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$I-$convergence of filters

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Abstract: In this paper, we have introduced the idea of $I-$convergence of filters and studied its various properties. We have proved the necessary and sufficient condition for a filter to be $I-$convergent.

Keywords: Ideal, filters, ideal convergence, admissible ideal, Hausdorff space.

1 Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by H. Fast [4] and I. J. Schoenberg [20]. Any convergent sequence is statistically convergent but the converse is not true [17]. Moreover, a statistically convergent sequence need not even be bounded [17]. Let $\mathbb{N}$ denotes the set of natural numbers. If $K \subset \mathbb{N}$, then K_n will denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ stands for the cardinality of K_n. The natural density of K is defined by

$$d(K) = \lim_{n} \frac{|K_n|}{n},$$

if the limit exists [5,16].

The concept of $I-$convergence of real sequences [6,7] is a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers. The notion of ideal convergence for single sequences was first defined and studied by Kostyrko et al. [6]. Mursaleen et al. [12] defined and studied the notion of ideal convergence in random $2-$normed spaces and construct some interesting examples. Several works on $I-$convergence and statistical convergence have been done in [1,3,6,7,8,11,12,13,14,15,19].

The idea of $I-$convergence of real sequences coincides with the idea of ordinary convergence if I is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if I is the ideal of subsets of $\mathbb{N}$ of natural density zero [9].

The idea of $I-$convergence has been extended from real number space to metric space [6] and to a normed linear space [18] in recent works.

Later B. K. Lahiri and P. Das [9] extended the idea of $I-$convergence to an arbitrary topological space and observed that the basic properties are preserved in a topological space. They also introduced [10] the idea of $I-$convergence of nets in a topological space and examined how far it affects the basic properties. We start with the following definitions.

Definition 1. *Let X be a non-empty set. Then a family $\mathscr{F} \subset 2^X$ is called a* **filter** *on X if*

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(i) $\emptyset \notin \mathscr{F}$,
(ii) $A, B \in \mathscr{F}$ *implies* $A \cap B \in \mathscr{F}$ *and*
(iii) $A \in \mathscr{F}, B \supset A$ *implies* $B \in \mathscr{F}$.

Definition 2. *Let X be a non-empty set. Then a family $I \subset 2^X$ is called an* **ideal** *of X if*

(i) $\emptyset \in I$,
(ii) $A, B \in I$ *implies* $A \cup B \in I$ *and*
(iii) $A \in I, B \subset A$ *implies* $B \in I$.

Definition 3. *Let X be a non-empty set. Then a filter $\mathscr{F}$ on X is said to be* **non-trivial** *if $\mathscr{F} \neq \{X\}$.*

Definition 4. *Let X be a non-empty set. Then an ideal I of X is said to be* **non-trivial** *if $I \neq \{\emptyset\}$ and $X \notin I$.*

Note 1. (i) $\mathscr{F} = \mathscr{F}(I) = \{A \subset X : X \setminus A \in I\}$ is a filter on X, called the **filter associated with the ideal** I.
(ii) $I = I(\mathscr{F}) = \{A \subset X : X \setminus A \in \mathscr{F}\}$ is an ideal of X, called the **ideal associated with the filter** $\mathscr{F}$.
(iii) A non-trivial ideal I is called **admissible** if I contains all the singleton sets.

Several examples of non-trivial admissible ideals have been considered in [6].

We give a brief discussion on $I-$convergence of topological spaces as given by [9].

Let (X, τ) stands for a topological space and I be a non-trivial ideal of the set of natural numbers $\mathbb{N}$.

Definition 5. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $I-$***convergent** *to $x_0 \in X$ if for any non-empty open set U containing x_0, $\{n \in \mathbb{N} : x_n \notin U\} \in I$.*
*In this case, we write $I - \lim x_n = x_0$ and x_0 is called the $I-$***limit** *of $\{x_n\}$.*

We mention below some usual properties of convergence in a topological space that are preserved in $I-$convergence.

Theorem 1. *If X is Hausdorff, then an $I-$convergent sequence has a unique $I-$limit.*

Proof. See [9].

Theorem 2. *If I is an admissible ideal and if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of distinct elements in a set $E \subset X$ which is $I-$convergent to $x_0 \in X$, then x_0 is a limit point of E.*

Proof. See [9].

Theorem 3. *A continuous function $g : X \to X$ preserves $I-$convergence.*

Proof. See [9].

Throughout this paper, $X = (X, \tau)$ will stand for a topological space and $I = I(\mathscr{F})$ will be the ideal of X associated with the filter $\mathscr{F}$ on X. Most of the work in this paper is inspired from [2,21].

2 $I-$convergence of filters

Definition 6. *A filter $\mathscr{F}$ on X is said to be $I-$***convergent** *to $x_0 \in X$ if for each nbd U of x_0, $\{y \in X : y \notin U\} \in I$. In this case, x_0 is called an $I-$***limit of** *$\mathscr{F}$ and is written as $I - \lim \mathscr{F} = x_0$.*

Example 1. Let $X = \{1,2,3\}$ and $\tau = \{\emptyset,\{1\},X\}$ be a topology on X. Let $\mathscr{F} = \{\{1\},\{1,2\},\{1,3\},X\}$. Then $I = \{\emptyset,\{2\},\{3\},\{2,3\}\}$. It is easy to see that $1,2$ and 3 are $I-$limits of $\mathscr{F}$.

Example 2. The nbd filter $\mathscr{U}_{x_0}$ at a point x_0 in X $I-$ converges to x_0. Because for each nbd U of x_0, $\{y \in X : y \notin U\} \in I$, as $I = I(\mathscr{U}_{x_0})$.

Example 3. Let $\mathscr{F}$ be a filter on an indiscrete space X. Then clearly, $\mathscr{F}$ will be $I-$convergent to each $x_0 \in X$ as X is the only nbd of each $x_0 \in X$ and $\{y \in X : y \notin X\} = \emptyset \in I$.

We now give the necessary and sufficient condition for a filter $\mathscr{F}$ to be $I-$convergent at some point.

Theorem 4. *A filter $\mathscr{F}$ on X is $I-$convergent to x_0 if and only if for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$.*

Proof. First suppose that $\mathscr{F}$ is $I-$convergent to x_0. This means that for each nbd U of x_0, $\{y \in X : y \notin U\} \in I$. We shall show that for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$. For this, let U be a nbd of x_0 and let $V \in \mathscr{P}(X)$ such that $U \cap V = \emptyset$. Then $V \subset X \setminus U$. Since U is a nbd of x_0 and $V \subset X \setminus U$, it follows that $V \subset \{y \in X : y \notin U\}$. Thus $V \in I$ and so $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$. Conversely, suppose for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$. We have to show that $\mathscr{F}$ is $I-$convergent to x_0. For this, let U be a nbd of x_0. Then by the given condition, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I \cdots (*)$. We claim that $\{y \in X : y \notin U\} \in I$. For this, let $z \in \{y \in X : y \notin U\}$. Then $z \notin U$. This implies that $U \cap \{z\} = \emptyset$. Thus $\{z\} \in \{V \in \mathscr{P}(X) : U \cap V = \emptyset\}$ and so by $(*)$, $\{z\} \in I$. Hence $\{y \in X : y \notin U\} \in I$. This proves that $\mathscr{F}$ is $I-$convergent to x_0.

We recall the following definition.

Definition 7. *A filter $\mathscr{F}$ on X is said to be* **finer** *than a filter $\mathscr{G}$ on X if $\mathscr{G} \subset \mathscr{F}$.*

Notation. In case more than one filter is involved, we use the notation $I(\mathscr{F})$ to denote the ideal associated with the corresponding filter $\mathscr{F}$.

Lemma 1. *Let $\mathscr{F}$ and $\mathscr{G}$ be two filters on X. Then $\mathscr{F} \subset \mathscr{G}$ if and only if $I(\mathscr{F}) \subset I(\mathscr{G})$.*

Proof. Proof is trivial.

We now show that an $I-$convergent filter $\mathscr{F}$ also satisfies some basic properties of filters.

Proposition 1. *If X is Hausdorff, then any $I-$convergent filter $\mathscr{F}$ on X has a unique $I-$limit.*

Proof. Suppose X is Hausdorff. Let $\mathscr{F}$ be an $I-$convergent filter on X. If possible, suppose x_0 and y_0 are two distinct $I-$limits of $\mathscr{F}$. Since X is Hausdorff, there exists two disjoint open sets U and V in X such that $x_0 \in U$ and $y_0 \in V$. Now, x_0 is $I-$limit of $\mathscr{F} \Rightarrow \{y \in X : y \notin U\} \in I$. Or, $\{y \in X : y \in U^c\} \in I$. Similarly, y_0 is $I-$limit of $\mathscr{F} \Rightarrow \{y \in X : y \in V^c\} \in I$. Further, $\{y \in X : y \in (U \cap V)^c\} \subset \{y \in X : y \in U^c\} \cup \{y \in X : y \in V^c\} \in I$. Thus we have $\{y \in X : y \in (U \cap V)^c\} \in I$. Since $X \notin I$, there exists $z \in X$ such that $z \notin (U \cap V)^c$. That is, $z \in U \cap V$, which is not possible as $U \cap V = \emptyset$. Therefore, our supposition is wrong. Hence $\mathscr{F}$ has a unique $I-$limit.

Note 2. The converse of above Proposition is given in Proposition $2 \cdot 19$.

Proposition 2. *Let $E \subset X$ and $\mathscr{F}$ be a filter on E which is $I-$convergent to $x_0 \in X$, where $I = I(\mathscr{F})$ is an admissible ideal of E. Then x_0 is a limit point of E. Conversely, if x_0 is a limit point of E, then there is a filter on $E \setminus \{x_0\}$ which is $I-$convergent to x_0, for some admissible ideal I of E.*

Proof. Let $\mathscr{F}$ be a filter on a set $E \subset X$ which is $I-$convergent to $x_0 \in X$, where $I = I(\mathscr{F})$ is an admissible ideal of E. To show that x_0 is a limit point of E, let U be an open set containing x_0. Since $I - lim\mathscr{F} = x_0$ in E, $\{y \in E : y \notin U\} \in I$ and so $\{y \in E : y \in U\} \notin I(\because I = I(\mathscr{F}))$. Since I is admissible, E is infinite and so we can choose $y_0 \in \{y \in E : y \in U\}$ such that $y_0 \neq x_0$. Then $y_0 \in U \cap (E \setminus \{x_0\})$. Thus x_0 is a limit point of E. Conversely, suppose x_0 is a limit point of E. Then for arbitrary nbd U of x_0,
$U \cap (E \setminus \{x_0\}) \neq \emptyset$. Let $\mathscr{F} = \{A \subset E \setminus \{x_0\} : A \supset U \cap (E \setminus \{x_0\})\}$. Then clearly, $\mathscr{F}$ is a non-empty family of subsets of $E \setminus \{x_0\}$.

(i) Clearly, $\emptyset \notin \mathscr{F}$.
(ii) Let $A_1, A_2 \in \mathscr{F}$. Then $A_1 \supset U \cap (E \setminus \{x_0\})$ and $A_2 \supset U \cap (E \setminus \{x_0\})$. Clearly, $A_1 \cap A_2 \supset U \cap (E \setminus \{x_0\})$ and so $A_1 \cap A_2 \in \mathscr{F}$.
(iii) Let $A \in \mathscr{F}$ and $B \supset A$.

Now, $A \in \mathscr{F}$ implies that $A \supset U \cap (E \setminus \{x_0\})$. Clearly, $B \supset U \cap (E \setminus \{x_0\})$ and so $B \in \mathscr{F}$. This proves that $\mathscr{F}$ is a filter on $E \setminus \{x_0\}$. Let $I = I(\mathscr{F})$ be the admissible ideal of E. We shall show that $I - \lim \mathscr{F} = x_0$. For this, let U be a nbd of x_0. We claim that $\{y \in E \setminus \{x_0\} : y \notin U\} \in I$. So, let $y \in E \setminus \{x_0\}$ such that $y \notin U$. Now $y \notin U \cap (E \setminus \{x_0\})$ implies that $\{y\} \notin \mathscr{F}$. Since I is admissible, $\{y\} \in I$. Thus $I - \lim \mathscr{F} = x_0$. Hence the proof.

We recall the following from [21]. Let X and Y be two topological spaces. Suppose that $\mathscr{F}$ is a filter on X and $f : X \to Y$ is a map. Then $f(\mathscr{F})$ is a filter on Y having for a base the sets $f(F), F \in \mathscr{F}$.

Proposition 3. *Let X and Y be two topological spaces and $f : X \to Y$ be a map. Let $\mathscr{F}$ be a filter on X. Then $f : X \to Y$ is continuous at $x_0 \in X$ if and only if $I_X - \lim \mathscr{F} = x_0$ in X implies $I_Y - \lim f(\mathscr{F}) = f(x_0)$, where $I_X = I_X(\mathscr{F})$, $f(\mathscr{F})$ is a filter on Y generated by the base $\{f(F) : F \in \mathscr{F}\}$ and $I_Y = I_Y(f(\mathscr{F}))$.*

Proof. First suppose that $f : X \to Y$ is continuous at x_0. Suppose $I_X - \lim \mathscr{F} = x_0$. Then for each nbd U of x_0, $\{W \in \mathscr{P}(X) : U \cap W = \emptyset\} \subset I_X$. We have to show that $I_Y - \lim f(\mathscr{F}) = f(x_0)$. For this, let V be a nbd of $f(x_0)$. We claim that $\{T \in \mathscr{P}(Y) : V \cap T = \emptyset\} \subset I_Y$. So, let $T \in \mathscr{P}(Y)$ such that $V \cap T = \emptyset$. Since f is continuous at x_0, for above nbd V of $f(x_0)$, there exists a nbd U of x_0 such that $f(U) \subset V$. Now, $V \cap T = \emptyset$ implies that $T \subset Y \setminus V \subset Y \setminus f(U) \cdots (*)$. Now, $U \cap (X \setminus U) = \emptyset$ implies that $X \setminus U \in I_X$ and so $U \in \mathscr{F}$. This further implies that $f(U) \in f(\mathscr{F})$. Thus $Y \setminus f(U) \in I_Y$. From $(*)$, $T \in I_Y$. Hence $I_Y - \lim f(\mathscr{F}) = f(x_0)$.

Conversely, suppose the condition holds. We have to show that $f : X \to Y$ is continuous at x_0. For this, let V be a nbd of $f(x_0)$ in Y. Since $I_X - \lim \mathscr{F} = x_0$, for each nbd U of x_0, $\{x \in X : x \notin U\} \in I_X \cdots (**)$. Also, $I_Y - \lim f(\mathscr{F}) = f(x_0)$ implies that for above nbd V of $f(x_0)$, $\{y \in Y : y \notin V\} \in I_Y \cdots (***)$. Thus clearly, for above nbd V of $f(x_0)$ in Y, there exists a nbd U of x_0 in X such that $f(U) \subset V$. For otherwise, if $f(U) \nsubseteq V$, then there exists $x \in U$ such that $f(x) \notin V$. From $(***)$, $f(x) \notin V$ implies that $\{f(x)\} \in I_Y$. This means that $\{x\} \in I_X$. That is, $x \notin U$, which is a contradiction. Hence f is continuous at x_0.

2.1 Characterization of closure

Proposition 4. *Let $E \subset X$. Then $x_0 \in \overline{E}$ if and only if there is a filter $\mathscr{F}$ on X such that $E \in \mathscr{F}$ and $I - \lim \mathscr{F} = x_0$.*

Proof. First suppose $x_0 \in \overline{E}$. Then each nbd of x_0 meets E. That is, $U \cap E \neq \emptyset, \forall\, U \in \mathscr{U}_{x_0}$, where $\mathscr{U}_{x_0}$ is the nbd system at x_0. Let $\mathscr{B} = \{U \cap E : U \in \mathscr{U}_{x_0}\}$. Then clearly, $\mathscr{B}$ is a non-empty family of non-empty subsets of X which is closed under finite intersection and so a filter base for some filter, say $\mathscr{F}$ on X.

Since $E \supset U \cap E, \forall\, U \in \mathscr{U}_{x_0}$, we have $E \in \mathscr{F}$. We shall show that $I-\lim \mathscr{F} = x_0$. For this, let U be a nbd of x_0. Since $U \supset U \cap E$, we have $U \in \mathscr{F}$. We claim that $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$. So, let $V \in \mathscr{P}(X)$ such that $U \cap V = \emptyset$. Now $U \cap V = \emptyset$ implies that $V \subset X \setminus U$. Now $U \in \mathscr{F}$ and $I = I(\mathscr{F})$ implies that $X \setminus U \in I$. Since I is an ideal, it is closed under subsets and so $V \in I$. Therefore, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$.

Conversely, suppose there is a filter $\mathscr{F}$ on X such that $E \in \mathscr{F}$ and $I-\lim \mathscr{F} = x_0$. To show that $x_0 \in \overline{E}$, let U be a nbd of x_0. Since $I-\lim \mathscr{F} = x_0$, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$. We claim that $U \in \mathscr{F}$. Since $U \cap (X \setminus U) = \emptyset$, we have $X \setminus U \in I$. Since $I = I(\mathscr{F})$, we have $U \in \mathscr{F}$. Now, $E, U \in \mathscr{F}$ and $\mathscr{F}$ is a filter implies that $U \cap E \in \mathscr{F}$ and so $U \cap E \neq \emptyset$. This proves that $x_0 \in \overline{E}$.

Proposition 5. *Let $\mathscr{F}$ be a filter on X such that $I-\lim \mathscr{F} = x_0$. Then every filter $\mathscr{F}'$ finer than $\mathscr{F}$ also $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Proof. Suppose $\mathscr{F}$ is a filter on X such that $I-\lim \mathscr{F} = x_0$. Let $\mathscr{F}'$ be an arbitrary filter on X such that $\mathscr{F}' \supset \mathscr{F}$. We claim that $I-\lim \mathscr{F}' = x_0$, where $I = I(\mathscr{F})$. For this, let U be a nbd of x_0. Since $I-\lim \mathscr{F} = x_0$, for above nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$. Thus it follows that for every nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$. Therefore, $I-\lim \mathscr{F}' = x_0$.

Remark. Let $\mathscr{F}$ be a filter on X and $\mathscr{F}'$ be another filter on X finer than $\mathscr{F}$. Then $I(\mathscr{F}')-\lim \mathscr{F}' = x_0$ need not imply that $I(\mathscr{F})-\lim \mathscr{F} = x_0$. Consider the example: Let $X = \{1,2,3\}$ and $\tau = \{\emptyset, \{2\}, \{1,2\}, X\}$ be a topology on X. Let $\mathscr{F} = \{\{2,3\}, X\}$ be a filter on X. Then $I(\mathscr{F}) = \{\emptyset, \{1\}\}$. It is easy to see that $I(\mathscr{F})-\lim \mathscr{F} = 3$. Let $\mathscr{F}' = \{\{2\}, \{1,2\}, \{2,3\}, X\}$. Then $I(\mathscr{F}') = \{\emptyset, \{1\}, \{3\}, \{1,3\}\}$. We can easily see that $1, 2$ and 3 are $I(\mathscr{F}')-$limits of $\mathscr{F}'$. Also, $I(\mathscr{F}')-\lim \mathscr{F} = 1$. Thus we observe that 1 and 2 are $I(\mathscr{F}')-$ limits of $\mathscr{F}'$ but not $I(\mathscr{F})-$ limits of $\mathscr{F}$.

Proposition 6. *Let $\mathscr{F}$ be a filter on X such that $I-\lim \mathscr{F} = x_0$. Then every filter $\mathscr{F}'$ on X coarser than $\mathscr{F}$ also $I-$converges to x_0, where $I = I(\mathscr{F})$.*

Proof. Suppose $\mathscr{F}$ is a filter on X such that $I-\lim \mathscr{F} = x_0$. Then for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I \cdots (*)$. Let $\mathscr{F}'$ be an arbitrary filter on X such that $\mathscr{F}' \subset \mathscr{F}$. We claim that $I-\lim \mathscr{F}' = x_0$, where $I = I(\mathscr{F})$. So, let U be a nbd of x_0. Then clearly by $(*)$, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I$. Therefore, $I-\lim \mathscr{F}' = x_0$, where $I = I(\mathscr{F})$.

Note 3. The above proposition need not be true if we replace $I(\mathscr{F})-\lim \mathscr{F}'$ by $I(\mathscr{F}')-\lim \mathscr{F}'$. Consider the example: Let $X = \{1,2,3\}$ and $\tau = \{\emptyset, \{2\}, X\}$ be a topology on X. Let $\mathscr{F} = \{\{2\}, \{1,2\}, \{2,3\}, X\}$ be a filter on X. Then $I(\mathscr{F}) = \{\emptyset, \{1\}, \{3\}, \{1,3\}\}$. Let $\mathscr{F}' = \{\{1,2\}, X\}$ be another filter on X. Then clearly, $\mathscr{F}' \subset \mathscr{F}$. Also, $I(\mathscr{F}') = \{\emptyset, \{3\}\}$. We can easily see that $I(\mathscr{F})-\lim \mathscr{F} = 1,2,3$ and $I(\mathscr{F}')-\lim \mathscr{F}' = 1,3$. Thus we observe that 2 is an $I(\mathscr{F})-$limit of $\mathscr{F}$ but it is not an $I(\mathscr{F}')-$limit of $\mathscr{F}'$.

Proposition 7. *Let $\mathscr{F}$ be a filter on X and $\mathscr{G}$ be any other filter on X finer than $\mathscr{F}$. Then $I(\mathscr{F})-\lim \mathscr{G} = x_0$ implies $I(\mathscr{G})-\lim \mathscr{G} = x_0$. But not conversely.*

Proof. Suppose $I(\mathscr{F})-\lim \mathscr{G} = x_0$. Then for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I(\mathscr{F})$. Since $\mathscr{F} \subset \mathscr{G}$, by Lemma $2 \cdot 7$, $I(\mathscr{F}) \subset I(\mathscr{G})$. Thus for each nbd U of x_0, $\{V \in \mathscr{P}(X) : U \cap V = \emptyset\} \subset I(\mathscr{G})$. Therefore, $I(\mathscr{G})-\lim \mathscr{G} = x_0$.

But converse need not be true. Consider the following example : Let $X = \{1,2,3\}$ and τ be the discrete topology on X. Let $\mathscr{F} = \{\{2,3\}, X\}$ be a filter on X. Then $I(\mathscr{F}) = \{\emptyset, \{1\}\}$. Let $\mathscr{G} = \{\{2\}, \{1,2\}, \{2,3\}, X\}$ be a filter on X finer than $\mathscr{F}$. Then $I(\mathscr{G}) = \{\emptyset, \{1\}, \{3\}, \{1,3\}\}$. We can easily see that $I(\mathscr{F})-\lim \mathscr{G} =$ nil and $I(\mathscr{G})-\lim \mathscr{G} = 2$. Thus we observe that 2 is an $I(\mathscr{G})-$limit of $\mathscr{G}$ but not an $I(\mathscr{F})-$limit of $\mathscr{G}$.

Proposition 8. *Let τ_1 and τ_2 be two topologies on X such that τ_1 is coarser than τ_2. Let $\mathscr{F}$ be a filter on X such that $I-\lim\mathscr{F}=x_0$ w.r.t τ_2. Then $I-\lim\mathscr{F}=x_0$ w.r.t τ_1. But the converse need not be true.*

Proof. Let U be a nbd of x_0 w.r.t τ_1. Since $\tau_1\subset\tau_2$, U is also a nbd of x_0 w.r.t τ_2. But $I-\lim\mathscr{F}=x_0$ w.r.t τ_2. Thus for above nbd U of x_0, $\{V\in\mathscr{P}(X):U\cap V=\emptyset\}\subset I$. Hence $I-\lim\mathscr{F}=x_0$ w.r.t τ_1 also. The converse is however not true. Consider the following example : Let $X=\{1,2,3\}$. Let τ_2 be the discrete topology on X and $\tau_1=\{\emptyset,\{2\},\{3\},\{2,3\},X\}$. Then $\tau_1\subset\tau_2$. Let $\mathscr{F}=\{\{1,2\},X\}$ be a filter on X. It is easy to see that $I-\lim\mathscr{F}=1$ w.r.t τ_1, but 1 is not an $I-\lim\mathscr{F}$ w.r.t τ_2.

Lemma 2. *Let $\mathscr{M}=\{\mathscr{G}:\mathscr{G}\ is\ a\ filter\ on\ X\}$. Then $\mathscr{F}=\cap_{\mathscr{G}\in\mathscr{M}}\mathscr{G}$ if and only if $I(\mathscr{F})=\cap_{\mathscr{G}\in\mathscr{M}}I(\mathscr{G})$.*

Proof. Suppose $\mathscr{F}=\cap_{\mathscr{G}\in\mathscr{M}}\mathscr{G}$. Then $A\in I(\mathscr{F})\Leftrightarrow X\setminus A\in\mathscr{F}=\cap_{\mathscr{G}\in\mathscr{M}}\mathscr{G}\Leftrightarrow X\setminus A\in\mathscr{G},\ \forall\ \mathscr{G}\in\mathscr{M}$ $\Leftrightarrow A\in I(\mathscr{G}),\ \forall\ \mathscr{G}\in\mathscr{M}\Leftrightarrow A\in\cap_{\mathscr{G}\in\mathscr{M}}I(\mathscr{G})$. Thus $I(\mathscr{F})=\cap_{\mathscr{G}\in\mathscr{M}}I(\mathscr{G})$.

Conversely, suppose $I(\mathscr{F})=\cap_{\mathscr{G}\in\mathscr{M}}I(\mathscr{G})$. Then $A\in\mathscr{F}\Leftrightarrow X\setminus A\in I(\mathscr{F})=\cap_{\mathscr{G}\in\mathscr{M}}I(\mathscr{G})\Leftrightarrow X\setminus A\in I(\mathscr{G}),\ \forall\ \mathscr{G}\in\mathscr{M}$ $\Leftrightarrow A\in\mathscr{G},\ \forall\ \mathscr{G}\in\mathscr{M}\Leftrightarrow A\in\cap_{\mathscr{G}\in\mathscr{M}}\mathscr{G}$. Thus $\mathscr{F}=\cap_{\mathscr{G}\in\mathscr{M}}\mathscr{G}$.

Proposition 9. *Let $\mathscr{M}$ be a collection of all those filters $\mathscr{G}$ on a space X which $I(\mathscr{G})-$converges to the same point $x_0\in X$. Then the intersection $\mathscr{F}$ of all the filters in $\mathscr{M}$ $I(\mathscr{F})-$converges to x_0.*

Proof. Here $\mathscr{M}=\{\mathscr{G}:\mathscr{G}\ is\ a\ filter\ on\ X\ such\ that\ I(\mathscr{G})-\lim\mathscr{G}=x_0\}$. Let $\mathscr{F}=\bigcap\{\mathscr{G}:\mathscr{G}\in\mathscr{M}\}$. We shall show that $I(\mathscr{F})-\lim\mathscr{F}=x_0$. For this, let U be a nbd of x_0(w.r.t $\mathscr{F}$). Then U is a nbd of x_0(w.r.t all $\mathscr{G}\in\mathscr{M}$). Since $I(\mathscr{G})-\lim\mathscr{G}=x_0,\forall\ \mathscr{G}\in\mathscr{M}$, it follows that $\{y\in X:y\notin U\}\in I(\mathscr{G}),\forall\ \mathscr{G}\in\mathscr{M}$. This implies that $\{y\in X:y\notin U\}\in\cap_{\mathscr{G}\in\mathscr{M}}I(\mathscr{G})=I(\mathscr{F})$. Hence $I(\mathscr{F})-\lim\mathscr{F}=x_0$. We are now in a position to prove the converse of Proposition $2\cdot 8$.

Proposition 10. *If every $I-$convergent filter $\mathscr{F}$ on X has a unique $I-$limit, then the space X is Hausdorff.*

Proof. Suppose every $I-$convergent filter $\mathscr{F}$ on X has a unique $I-$limit. We have to show that X is a Hausdorff space. Suppose not. This means that for any two distinct points x and y in X, there are open sets U and V in X containing x and y, respectively such that $U\cap V\neq\emptyset\cdots(*)$. Let $\mathscr{U}_x$ and $\mathscr{U}_y$ be the nbd filters at x and y, respectively. Then clearly by Example $2\cdot 3$, $\mathscr{U}_x$ $I(\mathscr{U}_x)-$converges to x and $\mathscr{U}_y$ $I(\mathscr{U}_y)-$converges to y. Now, since X is not Hausdorff, $\mathscr{U}_x\cup\mathscr{U}_y$ is a filter on X. This filter is clearly a filter base for some filter, say $\mathscr{F}$ on X such that $\mathscr{F}\supset\mathscr{U}_x$ and $\mathscr{F}\supset\mathscr{U}_y$. Since $\mathscr{U}_x$ $I(\mathscr{U}_x)-$converges to x, by Proposition $2\cdot 12$, $\mathscr{F}$ $I(\mathscr{U}_x)-$converges to x. Similarly, $\mathscr{F}$ $I(\mathscr{U}_y)-$converges to y. By Proposition $2\cdot 15$, $\mathscr{F}$ $I(\mathscr{F})-$converges to x and $\mathscr{F}$ $I(\mathscr{F})-$converges to y. That is, $I-lim\mathscr{F}=x$ and $I-lim\mathscr{F}=y$, where $I=I(\mathscr{F})$, which is a contradiction to the hypothesis. Hence X is Hausdorff.

Lemma 3. *If I_X is an ideal of $X=\prod_{\alpha\in\Lambda}X_\alpha$ associated with a filter $\mathscr{F}$ on X, then $I_X=\cap_{i=1}^{n}p_{\alpha_i}^{-1}(I_{X_{\alpha_i}})$, where $I_{X_{\alpha_i}}$ is an ideal of the factor space X_{α_i} associated with $p_{\alpha_i}(\mathscr{F})$.*

Proof. $t\in\cap_{i=1}^{n}p_{\alpha_i}^{-1}(I_{X_{\alpha_i}})\Leftrightarrow t\in p_{\alpha_i}^{-1}(I_{X_{\alpha_i}}),\forall\ i=1,2,\ldots,n\Leftrightarrow p_{\alpha_i}(t)\in I_{X_{\alpha_i}},\forall\ i=1,2,\ldots,n\Leftrightarrow p_{\alpha_i}(t)\in X_{\alpha_i}\setminus p_{\alpha_i}(\mathscr{F}),\forall\ i=1,2,\ldots,n\Leftrightarrow p_{\alpha_i}(t)\in p_{\alpha_i}(X\setminus\mathscr{F}),\forall\ i=1,2,\ldots,n\Leftrightarrow t\in X\setminus\mathscr{F}\Leftrightarrow t\in I_X$. Hence $I_X=\cap_{i=1}^{n}p_{\alpha_i}^{-1}(I_{X_{\alpha_i}})$.

Theorem 5. *A filter $\mathscr{F}$ I_X-converges to x in $X=\prod_{\alpha\in\Lambda}X_\alpha$ if and only if $p_\alpha(\mathscr{F})$ $I_{X_\alpha}-$converges to $p_\alpha(x),\forall\ \alpha$, where $I_X=I_X(\mathscr{F})$ and $I_{X_\alpha}=I_{X_\alpha}(p_\alpha(\mathscr{F}))$.*

Proof. Suppose $\mathscr{F}$ I_X-converges to x in $X=\prod_{\alpha\in\Lambda}X_\alpha$. Since each projection $p_\alpha:X\to X_\alpha$ is continuous at x in X, by Proposition $2\cdot 10$, we find that $p_\alpha(\mathscr{F})$ $I_{X_\alpha}-$converges to $p_\alpha(x)$ in $X_\alpha,\forall\ \alpha$. Conversely, suppose $p_\alpha(\mathscr{F})$ $I_{X_\alpha}-$converges to $p_\alpha(x)$ in $X_\alpha, \forall\ \alpha$. We have to show that $\mathscr{F}$ I_X-converges to x in X. For this, let $U = \cap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})$ be a basic nbd of x. This means that U_{α_i} is a nbd of $x_{\alpha_i} = p_{\alpha_i}(x)$, for $i = 1,2,\ldots,n$ in X_{α_i}. We claim that $\{y \in X : y \notin U\} \in I_X$. So, let $y \in X$ such that $y \notin U$. Now, $y \notin U \Rightarrow y \notin p_{\alpha_i}^{-1}(U_{\alpha_i})$, for some $i = 1,2,\ldots,n \Rightarrow p_{\alpha_i}(y) \notin U_{\alpha_i}$, for some $i = 1,2,\ldots,n$. Since $p_\alpha(\mathscr{F})$ $I_{X_\alpha}-$converges to $p_\alpha(x)$ in $X_\alpha, \forall\ \alpha$, we find that for each nbd U_α of $p_\alpha(x)$, $\{z_\alpha \in X_\alpha : z_\alpha \notin U_\alpha\} \in I_{X_\alpha}$. Thus $p_{\alpha_i}(y) \notin U_{\alpha_i}$ implies that $\{p_{\alpha_i}(y)\} \in I_{X_{\alpha_i}}, i = 1,2,\ldots n$. This further implies that $\{y\} \in \cap_{i=1}^{n} p_{\alpha_i}^{-1}(I_{X_{\alpha_i}})$. By above Lemma $2\cdot 20$, $I_X = \cap_{i=1}^{n} p_{\alpha_i}^{-1}(I_{X_{\alpha_i}})$. Thus $\{y\} \in I_X$. This proves the claim. Hence $\mathscr{F}$ I_X-converges to x in $X = \prod_{\alpha\in\Lambda} X_\alpha$.

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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