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Computation of growth rates of composite entire and meromorphic functions from the view point of their relative L^*-orders

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Abstract: In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using relative L^*-order and relative L^*-lower order as compared to their corresponding left and right factors.

Keywords: Entire function, meromorphic function, composition, growth, relative L^*-order, relative L^*-lower order, slowly changing function.

1 Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [2] and [7]. In the sequel we use the following notation : $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1,2,3,....$ and $\log^{[0]} x = x$.

The following definition is well known.

Definition 1. *The order ρ_f and lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r},$$

when f is meromorphic, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [5] defined it in the following way.

Definition 2. *[5] A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon(>0)$,*

$$\frac{1}{k^{\varepsilon}} \le \frac{L(kr)}{L(r)} \le k^{\varepsilon} \text{ for } r \ge r(\varepsilon) \text{ and}$$

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uniformly for $k\,(\geq 1)$. *If further,* $L\,(r)$ *is differentiable, the above condition is equivalent to*

$$\lim_{r\to\infty}\frac{rL'\,(r)}{L\,(r)}=0\,.$$

Somasundaram and Thamizharasi [6] introduced the notions of L-order and L-lower order for entire functions. The more generalised concept for L-order and L-lower order for entire and meromorphic functions are L^*-order and L^*-lower order respectively. Their definitions are as follows:

Definition 3. *[6] The* L^**-order* $\rho_f^{L^*}$ *and the* L^**-lower order* $\lambda_f^{L^*}$ *of an entire function* f *are defined as*

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log^{[2]}M_f\,(r)}{\log\left[re^{L(r)}\right]}\ \text{and}\ \lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log^{[2]}M_f\,(r)}{\log\left[re^{L(r)}\right]},$$

when f *is meromorphic, one can easily verify that*

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log T_f\,(r)}{\log\left[re^{L(r)}\right]}\ \text{and}\ \lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log T_f\,(r)}{\log\left[re^{L(r)}\right]}.$$

For an entire function g, the Nevanlinna's characteristic function $T_g\,(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|g(re^{i\theta})\right|d\theta$ where $\log^+ x=\max\,(0,\log x)$ for $x>0$.If g is non-constant then $T_g\,(r)$ is strictly increasing and continuous and its inverse $T_g^{-1}:(T_g\,(0)\,,\infty)\to(0,\infty)$ exists and is such that $\lim\limits_{s\to\infty}T_g^{-1}\,(s)=\infty$.

Lahiri and Banerjee [4] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows.

Definition 4. *[4] Let* f *be meromorphic and* g *be entire. The relative order of* f *with respect to* g *denoted by* $\rho_g\,(f)$ *is defined as*

$$\begin{aligned}\rho_g\,(f)&=\inf\left\{\mu>0:T_f\,(r)<T_g\,(r^{\mu})\ \text{for all sufficiently large}\ r\right\}\\&=\limsup_{r\to\infty}\frac{\log T_g^{-1}T_f\,(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [4] if $g\,(z)=\exp z$.
Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g\,(f)$ in the following manner :

$$\lambda_g\,(f)=\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f\,(r)}{\log r}.$$

In the line of Somasundaram and Thamizharasi [6] and Lahiri and Banerjee [4] one may define the relative L^*-order and relative L^*-lower order of a meromorphic function f with respect to an entire function g in the following manner.

Definition 5. *The relative* L^**-order* $\rho_g^{L^*}\,(f)$ *and the relative* L^**-lower order* $\lambda_g^{L^*}\,(f)$ *of a meromorphic function* f *with respect to an entire function* g *are defined by*

$$\rho_g^{L^*}\,(f)=\limsup_{r\to\infty}\frac{\log T_g^{-1}T_f\,(r)}{\log\left[re^{L(r)}\right]}\ \text{and}\ \lambda_g^{L^*}\,(f)=\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f\,(r)}{\log\left[re^{L(r)}\right]}.$$

In this paper we study some growth properties of composition of entire and meromorphic functions with respect to relative L^*-order and relative L^*-lower order as compared to the corresponding left and right factors.

2 Lemmas

In this section we present a lemma which will be needed in the sequel.

Lemma 1. *[1] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) \geq T_f\left(\exp(r)^{\mu}\right).$$

3 Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f be a meromorphic function and g, h be entire such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. Then for any $A > 0$,*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) + K(r,A;L)} = \infty,$$

where $0 < \mu < \rho_g$ and

$$K(r,A;L) = \begin{cases} 0 \text{ if } r^{\mu} = o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad \text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) \text{ otherwise}. \end{cases}$$

Proof. Let $0 < \mu < \mu' < \rho_g$. Since T_h^{-1} is an increasing functions, from the definition of relative L^*-lower order we obtain in view of Lemma 1, for a sequence of values of r tending to infinity that

$$\begin{aligned}
\log T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right) &\geq \log T_h^{-1} T_f\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right) \\
&\geq \left(\lambda_h^{L^*}(f) - \varepsilon\right).\log\left\{\exp\left(\exp\left(r^A\right)\right)^{\mu'}.\exp L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)\right\} \\
&\geq \left(\lambda_h^{L^*}(f) - \varepsilon\right).\left\{\left(\exp\left(r^A\right)\right)^{\mu'} + L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)\right\} \\
&\geq \left(\lambda_h^{L^*}(f) - \varepsilon\right).\left\{\left(\exp\left(r^A\right)\right)^{\mu'}\left(1 + \frac{L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)}{\left(\exp\left(r^A\right)\right)^{\mu'}}\right)\right\}.
\end{aligned}$$

$$\begin{aligned}
\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right) &\geq O(1) + \mu' \log\exp\left(r^A\right) + \log\left\{1 + \frac{L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)}{\left(\exp\left(r^A\right)\right)^{\mu'}}\right\} \\
&\geq O(1) + \mu' r^A + \log\left\{1 + \frac{L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)}{\left(\exp\left(r^A\right)\right)^{\mu'}}\right\} \\
&\geq O(1) + \mu' r^A + \log\left[1 + \frac{L\left(\exp\left(\exp\left(\mu' r^A\right)\right)\right)}{\exp\left(\mu' r^A\right)}\right] \\
&\geq O(1) + \mu' r^A + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) - \log\left[\exp\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\}\right] \\
&\quad + \log\left[1 + \frac{L\left(\exp\left(\exp\left(\mu' r^A\right)\right)\right)}{\exp\left(\mu' r^A\right)}\right]
\end{aligned}$$

$$\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right) \geq O(1) + \mu' r^A + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) + \log\left[\frac{1}{\exp\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\}} + \frac{L\left(\exp\left(\exp\left(\mu' r^A\right)\right)\right)}{\exp\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\}.\exp\left(\mu' r^A\right)}\right]$$
$$\geq O(1) + \mu' r^{(A-\mu)}.r^{\mu} + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right). \quad (1)$$

Again we have for all sufficiently large values of r that

$$\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) \leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left\{\exp\left(r^{\mu}\right)e^{L\left(\exp\left(r^{\mu}\right)\right)}\right\}$$
$$\leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\left\{\log\exp\left(r^{\mu}\right) + L\left(\exp\left(r^{\mu}\right)\right)\right\}$$
$$\leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\left\{r^{\mu} + L\left(\exp\left(r^{\mu}\right)\right)\right\}.$$

Thus,

$$\frac{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) - \left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\exp\left(r^{\mu}\right)\right)}{\left(\rho_h^{L^*}(f)+\varepsilon\right)} \leq r^{\mu}. \quad (2)$$

Now from (1) and (2) it follows for a sequence of values of r tending to infinity that

$$\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right) \geq O(1) + \left(\frac{\mu' r^{(A-\mu)}}{\rho_h^{L^*}(f)+\varepsilon}\right)\left[\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) - \left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\exp\left(r^{\mu}\right)\right)\right] + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right). \quad (3)$$

Therefore

$$\frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)} \geq \frac{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) + O(1)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)} + \frac{\mu' r^{(A-\mu)}}{\rho_h^{L^*}(f)+\varepsilon}\left\{1 - \frac{\left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\exp\left(r^{\mu}\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)}\right\}. \quad (4)$$

Again from (3) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)} \geq \frac{O(1) - \mu' r^{(A-\mu)} L\left(\exp\left(r^{\mu}\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)} + \frac{\left(\frac{\mu' r^{(A-\mu)}}{\rho_h^{L^*}(f)+\varepsilon}\right)\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)} + \frac{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)}.$$

Therefore

$$\frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)} \geq \frac{\frac{O(1) - \mu' r^{(A-\mu)} L\left(\exp\left(r^{\mu}\right)\right)}{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)}}{\frac{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)}{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)} + 1}$$
$$+ \frac{\left(\frac{\mu' r^{(A-\mu)}}{\rho_h^{L^*}(f)+\varepsilon}\right)\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)}{1 + \frac{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)}} + \frac{1}{1 + \frac{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)}{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)}}. \quad (5)$$

Case 1. If $r^{\mu} = o\left\{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\}$ then it follows from (4) that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)} = \infty\,.$$

Case 2. $r^{\mu} \neq o\left\{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\}$ then two sub cases may arise.

Case 2.1 *If* $L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) = o\left\{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)\right\}$, *then we get from* (5) *that*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} = \infty\,.$$

Case 2.2 *If* $L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right) \sim \log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)$ *then*

$$\lim_{r\to\infty} \frac{L\left\{\exp\left(\exp\left(\mu r^{A}\right)\right)\right\}}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)} = 1$$

and we obtain from (5) *that*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} = \infty\,.$$

Combining Case I and Case II we may obtain that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} = \infty\,,$$

where

$$K(r,A;L) = \begin{cases} 0 \text{ if } r^{\mu} = o\left\{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad \text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \text{ otherwise}\,. \end{cases}$$

This proves the theorem.

Theorem 2. *Let f be a meromorphic function and g,h be entire such that $\lambda_h^{L^*}(f) > 0$ and $\rho_h^{L^*}(g) < \infty$. Then for any $A > 0$*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_h^{-1} T_g\left(\exp\left(r^{\mu}\right)\right) + K(r,A;L)} = \infty\,,$$

where $0 < \mu < \rho_g$ *and*

$$K(r,A;L) = \begin{cases} 0 \textit{ if } r^{\mu} = o\left\{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad \textit{as } r\to\infty \\ L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \textit{ otherwise}\,. \end{cases}$$

The proof is omitted because it can be carried out in the line of Theorem 1.

Theorem 3. *Let f be meromorphic and g and h be any two entire functions such that $0 < \lambda_h^{L^*}(f\circ g) \leq \rho_h^{L^*}(f\circ g) < \infty$ and $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. If $L\left(r^A\right) = o\left\{\log T_h^{-1} T_f\left(r^A\right)\right\}$ as $r\to\infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)} \leq \liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right) + L\left(r^A\right)} \leq \frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)} \leq \limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right) + L\left(r^A\right)} \leq \frac{\rho_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}\,.$$

Proof. From the definition of relative L^*-order and relative L^*-lower order of a meromorphic function with respect to an entire function we have for arbitrary positive ε and for all sufficiently large values of r that

$$\begin{aligned}\log T_h^{-1}T_{f\circ g}(r) &\geq \left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\log\left\{re^{L(r)}\right\}\\ &\geq \left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\{\log r+L(r)\}\\ &\geq \left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\log r+\frac{1}{A}L\left(r^A\right)\right\}+\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\} \qquad (6)\end{aligned}$$

and

$$\begin{aligned}\log T_h^{-1}T_f\left(r^A\right) &\leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left\{r^Ae^{L\left(r^A\right)}\right\}\\ &\leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\left\{A\log r+L\left(r^A\right)\right\}.\end{aligned}$$

So we have,

$$\frac{\log T_h^{-1}T_f\left(r^A\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)} \leq \log r+\frac{1}{A}L\left(r^A\right). \qquad (7)$$

Now from (6) and (7) it follows for all sufficiently large values of r that

$$\log T_h^{-1}T_{f\circ g}(r) \geq \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}\log T_h^{-1}T_f\left(r^A\right)+\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\},$$

Therefore we write

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}\cdot\frac{\log T_h^{-1}T_f\left(r^A\right)}{\log T_h^{-1}T_f\left(r^A\right)+L\left(r^A\right)}+\frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log T_h^{-1}T_f\left(r^A\right)+L\left(r^A\right)}$$

Finally

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\frac{\lambda_h^{L^*}(f\circ g)-\varepsilon}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}}{1+\frac{L\left(r^A\right)}{\log T_h^{-1}T_f\left(r^A\right)}}+\frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\frac{L(r)}{L\left(r^A\right)}-\frac{1}{A}\right\}}{1+\frac{\log T_h^{-1}T_f\left(r^A\right)}{L\left(r^A\right)}}. \qquad (8)$$

Since $L\left(r^A\right)=o\left\{\log T_h^{-1}T_f\left(r^A\right)\right\}$ as $r\to\infty$, it follows from (8) that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}. \qquad (9)$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (9) that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\lambda_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}. \qquad (10)$$

Again for a sequence of values of r tending to infinity,

$$\log T_h^{-1} T_{f\circ g}(r) \leq \left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\log\left\{re^{L(r)}\right\}$$
$$\leq \left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\log r+\frac{1}{A}L\left(r^A\right)\right\}+\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\} \tag{11}$$

and for all sufficiently large values of r,

$$\log T_h^{-1} T_f\left(r^A\right) \geq \left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}$$
$$\geq \left(\lambda_h^{L^*}(f)-\varepsilon\right)\left\{A\log r+L\left(r^A\right)\right\}.$$

We obtain,

$$\frac{\log T_h^{-1} T_f\left(r^A\right)}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)} \geq \log r+\frac{1}{A}L\left(r^A\right). \tag{12}$$

Combining (11) and (12) we get for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \leq \frac{\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}\log T_h^{-1} T_f\left(r^A\right)+\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}.$$

That is,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} \leq \frac{\lambda_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}\cdot\frac{\log T_h^{-1} T_f\left(r^A\right)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)}+\frac{\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)}.$$

Therefore,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} \leq \frac{\frac{\lambda_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}}{1+\frac{L\left(r^A\right)}{\log T_h^{-1} T_f\left(r^A\right)}}+\frac{\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\frac{L(r)}{L\left(r^A\right)}-\frac{1}{A}\right\}}{1+\frac{\log T_h^{-1} T_f\left(r^A\right)}{L\left(r^A\right)}}. \tag{13}$$

As $L\left(r^A\right)=o\left\{\log T_h^{-1} T_f\left(r^A\right)\right\}$ as $r\to\infty$ we get from (13) that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} \leq \frac{\lambda_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}. \tag{14}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (14) that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} \leq \frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}. \tag{15}$$

Also for a sequence of values of r tending to infinity,

$$\log T_h^{-1} T_f\left(r^A\right) \leq \left(\lambda_h^{L^*}(f)+\varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}$$
$$\leq \left(\lambda_h^{L^*}(f)+\varepsilon\right)\left\{A\log r+L\left(r^A\right)\right\},$$

And we have,

$$i.e., \frac{\log T_h^{-1} T_f\left(r^A\right)}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)} \leq \log r + \frac{1}{A} L\left(r^A\right) . \tag{16}$$

Now from (6) and (16) we obtain for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \geq \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)} \log T_h^{-1} T_f\left(r^A\right) + \left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r) - \frac{1}{A} L\left(r^A\right)\right\}.$$

That is,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\lambda_h^{L^*}(f\circ g)-\varepsilon}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)} \cdot \frac{\log M_h^{-1} M_f\left(r^A\right)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} + \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)}.$$

So we have

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\frac{\lambda_h^{L^*}(f\circ g)-\varepsilon}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)}}{1+\frac{L\left(r^A\right)}{\log T_h^{-1} T_f\left(r^A\right)}} + \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\frac{L(r)}{L\left(r^A\right)}-\frac{1}{A}\right\}}{1+\frac{\log T_h^{-1} T_f\left(r^A\right)}{L\left(r^A\right)}} . \tag{17}$$

In view of the condition $L\left(r^A\right) = o\left\{\log T_h^{-1} T_f\left(r^A\right)\right\}$ as $r \to \infty$ we obtain from (17) that

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\lambda_h^{L^*}(f\circ g)-\varepsilon}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)} . \tag{18}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (18) that

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)} . \tag{19}$$

Also for all sufficiently large values of r,

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) &\leq \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\log\left\{re^{L(r)}\right\} \\ &\leq \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\{\log r + L(r)\} \\ &\leq \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\log r + \frac{1}{A}L\left(r^A\right)\right\} + \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r) - \frac{1}{A}L\left(r^A\right)\right\} . \end{aligned} \tag{20}$$

So from (12) and (20) it follows for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \leq \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)} \log T_h^{-1} T_f\left(r^A\right) + L\left(r^A\right) + \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r) - \frac{1}{A}L\left(r^A\right)\right\}.$$

That is,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} \leq \frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)} \cdot \frac{\log T_h^{-1} T_f\left(r^A\right)}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)} + \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log T_h^{-1} T_f\left(r^A\right)+L\left(r^A\right)}.$$

So we have,

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \leq \frac{\frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{A(\lambda_h^{L^*}(f)-\varepsilon)}}{1+\frac{L(r^A)}{\log T_h^{-1}T_f(r^A)}} + \frac{(\rho_h^{L^*}(f\circ g)+\varepsilon)\left\{\frac{L(r)}{L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log T_h^{-1}T_f(r^A)}{L(r^A)}}. \tag{21}$$

Using $L(r^A) = o\left\{\log T_h^{-1}T_f(r^A)\right\}$ as $r\to\infty$ we obtain from (21) that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \leq \frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{A(\lambda_h^{L^*}(f)-\varepsilon)}. \tag{22}$$

As $\varepsilon(>0)$ is arbitrary, it follows from (22) that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \leq \frac{\rho_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}. \tag{23}$$

Thus the theorem follows from $(10),(15),(19)$ and (23).

Similarly in view of Theorem 3 , we may state the following theorem without proof for the right factor g of the composite function $f\circ g$.

Theorem 4. *Let f be a meromorphic and g and h be any two entire functions such that $0<\lambda_h^{L^*}(f\circ g)\leq\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(g)\leq\rho_h^{L^*}(g)<\infty$. If $L(r^A)=o\left\{\log T_h^{-1}T_g(r^A)\right\}$ as $r\to\infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)} \leq \liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r^A)+L(r^A)} \leq \frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(g)} \leq \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r^A)+L(r^A)} \leq \frac{\rho_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(g)}.$$

Theorem 5. *Let f be a meromorphic and g and h be any two entire functions with $0<\rho_h^{L^*}(f\circ g)<\infty$ and $0<\rho_h^{L^*}(f)<\infty$. If $L(r^A)=o\left\{\log T_h^{-1}T_f(r^A)\right\}$ as $r\to\infty$ then for any positive number A,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \leq \frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)} \leq \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)}.$$

Proof. From the definition of $\rho_h^{L^*}(f)$, we get for a sequence of values of r tending to infinity that

$$\begin{aligned}\log T_h^{-1}T_f(r^A) &\geq \left(\rho_h^{L^*}(f)-\varepsilon\right)\log\left\{r^A e^{L(r^A)}\right\}\\ &\geq \left(\rho_h^{L^*}(f)-\varepsilon\right)\left\{A\log r+L(r^A)\right\}.\end{aligned}$$

That is,

$$\frac{\log T_h^{-1}T_f(r^A)}{A(\rho_h^{L^*}(f)-\varepsilon)} \geq \log r+\frac{1}{A}L(r^A). \tag{24}$$

Now from (20) and (24) it follows for a sequence of values of r tending to infinity that

$$\log T_h^{-1}T_{f\circ g}(r) \leq \frac{(\rho_h^{L^*}(f\circ g)+\varepsilon)}{A(\rho_h^{L^*}(f)-\varepsilon)}\log T_h^{-1}T_f(r^A)+\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A}L(r^A)\right\}.$$

That is,

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \leq \frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{A\left(\rho_h^{L^*}(f)-\varepsilon\right)}\cdot\frac{\log T_h^{-1}T_f(r^A)}{\log T_h^{-1}T_f(r^A)+L(r^A)}+\frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A}L(r^A)\right\}}{\log T_h^{-1}T_f(r^A)+L(r^A)}.$$

So we have

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \leq \frac{\frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{A\left(\rho_h^{L^*}(f)-\varepsilon\right)}}{1+\frac{L(r^A)}{\log T_h^{-1}T_f(r^A)}}+\frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\frac{L(r)}{L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log T_h^{-1}T_f(r^A)}{L(r^A)}}\,. \tag{25}$$

Using $L(r^A)=o\left\{\log T_h^{-1}T_f(r^A)\right\}$ as $r\to\infty$ we obtain from (25) that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \leq \frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{A\left(\rho_h^{L^*}(f)-\varepsilon\right)}\,. \tag{26}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from (26) that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \leq \frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}\,. \tag{27}$$

Again for a sequence of values of r tending to infinity,

$$\begin{aligned}\log T_h^{-1}T_{f\circ g}(r) &\geq \left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\log\left\{re^{L(r)}\right\}\\ &\geq \left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\{\log r+L(r)\}\\ &\geq \left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\log r+\frac{1}{A}L(r^A)\right\}+\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L(r^A)\right\}\end{aligned} \tag{28}$$

So combining (7) and (28) we get for a sequence of values of r tending to infinity that

$$\log T_h^{-1}T_{f\circ g}(r) \geq \frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}\log T_h^{-1}T_f(r^A)+\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L(r^A)\right\},$$

That is,

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \geq \frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}\cdot\frac{\log T_h^{-1}T_f(r^A)}{\log T_h^{-1}T_f(r^A)+L(r^A)}+\frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L(r^A)\right\}}{\log T_h^{-1}T_f(r^A)+L(r^A)}.$$

So we have,

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \geq \frac{\frac{\rho_h^{L^*}(f\circ g)-\varepsilon}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}}{1+\frac{L(r^A)}{\log T_h^{-1}T_f(r^A)}}+\frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\frac{L(r)}{L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log T_h^{-1}T_f(r^A)}{L(r^A)}}\,. \tag{29}$$

Since $L(r^A)=o\left\{\log T_h^{-1}T_f(r^A)\right\}$ as $r\to\infty$, it follows from (29) that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)} \geq \frac{\rho_h^{L^*}(f\circ g)-\varepsilon}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}\,. \tag{30}$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (30) that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)}\geq\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}. \tag{31}$$

Thus the theorem follows from (27) and (31).

Theorem 6. *Let f be a meromorphic and g and h be any two entire functions such that $0<\rho_h^{L^*}(f\circ g)<\infty$ and $0<\rho_h^{L^*}(g)<\infty$. If $L(r^A)=o\{\log T_h^{-1}T_g(r^A)\}$ as $r\to\infty$ then for any positive number A,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r^A)+L(r^A)}\leq\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)}\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r^A)+L(r^A)}.$$

The proof is omitted.

The following theorem is a natural consequence of Theorem 3 and Theorem 5.

Theorem 7. *Let f be a meromorphic and g and h be any two entire functions such that $0<\lambda_h^{L^*}(f\circ g)\leq\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$. If $L(r^A)=o\{\log T_h^{-1}T_f(r^A)\}$ as $r\to\infty$ then for any positive number A,*

$$\begin{aligned}\frac{\lambda_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}&\leq\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)}\leq\min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}\right\}\\&\leq\max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}\right\}\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r^A)+L(r^A)}\\&\leq\frac{\rho_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}.\end{aligned}$$

The proof is omitted.

Combining Theorem a4nd Theorem 6 we may state the following theorem.

Theorem 8. *Let f be a meromorphic and g and h be any two entire functions such that $0<\lambda_h^{L^*}(f\circ g)\leq\rho_h^{L^*}(f\circ g)<\infty$, and $0<\lambda_h^{L^*}(g)\leq\rho_h^{L^*}(g)<\infty$. If $L(r^A)=o\{\log M_h^{-1}M_g(r^A)\}$ as $r\to\infty$ then for any positive number A,*

$$\begin{aligned}\frac{\lambda_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)}&\leq\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r^A)+L(r^A)}\leq\min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)}\right\}\\&\leq\max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)}\right\}\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r^A)+L(r^A)}\\&\leq\frac{\rho_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(g)}.\end{aligned}$$

Theorem 9. *Let f be a meromorphic and g and h be any two entire functions such that $\rho_h^{L^*}(f)<\infty$. Also let g be entire.If $\lambda_h^{L^*}(f\circ g)=\infty$ then*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)}=\infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold.Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity

$$\log T_h^{-1} T_{f\circ g}(r) \leq \beta \log T_h^{-1} T_f(r). \tag{32}$$

Again from the definition of $\rho_h^{L^*}(f)$ it follows for all sufficiently large values of r that

$$\log T_h^{-1} T_f(r) \leq \left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left(re^{L(r)}\right). \tag{33}$$

Thus from (32) and (33) we have for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \leq \beta \left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left\{re^{L(r)}\right\}.$$

That is,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log\left(re^{L(r)}\right)} \leq \frac{\beta\left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left\{re^{L(r)}\right\}}{\log\left\{re^{L(r)}\right\}}.$$

So we have,

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log r} = \lambda_h^{L^*}(f\circ g) < \infty.$$

This is a contradiction.

This proves the theorem.

Remark. Theorem 9 is also valid with "limit superior" instead of "limit" if $\lambda_h^{L^*}(f\circ g) = \infty$ is replaced by $\rho_h^{L^*}(f\circ g) = \infty$ and the other conditions remaining the same.

Corollary 1. *Under the assumptions of Theorem 9 or Remark 3,*

$$\limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} = \infty.$$

Proof. From Theorem 9 or Remark 3 we obtain for all sufficiently large values of r and for $K > 1$ that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) &> K \log T_h^{-1} T_f(r) \\ &> \left\{T_h^{-1} T_f(r)\right\}^K, \end{aligned}$$

from which the corollary follows.

Theorem 10. *Let f be a meromorphic and g and h be any two entire functions such that $\rho_h^{L^*}(g) < \infty$. Also let g be entire.If $\lambda_h^{L^*}(f\circ g) = \infty$ then*

$$\lim_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_g(r)} = \infty.$$

We omit the proof of Theorem 10 because it can be carried out in the line of Theorem 9.

Remark. Theorem 10 is also valid with "limit superior" instead of "limit" if $\lambda_h^{L^*}(f\circ g) = \infty$ is replaced by $\rho_h^{L^*}(f\circ g) = \infty$ and the other conditions remaining the same.

Corollary 2. *Under the assumptions of Theorem 10 or Remark 3,*

$$\limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_g(r)} = \infty.$$

The proof is omitted because it may be carried out in the line of Corollary 1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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