^4 - 0.08893879x^5 \\ &- 0.032026033x^6 - 0.0086044727x^7 - 0.00149894248x^8 - 0.0001249963548x^9. \end{aligned} \tag{25}$$

where $y_1(x), y_2(x), y_3(x)$ and are the power series solutions of differential-algebraic equation (DAE), $y_1^*(x), y_2^*(x), y_3^*(x)$ are the Laguerre Polynomials Approximations of $y_1(x), y_2(x), y_3(x)$. If we use the algorithm to get tables in section 3.2, we obtain Table 1, Table 2, Table 3.
Let's show the graphs of $y_1(x)$, $y_2(x)$, $y_3(x)$ and their the Laguerre polynomials approximations.

Example 2. We consider the following differential-algebraic equation

$$\begin{aligned} y_1'(x) + 2y_3(x) + xy_4'(x) &= 3x + 2sinx \\ y_1(x) + x^2 y_2'(x) - xy_4(x) &= 3 - 3x + x^2 e^x \\ y_2'(x) - e^x y_4'(x) + y_4(x) &= 3 + x \\ xy_1(x) + 3y_2(x) - x^2 y_4(x) &= -3x^2 + 3x + 6 + 3e^x \end{aligned} \tag{26}$$

x	$y_1(x)$	$y_1^*(x)$	$\lvert y_1(x) - y_1^*(x)\rvert$
-1.0	0.3678794412	0.3678794411	$1.000000000 \times 10^{-10}$
-0.8	0.4493289641	0.4493289643	$1.896863000 \times 10^{-10}$
-0.6	0.5488116361	0.5488116363	$1.290199700 \times 10^{-10}$
-0.4	0.6703200460	0.6703200456	$4.718168793 \times 10^{-10}$
-0.2	0.8187307531	0.8187307416	$1.156724894 \times 10^{-8}$
0.0	1.0000000000	0.9999998888	$1.112000000 \times 10^{-7}$
0.2	1.2214027580	1.2214020540	$7.032885759 \times 10^{-7}$
0.4	1.4918246980	1.4918213440	0.000003354417429
0.6	1.8221188000	1.8221057850	0.000013013501880
0.8	2.2255409280	2.2254977810	0.000043148036910
1.0	2.7182818280	2.7181554340	0.000126393845300

Table 1: Numerical solution of $y_1(x)$ in equation (21).

x	$y_2(x)$	$y_2^*(x)$	$\lvert y_2(x) - y_2^*(x)\rvert$
-1.0	0.3678794412	0.3678794411	$1.000000000 \times 10^{-10}$
-0.8	0.4493289641	0.4493289643	$1.896863000 \times 10^{-10}$
-0.6	0.5488116361	0.5488116363	$1.290199700 \times 10^{-10}$
-0.4	0.6703200460	0.6703200456	$4.718168793 \times 10^{-10}$
-0.2	0.8187307531	0.8187307416	$1.156724894 \times 10^{-8}$
0.0	1.0000000000	0.9999998888	$1.112000000 \times 10^{-7}$
0.2	1.2214027580	1.2214020540	$7.032885759 \times 10^{-7}$
0.4	1.4918246980	1.4918213440	0.000003354417429
0.6	1.8221188000	1.8221057850	0.000013013501880
0.8	2.2255409280	2.2254977810	0.000043148036910
1.0	2.7182818280	2.7181554340	0.000126393845300

Table 2: Numerical solution of $y_2(x)$ in equation (21).

x	$y_3(x)$	$y_3^*(x)$	$\lvert y_3(x) - y_3^*(x)\rvert$
-1.0	-0.1226264804	-0.1226264800	$4.000000000 \times 10^{-10}$
-0.8	-0.1604746300	-0.1604746299	$9.915000000 \times 10^{-11}$
-0.6	-0.2110813985	-0.2110813932	$5.260145000 \times 10^{-9}$
-0.4	-0.2793000192	-0.2792997037	$3.152673607 \times 10^{-7}$
-0.2	-0.3721503423	-0.3721442363	0.0000061060307
0.0	-0.5000000000	-0.4999374461	0.0000625539000
0.2	-0.6785570878	-0.6781267279	0.0004303599921
0.4	-0.9323904362	-0.9301286035	0.0022618327210
0.6	-1.301513429	-1.2916873510	0.0098260777030
0.8	-1.854617440	-1.8173902010	0.0372272395200
1.0	-2.718281828	-2.5901591660	0.1281226613000

Table 3: Numerical solution of $y_3(x)$ in equation (21).

with initial conditions

$$\begin{aligned} &y_1(-1) = 4, y_2(-1) = e^{-1} + 2, \\ &y_3(-1) = sin(-1), y_4(-1) = 2 \end{aligned} \tag{27}$$

The system (26) has exact solution

$$\begin{aligned} &y_1(x) = 3 + x^2, y_2(x) = e^x + 2, \\ &y_3(x) = sin(x), y_4(x) = x + 3 \end{aligned} \tag{28}$$

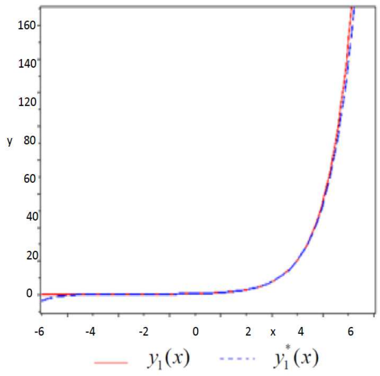

Fig. 2: Graphs of $y_1(x)$ and its Laguerre polynomials approximation.

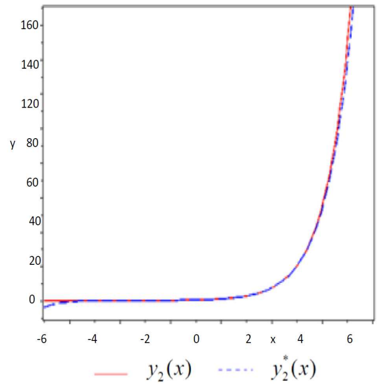

Fig. 3: Graphs of $y_2(x)$ and its Laguerre polynomials approximation.

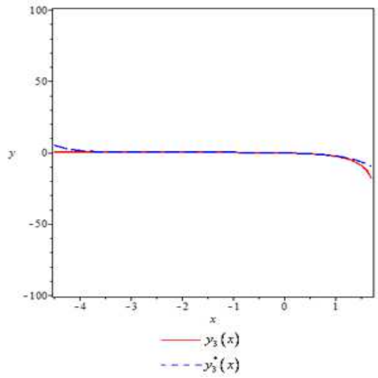

Fig. 4: Graphs of $y_3(x)$ and its Laguerre polynomials approximation.

[16]. If the method is applied to the equation (26) we have

$$\begin{aligned}
y_1(x) &= 3+x^2,\\
y_2(x) &= 2.999999889+0.9999988749x+0.4999948754x^2+0.1666527932x^3+0.04164190895x^4\\
&\quad +0.008302834609x^5+0.001362516449x^6+0.0001824798815x^7+0.00001824798815x^8\\
&\quad +0.000001013777120x^9,\\
y_3(x) &= -2.167\times 10^{-7}+0.9999978499x-0.0000095821x^2-0.1666919013x^3-0.00004343094x^4\\
&\quad +0.008282413080x^5-0.0000409950404x^6-0.0002205599479x^7-0.00000746946128x^8\\
&\quad +0.000001488928312x^9,\\
y_4(x) &= 3+x.
\end{aligned} \tag{29}$$

x	$y_1(x)$	$y_1^*(x)$	$\lvert y_1(x)-y_1^*(x)\rvert$
-1.0	4.00	4.00	0
-0.8	3.64	3.64	0
-0.6	3.36	3.36	0
-0.4	3.16	3.16	0
-0.2	3.04	3.04	0
0.0	3.00	3.00	0
0.2	3.04	3.04	0
0.4	3.16	3.16	0
0.6	3.36	3.36	0
0.8	3.64	3.64	0
1.0	4.00	4.00	0

Table 4: Numerical solution of $y_1(x)$ in equation (26).

x	$y_2(x)$	$y_2^*(x)$	$\lvert y_2(x)-y_2^*(x)\rvert$
-1.0	2.367879441	2.367879159	2.819000000×10^{-7}
-0.8	2.449328964	2.449328769	1.947611187×10^{-7}
-0.6	2.548811636	2.548811508	1.282216435×10^{-7}
-0.4	2.670320046	2.670319966	7.976767376×10^{-8}
-0.2	2.818730753	2.818730697	5.725070189×10^{-8}
0.0	3.000000000	2.999999863	1.370000000×10^{-7}
0.2	3.221402758	3.221402035	7.219361628×10^{-7}
0.4	3.491824698	3.491821320	0.000003378217564
0.6	3.822118800	3.822105744	0.000013054885110
0.8	4.225540928	4.225497709	0.000043220765500
1.0	4.718281828	4.718155315	0.000126512502700

Table 5: Numerical solution of $y_2(x)$ in equation (26).

Then, if we use above algorithm in section 3.2, we obtain

$$\begin{aligned}
y_1^*(x) &= 3+x^2,\\
y_2^*(x) &= 2.999999863+0.999998942x+0.499994717x^2+0.166652808x^3+0.041641893x^4\\
&\quad +0.0083028344x^5+0.00136251645x^6+0.000182479882x^7+0.00001824798818x^8\\
&\quad +0.00000101377712x^9,\\
y_3^*(x) &= -2.419\times10^{-7}+0.999998003x-0.000009566x^2-0.166691887x^3-0.000043426x^4\\
&\quad +0.0082824132x^5-0.00004099502x^6-0.000220559949x^7-0.0000074694613x^8\\
&\quad +0.000001488928312x^9,\\
y_4^*(x) &= 3+x.
\end{aligned} \tag{30}$$

where $y_1(x), y_2(x), y_3(x), y_4(x)$ and are the power series solutions of differential-algebraic equation (DAE), $y_1^*(x), y_2^*(x), y_3^*(x), y_4^*(x)$ are the Laguerre Polynomials Approximations of $y_1(x), y_2(x), y_3(x), y_4(x)$. If we use the algorithm to get tables in section 3.2, we obtain Table 4, Table 5, Table 6, Table 7.

Let's show the graphs of $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ and their the Laguerre Polynomials Approximations,

x	$y_3(x)$	$y_3^*(x)$	$\lvert y_3(x)-y_3^*(x)\rvert$
-1.0	-0.8414709848	-0.8414711567	$1.719000000 \times 10^{-7}$
-0.8	-0.7173560909	-0.7173562337	$1.427799312 \times 10^{-7}$
-0.6	-0.5646424734	-0.5646425873	$1.137898990 \times 10^{-7}$
-0.4	-0.3894183423	-0.3894184283	$8.604245128 \times 10^{-8}$
-0.2	-0.1986693308	-0.1986694097	$7.891822208 \times 10^{-8}$
0.0	0.0000000000	-2.4190×10^{-7}	$2.419000000 \times 10^{-7}$
0.2	0.1986693308	0.1986680165	0.000001314410584
0.4	0.3894183423	0.3894123143	0.000006027910324
0.6	0.5646424734	0.5646198835	0.00002258979314
0.8	0.7173560909	0.7172839313	0.00007215961242
1.0	0.8414709848	0.8412677598	0.0002032250020

Table 6: Numerical solution of $y_3(x)$ in equation (26).

x	$y_4(x)$	$y_4^*(x)$	$\lvert y_4(x)-y_4^*(x)\rvert$
-1.0	2.0	2.0	0
-0.8	2.2	2.2	0
-0.6	2.4	2.4	0
-0.4	2.6	2.6	0
-0.2	2.8	2.8	0
0.0	3.0	3.0	0
0.2	3.2	3.2	0
0.4	3.4	3.4	0
0.6	3.6	3.6	0
0.8	3.8	3.8	0
1.0	4	4	0

Table 7: Numerical solution of $y_4(x)$ in equation (26).

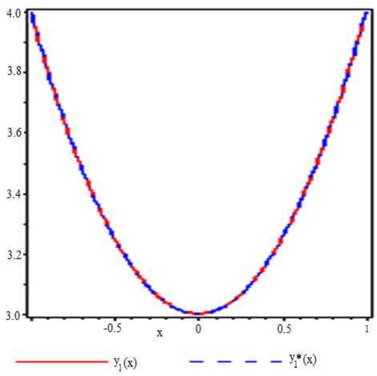

Fig. 5: Graphs of $y_1(x)$ and its Laguerre Polynomials Approximation.

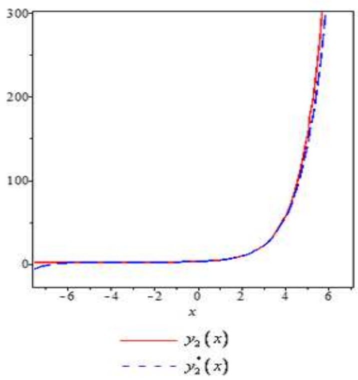

Fig. 6: Graphs of $y_2(x)$ and its Laguerre Polynomials Approximation.

5 Conclusion

Laguerre Polynomials Approximation has proposed for solving differential-algebraic equations in this study. The computations associated with the example discussed above were performed by using Maple 17 [26]. Results show the advantages of the method.

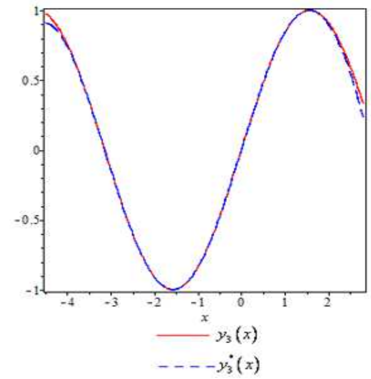

Fig. 7: Graphs of $y_3(x)$ and its Laguerre Polynomials Approximation.

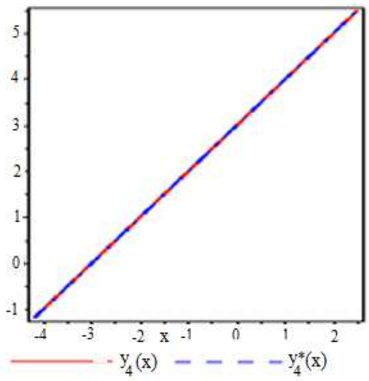

Fig. 8: Graphs of $y_4(x)$ and its Laguerre Polynomials Approximation.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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