Communication in Mathematical Modeling and Applications
http://ntmsci.com/cmma

Hermite-Hadamard-Fejer type inequalities for p-convex functions via fractional integrals

Mehmet Kunt[1] *and Imdat Iscan*[2]

[1]Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, Trabzon, Turkey
[2]Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Giresun, Turkey

Received: 15 September 2016, Accepted: 23 December 2016
Published online: 7 January 2017.

Abstract: In this paper, firstly, Hermite-Hadamard-Fejer type inequalities for p-convex functions in fractional integral forms are built. Secondly, an integral identity and some Hermite-Hadamard-Fejer type integral inequalities for p-convex functions in fractional integral forms are obtained. Finally, some Hermite-Hadamard and Hermite-Hadamard-Fejer inequalities for convex, harmonically convex and p-convex functions are given. Many results presented here for p-convex functions provide extensions of others given in earlier works for convex and harmonically convex and p-convex fuctions.

Keywords: Hermite-Hadamard inequalities, Hermite-Hadamard-Fejer inequalities, Fractional integral, convex functions, harmonically convex functions, p-convex functions.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature as Hermite-Hadamard's inequality [7,8].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [5], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1):

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \le \int_a^b f(x)w(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{2}$$

holds, where $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve, and extend the inequalities (1) and (2) see [1,2,4,6,9,12,13,17,18,21,22,23,24,26].

* Corresponding author e-mail: mkunt@ktu.edu.tr

We will now give definitions of the right-hand side and left-hand side Riemann-Liouville fractional integrals which are used throughout this paper.

Definition 1.[20]. *Let $f \in L[a,b]$. The right-hand side and left-hand side Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt,\ x > a \ \text{ and } \ J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt,\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t} t^{\alpha-1} dt$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [3,10,11,14,15,19,25,27,28].

In [13], İşcan gave the definition of harmonically convex function and established following Hermite-Hadamard type inequality for harmonically convex functions as follows.

Definition 2. *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq t f(y) + (1-t) f(x) \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (3) is reversed, then f is said to be harmonically concave.

Theorem 2.[13]. *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities holds.*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}. \tag{4}$$

In [2], Chan and Wu presented Hermite-Hadamard-Fejér inequality for harmonically convex functions as follows.

Theorem 3. *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ and $w : [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{w(x)}{x^2} dx \leq \int_a^b \frac{f(x) w(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{w(x)}{x^2} dx. \tag{5}$$

In [25], Sarıkaya et al. presented Hermite-Hadamard inequality for convex functions via fractional integrals as follows.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a) + f(b)}{2} \tag{6}$$

with $\alpha > 0$.

In [14], İşcan and Wu presented Hermite-Hadamard inequality for harmonically convex functions via fractional integrals as follows.

Theorem 5. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a,b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{1/a-}(f\circ g)(1/b)+J^{\alpha}_{1/b+}(f\circ g)(1/a)\right] \leq \frac{f(a)+f(b)}{2} \tag{7}$$

with $\alpha > 0$ and $g(x) = \frac{1}{x}$, $x \in \left[\frac{1}{b},\frac{1}{a}\right]$.

In [15], İşcan presented Hermite-Hadamard-Fejer inequality for convex functions via fractional integrals as follows:

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a convex function with $a < b$ and $f \in L[a,b]$. If w is nonnegative, integrable and symmetric to $(a+b)/2$, then the following inequalities for fractional integrals holds.*

$$f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{a+}w(b)+J^{\alpha}_{b-}w(a)\right] \leq \left[J^{\alpha}_{a+}(fw)(b)+J^{\alpha}_{b-}(fw)(a)\right] \leq \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a+}w(b)+J^{\alpha}_{b-}w(a)\right] \tag{8}$$

with $\alpha > 0$.

In [19], İşcan and Kunt presented Hermite-Hadamard-Fejer inequality for harmonically convex functions via fractional integrals as follows:

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be a harmonically convex function with $a < b$ and $f \in L[a,b]$. If $w : [a,b] \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a+b$, then the following inequalities for fractional integrals holds.*

$$f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{1/b+}(w\circ g)(1/a)+J^{\alpha}_{1/a-}(w\circ g)(1/b)\right] \leq \left[J^{\alpha}_{1/b+}(fw\circ g)(1/a)+J^{\alpha}_{1/a-}(fw\circ g)(1/b)\right] \tag{9}$$

$$\leq \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(w\circ g)(1/a)+J^{\alpha}_{1/a-}(w\circ g)(1/b)\right]$$

with $\alpha > 0$ and $g(x) = \frac{1}{x}$, $x \in \left[\frac{1}{b},\frac{1}{a}\right]$.

In [29], Zhang and Wan gave the definition of p-convex function on $I \subset \mathbb{R}$, in [18], İşcan gave a different definition of p-convex function on $I \subset (0,\infty)$ as follows.

Definition 3. *Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be p-convex, if*

$$f\left(\left[tx^{p}+(1-t)y^{p}\right]^{1/p}\right) \leq tf(x)+(1-t)f(y) \tag{10}$$

for all $x,y \in I$ and $t \in [0,1]$.

It can be easily seen that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

In [6, Theorem 5], if we take $I \subset (0,\infty)$, $p \in \mathbb{R}\backslash\{0\}$ and $h(t) = t$, then we have the following theorem.

Theorem 8. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $a,b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities holds.*

$$f\left(\left[\frac{a^{p}+b^{p}}{2}\right]^{1/p}\right) \leq \frac{p}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2}. \tag{11}$$

For some results related to p-convex functions and its generalizations, we refer the reader to see [6,16,17,18,21,22,29].

In this paper, we built Hermite-Hadamard-Fejer type inequalities for p-convex functions in fractional integral forms. We obtain an integral identity and some Hermite-Hadamard-Fejer type integral inequalities for p-convex functions in fractional integral forms. We give some Hermite-Hadamard and Hermite-Hadamard-Fejer inequalities for convex, harmonically convex and p-convex functions.

2 Main results

Throughout this section, $\|w\|_\infty = \sup_{t\in[a,b]} |w(t)|$, for the continuous function $w : [a,b] \to \mathbb{R}$.

Definition 4. *Let $p \in \mathbb{R}\setminus\{0\}$. A function $w : [a,b] \subset (0,\infty) \to \mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$ if*

$$w(x) = w\left(\left[a^p+b^p-x^p\right]^{1/p}\right)$$

holds for all $x \in [a,b]$.

Lemma 1. *Let $p \in \mathbb{R}\setminus\{0\}$, $\alpha > 0$ and $w : [a,b] \subset (0,\infty) \to \mathbb{R}$ is integrable, p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then*

(i) *If $p > 0$,*

$$J^{\alpha}_{a^p+}(w\circ g)(b^p) = J^{\alpha}_{b^p-}(w\circ g)(a^p) = \frac{1}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p) + J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]$$

with $g(x) = x^{\frac{1}{p}}$, $x \in [a^p,b^p]$,

(ii) *If $p < 0$,*

$$J^{\alpha}_{b^p+}(w\circ g)(a^p) = J^{\alpha}_{a^p-}(w\circ g)(b^p) = \frac{1}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p) + J^{\alpha}_{a^p-}(w\circ g)(b^p)\right]$$

with $g(x) = x^{\frac{1}{p}}$, $x \in [b^p,a^p]$.

Proof. (i) Let $p > 0$. Since w is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, using Definition 4 we have $w\left(x^{1/p}\right) = w\left(\left[a^p+b^p-x\right]^{1/p}\right)$ for all $x \in [a^p,b^p]$. Hence in the following integral setting $t = a^p+b^p-x$ and $dt = -dx$ gives

$$\begin{aligned} J^{\alpha}_{a^p+}(w\circ g)(b^p) &= \frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-x)^{\alpha-1}(w\circ g)(x)\,dx = \frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-x)^{\alpha-1}w\left(x^{1/p}\right)dx \\ &= \frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-x)^{\alpha-1}w\left(\left[a^p+b^p-x\right]^{1/p}\right)dx = \frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(x-a^p)^{\alpha-1}w\left(x^{1/p}\right)dx = J^{\alpha}_{b^p-}(w\circ g)(a^p). \end{aligned}$$

This completes the proof of (i).

(ii) The proof is similar with (i).

Theorem 9. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\setminus\{0\}$, $\alpha > 0$ and $a,b \in I$ with $a < b$. If $f \in L[a,b]$ and $w : [a,b] \to \mathbb{R}$ is nonnegative, integrable and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then the following inequalities for fractional integrals holds.*

(i) *If $p > 0$,*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\left[J^{\alpha}_{a^p+}(w\circ g)(b^p) + J^{\alpha}_{b^p-}(w\circ g)(a^p)\right] \le \left[J^{\alpha}_{a^p+}(fw\circ g)(b^p) + J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right] \quad (12)$$

$$\leq \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]$$

with $g(x)=x^{1/p}$, $x\in[a^p,b^p]$,

(ii) *If* $p<0$,

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right]\leq\left[J^{\alpha}_{b^p+}(fw\circ g)(a^p)+J^{\alpha}_{a^p-}(fw\circ g)(b^p)\right] \tag{13}$$

$$\leq \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right]$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof. (i) Let $p>0$. Since $f:I\subset(0,\infty)\to\mathbb{R}$ is a p-convex function, we have, for all $x,y\in I$ (with $t=\frac{1}{2}$ in the inequality (10))

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right)\leq\frac{f(x)+f(y)}{2}.$$

Choosing $x=[ta^p+(1-t)b^p]^{1/p}$ and $y=[tb^p+(1-t)a^p]^{1/p}$, we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq\frac{f\left([ta^p+(1-t)b^p]^{1/p}\right)+f\left([tb^p+(1-t)a^p]^{1/p}\right)}{2}. \tag{14}$$

Multiplying both sides of (14) by $2t^{\alpha-1}w\left([ta^p+(1-t)b^p]^{1/p}\right)$ and integrating with respect to t over $[0,1]$, using Lemma 1-i, we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]\leq\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]$$

the left hand side of (12). For the proof of the second inequality in (12) we first note that if f is a p-convex function, then, for all $t\in[0,1]$, it yields

$$\frac{f\left([ta^p+(1-t)b^p]^{1/p}\right)+f\left([tb^p+(1-t)a^p]^{1/p}\right)}{2}\leq\frac{f(a)+f(b)}{2}. \tag{15}$$

Multiplying both sides of (15) by $2t^{\alpha-1}w\left([ta^p+(1-t)b^p]^{1/p}\right)$ and integrating with respect to t over $[0,1]$, using Lemma 1-i, we get

$$\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\leq\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]$$

the right hand side of (12). This completes the proof of i.

(ii) The proof is similar with i.

Remark. In Theorem 9, one can see the following.

(1) If one takes $p=1$, one has (8),
(2) If one takes $p=1$ and $w(x)=1$, one has (6),
(3) If one takes $p=1$ and $\alpha=1$, one has (2),

(4) If one takes $p=1$, $\alpha=1$ and $w(x)=1$, one has (1),
(5) If one takes $p=-1$, one has (9),
(6) If one takes $p=-1$ and $w(x)=1$, one has (7),
(7) If one takes $p=-1$ and $\alpha=1$, one has (5),
(8) If one takes $p=-1$, $\alpha=1$ and $w(x)=1$, one has (4),
(9) If one takes $\alpha=1$ and $w(x)=1$, one has (11).

Lemma 2. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}\backslash\{0\}$ and $\alpha>0$. If $f'\in L[a,b]$ and $w:[a,b]\to\mathbb{R}$ is integrable and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then the following equalities for fractional integrals holds.*

(i) *If $p>0$,*

$$\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]-\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right] \tag{16}$$
$$=\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left[\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\,ds-\int_{t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\,ds\right](f\circ g)'(t)\,dt$$

with $g(x)=x^{1/p}$, $x\in[a^p,b^p]$,

(ii) *If $p<0$,*

$$\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right]-\left[J^{\alpha}_{b^p+}(fw\circ g)(a^p)+J^{\alpha}_{a^p-}(fw\circ g)(b^p)\right] \tag{17}$$
$$=\frac{1}{\Gamma(\alpha)}\int_{b^p}^{a^p}\left[\int_{b^p}^{t}(a^p-s)^{\alpha-1}(w\circ g)(s)\,ds-\int_{t}^{a^p}(s-b^p)^{\alpha-1}(w\circ g)(s)\,ds\right](f\circ g)'(t)\,dt$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof. (i) Let $p>0$. It suffices to note that

$$I=\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left[\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\,ds-\int_{t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\,ds\right](f\circ g)'(t)\,dt \tag{18}$$
$$=\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left[\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\,ds\right](f\circ g)'(t)\,dt-\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left[\int_{t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\,ds\right](f\circ g)'(t)\,dt$$
$$=I_1-I_2.$$

By integration by parts and using Lemma 1-i, we have

$$I_1=\frac{1}{\Gamma(\alpha)}(f\circ g)(t)\left(\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\,ds\right)\Bigg|_{a^p}^{b^p}-\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-t)^{\alpha-1}(fw\circ g)(t)\,dt \tag{19}$$
$$=f(b)\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-s)^{\alpha-1}(w\circ g)(s)\,ds-\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-t)^{\alpha-1}(fw\circ g)(t)\,dt$$
$$=\frac{f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]-J^{\alpha}_{a^p+}(fw\circ g)(b^p)$$

and similarly

$$I_2=\frac{1}{\Gamma(\alpha)}(f\circ g)(t)\left(\int_{t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\,ds\right)\Bigg|_{a^p}^{b^p}+\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(t-a^p)^{\alpha-1}(fw\circ g)(t)\,dt \tag{20}$$
$$=-f(a)\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\,ds+\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(t-a^p)^{\alpha-1}(fw\circ g)(t)\,dt$$
$$=-\frac{f(a)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]+J^{\alpha}_{b^p-}(fw\circ g)(a^p).$$

A combination of (18), (19) and (20) we have (16). This completes the proof of (i).

(ii) The proof is similar with (i).

Remark. In Lemma 2, one can see the following.

(1) If one takes $p=1$, one has [15, Lemma 2.4],
(2) If one takes $p=1$ and $w(x)=1$, one has [25, Lemma 2],
(3) If one takes $p=1$ and $\alpha=1$, one has [24, Lemma 2.6],
(4) If one takes $p=1$, $\alpha=1$ and $w(x)=1$, one has [4, Lemma 2.1],
(5) If one takes $p=-1$, one has [19, Lemma 3],
(6) If one takes $p=-1$ and $w(x)=1$, one has [14, Lemma 3],
(7) If one takes $p=-1$, $\alpha=1$ and $w(x)=1$, one has [13, 2.5. Lemma],
(8) If one takes $\alpha=1$ and $w(x)=1$, one has [22, Lemma 2.4].

Theorem 10. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $|f'|$ is p-convex function on $[a,b]$ for $p\in\mathbb{R}\backslash\{0\}$ and $\alpha>0$, $w:[a,b]\to\mathbb{R}$ is continuous and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then the following inequality for fractional integrals holds.*

(i) *If $p>0$,*

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]-\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\right|$$

$$\le\frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left[C_1(\alpha,p)\left|f'(a)\right|+C_2(\alpha,p)\left|f'(b)\right|\right]$$

where

$$C_1(\alpha,p)=\int_0^1\frac{\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}udu\quad\text{and}\quad C_2(\alpha,p)=\int_0^1\frac{\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}(1-u)du$$

with $g(x)=x^{1/p}$, $x\in[a^p,b^p]$,

(ii) *If $p<0$,*

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right]-\left[J^{\alpha}_{b^p+}(fw\circ g)(a^p)+J^{\alpha}_{a^p-}(fw\circ g)(b^p)\right]\right|$$

$$\le\frac{\|w\|_{\infty}(a^p-b^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left[C_3(\alpha,p)\left|f'(a)\right|+C_4(\alpha,p)\left|f'(b)\right|\right]$$

where

$$C_3(\alpha,p)=\int_0^1\frac{-\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}udu\quad\text{and}\quad C_4(\alpha,p)=\int_0^1\frac{-\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}(1-u)du$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof. (i) Let $p>0$. Using Lemma 2-i, it follows that

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]-\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\right| \tag{21}$$
$$\le\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left|\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)ds-\int_{t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)ds\right|\left|(f\circ g)'(t)\right|dt.$$

Since w is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, using Definition 4 we have $w\left(x^{1/p}\right)=w\left(\left[a^p+b^p-x\right]^{1/p}\right)$ for all $x\in\left[a^p,b^p\right]$,

$$\begin{aligned}
&\left|\int_{a^p}^{t}\left(b^p-s\right)^{\alpha-1}(w\circ g)(s)\,ds-\int_{t}^{b^p}\left(s-a^p\right)^{\alpha-1}(w\circ g)(s)\,ds\right| \qquad (22)\\
&=\left|\int_{a^p+b^p-t}^{b^p}\left(s-a^p\right)^{\alpha-1}(w\circ g)(s)\,ds+\int_{b^p}^{t}\left(s-a^p\right)^{\alpha-1}(w\circ g)(s)\,ds\right|\\
&=\left|\int_{a^p+b^p-t}^{t}\left(s-a^p\right)^{\alpha-1}(w\circ g)(s)\,ds\right|\leq\begin{cases}\int_{t}^{a^p+b^p-t}\left|\left(s-a^p\right)^{\alpha-1}(w\circ g)(s)\right|ds\ , & t\in\left[a^p,\frac{a^p+b^p}{2}\right]\\ \int_{a^p+b^p-t}^{t}\left|\left(s-a^p\right)^{\alpha-1}(w\circ g)(s)\right|ds\ , & t\in\left[\frac{a^p+b^p}{2},b^p\right]\end{cases}.
\end{aligned}$$

A combination of (21) and (22), we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha}(w\circ g)\left(b^p\right)+J_{b^p-}^{\alpha}(w\circ g)\left(a^p\right)\right]-\left[J_{a^p+}^{\alpha}(fw\circ g)\left(b^p\right)+J_{b^p-}^{\alpha}(fw\circ g)\left(a^p\right)\right]\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_{t}^{a^p+b^p-t}\left|\left(s-a^p\right)^{\alpha-1}(w\circ g)(s)\right|ds\right)\left|(f\circ g)'(t)\right|dt\\ +\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}\left|\left(s-a^p\right)^{\alpha-1}(w\circ g)(s)\right|ds\right)\left|(f\circ g)'(t)\right|dt\end{array}\right]\\
&\leq\frac{\|w\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_{t}^{a^p+b^p-t}\left(s-a^p\right)^{\alpha-1}ds\right)\left|(f\circ g)'(t)\right|dt\\ +\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}\left(s-a^p\right)^{\alpha-1}ds\right)\left|(f\circ g)'(t)\right|dt\end{array}\right]\\
&\leq\frac{\|w\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_{t}^{a^p+b^p-t}\left(s-a^p\right)^{\alpha-1}ds\right)\frac{1}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right|dt\\ +\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}\left(s-a^p\right)^{\alpha-1}ds\right)\frac{1}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right|dt\end{array}\right]\\
&\leq\frac{\|w\|_{\infty}}{\Gamma(\alpha+1)}\left[\begin{array}{l}\int_{a^p}^{\frac{a^p+b^p}{2}}\frac{\left(b^p-t\right)^{\alpha}-\left(t-a^p\right)^{\alpha}}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right|dt\\ +\int_{\frac{a^p+b^p}{2}}^{b^p}\frac{\left(t-a^p\right)^{\alpha}-\left(b^p-t\right)^{\alpha}}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right|dt\end{array}\right]
\end{aligned}$$

Setting $t=ua^p+(1-u)b^p$ and $dt=\left(a^p-b^p\right)du$ gives

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha}(w\circ g)\left(b^p\right)+J_{b^p-}^{\alpha}(w\circ g)\left(a^p\right)\right]-\left[J_{a^p+}^{\alpha}(fw\circ g)\left(b^p\right)+J_{b^p-}^{\alpha}(fw\circ g)\left(a^p\right)\right]\right| \qquad (23)\\
&\leq\frac{\|w\|_{\infty}\left(b^p-a^p\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{l}\int_{0}^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|du\\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|du\end{array}\right]\\
&=\frac{\|w\|_{\infty}\left(b^p-a^p\right)^{\alpha+1}}{\Gamma(\alpha+1)}\int_{0}^{1}\frac{\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|du.
\end{aligned}$$

Since $|f'|$ is p-convex function on $[a,b]$, we have

$$\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|\leq u\left|f'(a)\right|+(1-u)\left|f'(b)\right|. \qquad (24)$$

A combination of (23) and (24), we have

$$\left|\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha}(w\circ g)\left(b^p\right)+J_{b^p-}^{\alpha}(w\circ g)\left(a^p\right)\right]-\left[J_{a^p+}^{\alpha}(fw\circ g)\left(b^p\right)+J_{b^p-}^{\alpha}(fw\circ g)\left(a^p\right)\right]\right|$$

$$\leq \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)} \left[\begin{array}{c} \int_0^1 \frac{\left|(1-u)^\alpha-u^\alpha\right|}{p[ua^p+(1-u)b^p]^{1-(1/p)}} udu \left|f'(a)\right| \\ +\int_0^1 \frac{\left|(1-u)^\alpha-u^\alpha\right|}{p[ua^p+(1-u)b^p]^{1-(1/p)}} (1-u)du\left|f'(b)\right| \end{array} \right]$$

$$= \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)} \left[C_1(\alpha,p)\left|f'(a)\right| + C_2(\alpha,p)\left|f'(b)\right|\right].$$

This completes the proof of (i).

(ii) The proof is similar with (i).

Remark. In Theorem 10, one can see the following.

(1) If one takes $p=1$, one has [15, Theorem 2.6],
(2) If one takes $p=1$ and $w(x)=1$, one has [25, Theorem 3],
(3) If one takes $p=1$, $\alpha=1$ and $w(x)=1$, one has [4, Theorem 2.2],
(4) If one takes $\alpha=1$ and $w(x)=1$, one has [22, Teheorem 3.1].

Corollary 1. *In Theorem 10, one can see the following.*

(1) *If one takes $p=1$ and $\alpha=1$, one has the following Hermite-Hadamard-Fejer inequality for convex functions:*

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \frac{1}{b-a}\int_a^b f(x)w(x)dx\right| \leq \frac{\|w\|_\infty (b-a)}{2}\left[C_1(1,1)\left|f'(a)\right| + C_2(1,1)\left|f'(b)\right|\right],$$

(2) *If one takes $p=-1$, one has the following Hermite-Hadamard-Fejer inequality for harmonically convex functions via fractional integrals:*

$$\left|\frac{f(a)+f(b)}{2}\left[J^\alpha_{1/b+}(w\circ g)(1/a) + J^\alpha_{1/a-}(w\circ g)(1/b)\right] - \left[J^\alpha_{1/b+}(fw\circ g)(1/a) + J^\alpha_{1/a-}(fw\circ g)(1/b)\right]\right|$$
$$\leq \frac{\|w\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha \left[C_3(\alpha,-1)\left|f'(a)\right| + C_4(\alpha,-1)\left|f'(b)\right|\right],$$

(3) *If one takes $p=-1$, $\alpha=1$ and $w(x)=1$, one has the following Hermite-Hadamard inequality for harmonically convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \left(\frac{b-a}{ab}\right)\left[C_3(1,-1)\left|f'(a)\right| + C_4(1,-1)\left|f'(b)\right|\right],$$

(4) *If one takes $p=-1$ and $\alpha=1$, one has the following Hermite-Hadamard-Fejer inequality for harmonically convex functions:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^2}dx - \int_a^b \frac{f(x)w(x)}{x^2}dx\right| \leq \frac{\|w\|_\infty (b-a)^2}{2}\left[C_3(1,-1)\left|f'(a)\right| + C_4(1,-1)\left|f'(b)\right|\right],$$

(5) *If one takes $p=-1$ and $w(x)=1$, one has the following Hermite-Hadamard inequality for harmonically convex functions via fractional integrals.*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2\left(\frac{1}{b}-\frac{1}{a}\right)^\alpha}\left[J^\alpha_{1/b+}(f\circ g)(1/a) + J^\alpha_{1/a-}(f\circ g)(1/b)\right]\right|$$
$$\leq \left(\frac{b-a}{ab}\right)\left[C_3(\alpha,-1)\left|f'(a)\right| + C_4(\alpha,-1)\left|f'(b)\right|\right].$$

Theorem 11. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ and $a < b$. If $|f'|^q$, $q \geq 1$, is p-convex function on $[a,b]$ for $p \in \mathbb{R} \backslash \{0\}$, $\alpha > 0$, $w : [a,b] \to \mathbb{R}$ is continuous and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then the following inequality for fractional integrals holds.*

(i) *If $p > 0$,*

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]-\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\right|$$
$$\leq \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)} C_5^{1-\frac{1}{q}}(\alpha,p)\left[C_1(\alpha,p)\left|f'(a)\right|^q + C_2(\alpha,p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}$$

where $C_1(\alpha,p)$, $C_2(\alpha,p)$ are the same in Theorem 10,

$$C_5(\alpha,p) = \int_0^1 \frac{\left|(1-u)^\alpha - u^\alpha\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du$$

with $g(x) = x^{1/p}$, $x \in [a^p, b^p]$,

(ii) *If $p < 0$,*

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right]-\left[J^{\alpha}_{b^p+}(fw\circ g)(a^p)+J^{\alpha}_{a^p-}(fw\circ g)(b^p)\right]\right|$$
$$\leq \frac{\|w\|_\infty (a^p-b^p)^{\alpha+1}}{\Gamma(\alpha+1)} C_6^{1-\frac{1}{q}}(\alpha,p)\left[C_3(\alpha,p)\left|f'(a)\right|^q + C_4(\alpha,p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}$$

where $C_3(\alpha,p)$, $C_4(\alpha,p)$ are the same in Theorem 10,

$$C_6(\alpha,p) = \int_0^1 \frac{-\left|(1-u)^\alpha - u^\alpha\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du$$

with $g(x) = x^{1/p}$, $x \in [b^p, a^p]$.

Proof. (i) Let $p > 0$. Using (23), power mean inequality and the p-convexity of $|f'|^q$it follows that

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]-\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\right|$$
$$\leq \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1 \frac{\left|(1-u)^\alpha - u^\alpha\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|du$$
$$\leq \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^1 \frac{\left|(1-u)^\alpha - u^\alpha\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \frac{\left|(1-u)^\alpha - u^\alpha\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|^q du\right)^{\frac{1}{q}}$$
$$\leq \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^1 \frac{\left|(1-u)^\alpha - u^\alpha\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\left(\int_0^1 \frac{\left|(1-u)^\alpha-u^\alpha\right|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}udu\right)\left|f'(a)\right|^q \\ +\left(\int_0^1 \frac{\left|(1-u)^\alpha-u^\alpha\right|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}(1-u)du\right)\left|f'(b)\right|^q\end{array}\right)^{\frac{1}{q}}$$
$$= \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)} C_5^{1-\frac{1}{q}}(\alpha,p)\left(C_1(\alpha,p)\left|f'(a)\right|^q + C_2(\alpha,p)\left|f'(b)\right|^q\right)^{\frac{1}{q}}.$$

This completes the proof of (i).

(ii) The proof is similar with (i).

Remark. In Theorem 11, one can see the following.

(1) If one takes $p=1$, one has [15, Theorem 2.8],
(2) If one takes $p=1$, $\alpha=1$ and $w(x)=1$, one has [23, Theorem 1],
(3) If one takes $p=-1$ and $w(x)=1$, one has [14, Theorem 5],
(4) If one takes $p=-1$, $\alpha=1$ and $w(x)=1$, one has [13, 2.6. Theorem],
(5) If one takes $\alpha=1$ and $w(x)=1$, one has [22, Theorem 3.2].

Corollary 2. *In Theorem 11, one can see the following.*

(1) *If one takes $p=1$ and $w(x)=1$, one has the following Hermite-Hadamard inequality for convex functions via fractional integrals:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)-J_{b-}^{\alpha}f(a)\right]\right|\leq\frac{(b-a)}{2}C_5^{1-\frac{1}{q}}(\alpha,1)\left[C_1(\alpha,1)\left|f'(a)\right|^q+C_2(\alpha,1)\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

(2) *If one takes $p=1$ and $\alpha=1$, one has the following Hermite-Hadamard-Fejer inequality for convex functions:*

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b f(x)w(x)\,dx\right|\leq\frac{\|w\|_\infty(b-a)}{2}C_5^{1-\frac{1}{q}}(1,1)\left[C_1(1,1)\left|f'(a)\right|^q+C_2(1,1)\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

(3) *If one takes $p=-1$, one has the following Hermite-Hadamard-Fejer inequality for harmonically convex functions via fractional integrals:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}(w\circ g)(1/a)+J_{1/a-}^{\alpha}(w\circ g)(1/b)\right]-\left[J_{1/b+}^{\alpha}(fw\circ g)(1/a)+J_{1/a-}^{\alpha}(fw\circ g)(1/b)\right]\right|\\&\leq\frac{\|w\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}C_6^{1-\frac{1}{q}}(\alpha,-1)\left[C_3(\alpha,-1)\left|f'(a)\right|^q+C_4(\alpha,-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}},\end{aligned}$$

(4) *If one takes $p=-1$ and $\alpha=1$, one has the following Hermite-Hadamard-Fejer inequality for harmonically convex functions:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{w(x)}{x^2}dx-\int_a^b\frac{f(x)w(x)}{x^2}dx\right|\leq\frac{\|w\|_\infty(b-a)^2}{2}C_6^{1-\frac{1}{q}}(1,-1)\left[C_3(1,-1)\left|f'(a)\right|^q+C_4(1,-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}.$$

Theorem 12. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $|f'|^q$, $q>1$, is p-convex function on $[a,b]$ for $p\in\mathbb{R}\setminus\{0\}$, $\alpha>0$, $\frac{1}{q}+\frac{1}{r}=1$, $w:[a,b]\to\mathbb{R}$ is continuous and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then the following inequality for fractional integrals holds.*

(i) *If $p>0$,*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha}(w\circ g)(b^p)+J_{b^p-}^{\alpha}(w\circ g)(a^p)\right]-\left[J_{a^p+}^{\alpha}(fw\circ g)(b^p)+J_{b^p-}^{\alpha}(fw\circ g)(a^p)\right]\right|\\&\leq\frac{\|w\|_\infty(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}C_7^{\frac{1}{r}}(\alpha,p,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\end{aligned}$$

where

$$C_7(\alpha,p,r)=\int_0^1\left(\frac{\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\right)^r du$$

with $g(x)=x^{1/p}$, $x\in[a^p,b^p]$,

(ii) *If* $p<0$,

$$\left|\frac{f(a)+f(b)}{2}\left[J_{b^p+}^{\alpha}(w\circ g)(a^p)+J_{a^p-}^{\alpha}(w\circ g)(b^p)\right]-\left[J_{b^p+}^{\alpha}(fw\circ g)(a^p)+J_{a^p-}^{\alpha}(fw\circ g)(b^p)\right]\right|$$
$$\leq \frac{\|w\|_{\infty}(a^p-b^p)^{\alpha+1}}{\Gamma(\alpha+1)}C_8^{\frac{1}{r}}(\alpha,p,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

where

$$C_8(\alpha,p,r)=\int_0^1\left(\frac{-\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\right)^r du$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof. (i) Let $p>0$. Using (23), Hölder's inequality and the p-convexity of $|f'|^q$ it follows that

$$\left|\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha}(w\circ g)(b^p)+J_{b^p-}^{\alpha}(w\circ g)(a^p)\right]-\left[J_{a^p+}^{\alpha}(fw\circ g)(b^p)+J_{b^p-}^{\alpha}(fw\circ g)(a^p)\right]\right|$$
$$\leq \frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1\frac{\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|du$$
$$\leq \frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^1\left(\frac{\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\right)^r du\right)^{\frac{1}{r}}$$
$$\times\left(\int_0^1\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|^q du\right)^{\frac{1}{q}}$$
$$\leq \frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^1\left(\frac{\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\right)^r du\right)^{\frac{1}{r}}$$
$$\times\left(\int_0^1 u\left|f'(a)\right|^q+(1-u)\left|f'(b)\right|^q du\right)^{\frac{1}{q}}$$
$$= \frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^1\left(\frac{\left|(1-u)^{\alpha}-u^{\alpha}\right|}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\right)^r du\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$
$$= \frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}C_7^{\frac{1}{r}}(\alpha,p,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

This completes the proof of (i).

(ii) The proof is similar with (i).

Remark. In Theorem 12, one can see the following.

(1) If one takes $p=1$, one has [15, Theorem 2.9-i],
(2) If one takes $p=1$, $\alpha=1$ and $w(x)=1$, one has [4, Theorem 2.3],
(3) If one takes $p=1$ and $\alpha=1$, one has [24, Theorem 2.8].

Corollary 3. *In Theorem 12, one can see the following.*

(1) *If one takes $p=1$ and $w(x)=1$, one has the following Hermite-Hadamard inequality for convex functions via fractional integrals:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)-J_{b-}^{\alpha}f(a)\right]\right|\leq\frac{b-a}{2}C_7^{\frac{1}{r}}(\alpha,1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

(2) *If one takes* $p=-1$ *and* $w(x)=1$*, one has the following Hermite-Hadamard inequality for harmonically convex functions via fractional integrals:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\frac{1}{b}-\frac{1}{a}\right)^{\alpha}}\left[J^{\alpha}_{1/b+}(f\circ g)(1/a)+J^{\alpha}_{1/a-}(f\circ g)(1/b)\right]\right|\leq\left(\frac{b-a}{ab}\right)C_8^{\frac{1}{r}}(\alpha,-1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

(3) *If one takes* $p=-1$, $\alpha=1$ *and* $w(x)=1$*, one has the following Hermite-Hadamard inequality for harmonically convex functions:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\left(\frac{b-a}{ab}\right)C_8^{\frac{1}{r}}(1,-1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

(4) *If one takes* $p=-1$*, one has the following Hermite-Hadamard-Fejer inequality for harmonically convex functions via fractional integrals:*

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(w\circ g)(1/a)+J^{\alpha}_{1/a-}(w\circ g)(1/b)\right]-\left[J^{\alpha}_{1/b+}(fw\circ g)(1/a)+J^{\alpha}_{1/a-}(fw\circ g)(1/b)\right]\right|$$

$$\leq\frac{\|w\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}C_8^{\frac{1}{r}}(\alpha,-1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

(5) *If one takes* $p=-1$ *and* $\alpha=1$*, one has the following Hermite-Hadamard-Fejer inequality for harmonically convex functions.*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{w(x)}{x^2}dx-\int_a^b\frac{f(x)w(x)}{x^2}dx\right|\leq\frac{\|w\|_{\infty}(b-a)^2}{2}C_8^{\frac{1}{r}}(1,-1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

(6) *If one takes* $\alpha=1$ *and* $w(x)=1$*, one has the following Hermite-Hadamard inequality for* p*-convex functions.*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\begin{cases}\frac{(b^p-a^p)}{2}C_7^{\frac{1}{r}}(1,p,r)\ ,\ p>0\\ \frac{(a^p-b^p)}{2}C_8^{\frac{1}{r}}(1,p,r)\ ,\ p<0\end{cases}.$$

3 Conclusion

In Theorem 9, Hermite-Hadamard-Fejer type inequalities for p-convex functions in fractional integral forms are built. In Lemma 2, an integral identity and in Theorem 10, Theorem 11 and Theorem 12, some Hermite-Hadamard-Fejer type integral inequalities for p-convex functions in fractional integral forms are obtained. In Corollary 1, Corollary 2 and Corollary 3, some Hermite-Hadamard and Hermite-Hadamard-Fejer inequalities for convex, harmonically convex and p-convex functions are given. Some results presented Remark 2, Remark 2 and Remark 2, provide extensions of others given in earlier works for convex, harmonically convex and p-convex fuctions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite Hadamard Fejér inequalities, Computers and Mathematics with Applications 58 (2009), 1869 1877.

[2] F. Chen and S. Wu, Fejér and Hermite-Hadamard type inqequalities for harmonically convex functions, Jurnal of applied Mathematics, volume 2014, article id:386806.

[3] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, Ann. Funct. Anal. 1(1) (2010), 51-58.

[4] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, App. Math. Lett., 11(5) (1998), 91-95.

[5] L. Fejér, Uber die Fourierreihen, II, Math. Naturwise. Anz Ungar. Akad., Wiss, 24 (1906), 369-390, (in Hungarian).

[6] Z. B. Fang, R. Shi, On the (p,h)-convex function and some integral inequalities, J. Inequal. Appl., 2014 (45) (2014), 16 pages.

[7] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann, J. Math. Pures Appl., 58 (1893), 171-215.

[8] Ch. Hermite, Sur deux limites d'une intégrale définie, Mathesis, 3 (1883), 82–83.

[9] İ. İşcan, New estimates on generalization of some integral inequalities for *s*-convex functions and their applications, Int. J. Pure Appl. Math., 86(4) (2013), 727-746.

[10] İ. İşcan, Generalization of different type integral inequalitiesfor *s*-convex functions via fractional integrals, Applicable Analysis, 2013. doi: 10.1080/00036811.2013.851785.

[11] İ. İşcan, On generalization of different type integral inequalities for *s*-convex functions via fractional integrals, Mathematical Sciences and Applications E-Notes, 2(1) (2014), 55-67.

[12] İ. İşcan, Some new general integral inequalities for *h*-convex and *h*-concave functions, Adv. Pure Appl. Math. 5(1) (2014), 21-29. doi: 10.1515/apam-2013-0029.

[13] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacet. J. Math. Stat., 43 (6) (2014), 935-942.

[14] İ. İşcan, S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, Appl. Math. Comput., 238 (2014) 237-244.

[15] İ. İşcan, Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals, Studia Universitatis Babeş-Bolyai Mathematica, vol. 60, no. 3, (2015) 355-366.

[16] İ. İşcan, Ostrowski type inequalities for p-convex functions, doi:10.13140/RG.2.1.1028.5209, Available online at https://www.researchgate.net/publication/299593487.

[17] İ. İşcan, Hermite-Hadamard type inequalities for p-convex functions, doi:10.13140/RG.2.1.2339.2404. Available online at https://www.researchgate.net/publication/299594155.

[18] İ. İşcan, Hermite-Hadamard and Simpson-like type inequalities for differantiable *p*-quasi-convex functions, doi:10.13140/RG.2.1.2589.4801, Available online at https://www.researchgate.net/publication/299610889.

[19] İ. İşcan, M. Kunt, Hermite-Hadamard-Fejér type inequalities for harmonically convex functions via fractional integrals, RGMIA, 18(2015), Article 107, 16 pp.

[20] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations. Elsevier, Amsterdam (2006).

[21] M. V. Mihai, M. A. Noor, K. I. Noor, M. U. Awan, New estimates for trapezoidal like inequalities via differentiable (p,h)-convex functions, doi:10.13140/RG.2.1.5106.5046, Available online at https://www.researchgate.net/publication/282912293

[22] M. A. Noor, K. I. Noor, M. V. Mihai, M. U. Awan, Hermite-Hadamard inequalities for differentiable *p*-convex functions using hypergeometric functions, doi:10.13140/RG.2.1.2485.0648, Available online at https://www.researchgate.net/publication/282912282.

[23] Pearce, C.E.M., Pecaric, J., Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2)(2000), 51–55.

[24] M.Z. Sarıkaya, On new Hermite Hadamard Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai Math. 57(3) (2012), 377–386.

[25] M.Z. Sarıkaya, E. Set, H. Yaldız and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, 57(9) (2013), 2403-2407.

[26] K.-L. Tseng, G.-S. Yang and K.-C. Hsu, Some inequalities for differentiable mappings and applications to Fejér inequality and weighted trapezoidal formula, Taiwanese journal of Mathematics, 15(4) (2011), 1737-1747.

[27] J. Wang, X. Li, M. Fečkan and Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, Appl. Anal., 92(11) (2012), 2241-2253. doi:10.1080/00036811.2012.727986

[28] J. Wang, C. Zhu and Y. Zhou, New generalized Hermite-Hadamard type inequalities and applications to special means, J. Inequal. Appl., 2013(325) (2013), 15 pages.

[29] K. S. Zhang, J. P. Wan, p-convex functions and their properties, Pure Appl. Math. 23(1) (2007), 130-133.