Communication in Mathematical Modeling and Applications
http://ntmsci.com/cmma

Numerical solution of the neutral functional-differential equations with proportional delays via collocation method based on Hermite polynomials

Birol Ibis[1], *Mustafa Bayram*[2]

[1]Department of Basic Sciences, Turkish Air Force Academy, Istanbul,Turkey
[2]Department of Computer Engineering, Faculty of Engineering and Architecture, Istanbul Gelisim University, Istanbul,Turkey

Received: 17 December 2016, Accepted: 19 December 2016
Published online: 20 December 2016.

Abstract: In this paper, a collocation method based on Hermite polynomials is presented for the numerical solution of the neutral functional-differential equations (NFDEs) with proportional delays. By using Hermite polynomials and collocation points, NFDEs and the given conditions are transformed into matrix equation which corresponds to a system of linear algebraic equations with unknown Hermite coefficients. Hence, by solving this system, the unknown Hermite coefficients are computed. In addition, some numerical examples are given and comparisons with other methods are made in order to demonstrate the validity and applicability of the proposed method.

Keywords: Hermite collocation method (HCM), collocation points, Hermite polynomials, neutral functional-differential equations (NFDEs).

1 Introduction

Many problems in mechanical engineering, physics, biology, chemistry, control theory, fluid mechanics, signal processing, viscoelasticity, electromagnetism, electrochemistry, thermal engineering and many other physical processes are modeled by ordinary, partial or fractional differential equations. Since in many cases to find an exact solution of these equations is difficult, approximate or numerical solution methods are used [1,2,3,4,5,6,7,8,9].

The Neutral functional-differential equations (NFDEs) with proportional delays are one of the important classes of delay differential equations and these equations arise in modeling of various phenomena in science and engineering [10,11,12, 13,14]. NFDEs have been investigated by many authors and various analytical and numerical methods have been developed, some of which are Legendre-Gauss collocation method [15], homotopy perturbation method (HPM) [16], variational iteration method (VIM) [17], segmented Tau approximation[18], Adams predictor-corrector method [19,20], reproducing kernel Hilbert space method(RKHSM) [21], one-leg θ-methods [22,23], continuous Runge-Kutta method (RKTM)[24] and waveform relaxation methods [25].

In this paper, we develop Hermite collocation method to solve the following NFDEs with proportional delays.

$$y^{(n)}(t) = \lambda(t)y(t) + \sum_{k=0}^{n} \beta_k(t)y^{(k)}(q_k t) + g(t),\ t \geq 0 \tag{1}$$

with the initial conditions

$$y^{(i)}(0) = c_i, \quad i = 0,1,2,\ldots,n-1 \tag{2}$$

* Corresponding author e-mail: bibis@hho.edu.tr

where $\lambda(t)$ and $\beta_k(t)$ are given analytical functions, and q_n, c_i are appropriate constants with $0 < q_n < 1$.

2 Hermite collocation method (HCM)

The main idea of the collocation method is to seek the unknown solution $y(t)$ in the form of a linear combination of some basis functions with unknown coefficients. Here, basis functions can be preferred as orthogonal polynomials according to their particular properties, which make them especially ideal for a problem under consideration. In recent years, the various collocation methods have been studied by many authors to obtain solutions of problems arising in different fields of science and engineering [26,27,28,29,30,31,32,33,34,35,36,37,38].

3 Hermite polynomials

The explicit form of well-known Hermite polynomials of n-th degree is defined as:

$$H_n(t) = n! \cdot \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k! \cdot (n-2k)!} (2t)^{n-2k}, \ n \in N. \tag{3}$$

The first few Hermite polynomials are

$$H_0(t) = 1, H_1(t) = 2t, H_2(t) = t^2 - 1, H_3(t) = t^3 - 3t, H_4(t) = t^4 - 6t^2 + 3.$$

In practice, the Hermite polynomials can be computed using the following recurrence relations for $n \in N^+$.

$$H_{n+1}(t) = 2tH_n(t) - 2nH_{n-1}(t) \tag{4}$$

$$H_n'(t) = 2nH_{n-1}(t) \tag{5}$$

where $H_0(t) = 1$ and $H_1(t) = 2t$. If we present the Hermite polynomial as a vector in the form

$$H(t) = [H_0(t), H_1(t), \cdots, H_N(t)],$$

then the derivative of the $H(t)$, using (5), can be denoted in the matrix form by

$$H'(t)^T = MH(t)^T \tag{6}$$

where

$$H(t) = \left[H_0(t) \ H_1(t) \ \cdots \ H_{N-1}(t) \ H_N(t) \right]$$

$$H'(t) = \left[H_0'(t) \ H_1'(t) \ \cdots \ H_{N-1}'(t) \ H_N'(t) \right]$$

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 2\cdot 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\cdot 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2\cdot(N-1) & 0 & 0 \\ 0 & 0 & \cdots & 0 & 2\cdot N & 0 \end{bmatrix}_{(N+1)\times(N+1)}.$$

Accordingly, the k-th derivative with respect to t of $H(t)$ can be obtained by

$$H'(t)^T = MH(t)^T, \Rightarrow H'(t) = H(t)M^T,$$

$$H''(t) = H'(t)M^T = H(t)\left(M^T\right)^2,$$
$$H'''(t) = H'(t)\left(M^T\right)^2 = H(t)\left(M^T\right)^3$$
$$\vdots$$
$$H^{(k)}(t) = H^{(k-1)}(t)\left(M^T\right)^{k-1} = H(t)\left(M^T\right)^k \tag{7}$$

where M is the Hermite operational matrix of derivative.

4 Method for solution

In this section, we use the collocation method based on Hermite polynomial to solve numerically the NFDEs. We suppose that the solution of (1) can be expanded in Hermite polynomials:

$$y(t) \cong \sum_{j=0}^{\infty} a_j H_j(t). \tag{8}$$

A finite expansion in the first *(N+1)*-terms Hermite polynomials is

$$y_N(t) \cong \sum_{j=0}^{N} a_j H_j(t) = H(t)A \tag{9}$$

where the Hermite vector $H(t)$ and the Hermite coefficient vector A are given by

$$\begin{aligned} H(t) &= [H_0(t) \quad H_1(t) \quad \cdots \quad H_N(t)] \\ A^T &= [a_0 \quad a_1 \quad \cdots \quad a_N] \end{aligned} \tag{10}$$

respectively. From (7), the kth derivative of $y(t)$ can be expressed in the matrix form by

$$y_N^{(k)}(t) = H^{(k)}(t)A. \tag{11}$$

By the help of relations (7) and (11), we get

$$y_N^{(k)}(t) = H(t)\left(M^T\right)^k A. \tag{12}$$

By substituting (9) and (12)into (1), we get

$$H(t)\left(M^T\right)^n A = \lambda(t)H(t)A + \sum_{k=0}^{n} \beta_k(t)H(q_k t)\left(M^T\right)^k A + g(t). \tag{13}$$

To find the unknown Hermite coefficient, the collocation points $t_i = i/N, i = 0,1,2,...,N$ are put into (13) and the systems of the matrix equations are obtained as

$$H(t_i)\left(M^T\right)^n A = \lambda(t)H(t_i)A + \sum_{k=0}^{n} \beta_k(t_i)H(q_k t_i)\left(M^T\right)^k A + g(t_i). \tag{14}$$

This system can be rescripted as follows

$$\left\{ H_1\left(M^T\right)^n - \lambda H_1 - \sum_{k=0}^{n} \beta_k H_{q_k}\left(M^T\right)^k \right\} A = G \tag{15}$$

where

$$G=\begin{bmatrix} g(t_0)\\ g(t_1)\\ \vdots\\ g(t_N)\end{bmatrix},\lambda=\begin{bmatrix}\lambda(t_0) & 0 & \cdots & 0\\ 0 & \lambda(t_1) & \ddots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda(t_N)\end{bmatrix},\beta_k=\begin{bmatrix}\beta_k(t_0) & 0 & \cdots & 0\\ 0 & \beta_k(t_1) & \ddots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \beta_k(t_N)\end{bmatrix},$$

$$H_1=\begin{bmatrix} H_0(t_0) & H_1(t_0) & \cdots & H_N(t_0)\\ H_0(t_1) & H_1(t_1) & \cdots & H_N(t_1)\\ \vdots & \vdots & \ddots & \vdots\\ H_0(t_N) & H_1(t_N) & \cdots & H_N(t_N)\end{bmatrix},H_{q_k}=\begin{bmatrix} H_0(q_kt_0) & H_1(q_kt_0) & \cdots & H_N(q_kt_0)\\ H_0(q_kt_1) & H_1(q_kt_1) & \cdots & H_N(q_kt_1)\\ \vdots & \vdots & \ddots & \vdots\\ H_0(q_kt_N) & H_1(q_kt_N) & \cdots & H_N(q_kt_N)\end{bmatrix}.$$

Now, the fundemantal matrix equation (15) corresponding to (1) can be written as follows

$$WA=G \ or \ [W;G] \tag{16}$$

where

$$W=\left\{H_1\left(M^T\right)^n-\lambda H_1-\sum_{k=0}^{n}\beta_k H_{q_k}\left(M^T\right)^k\right\}.$$

Thus, (1)is transformed into matrix equation which corresponds to a system of (N+1) linear algebraic equations with unknown Hermite coefficients which can be written in augmented matrix form

$$[W;G]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(t_0)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(t_0)\\ \vdots & \vdots & \ddots & \vdots & ; & \vdots\\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & g(t_0)\end{bmatrix}. \tag{17}$$

Using (9) and (11) at $t=0$, initial conditions given in (2) can be written in the form of a matrix representation as

$$H(0)\left(M^t\right)^i A=[c_i],\ i=0,1,2,\dots,n-1. \tag{18}$$

Thus, the matrix form of (2) is:

$$U_iA=[c_i] \ or \ [U_i;c_i],\ \ i=0,1,2,\dots,N-1 \tag{19}$$

where

$$U_i=H(0)\left(M^t\right)^i=[u_{i0}\ \ u_{i1}\ \ \dots\ \ u_{iN}],i=0,1,2,\dots,n-1.$$

Finally, by replacing the last rows of the augmented matrix (17) by the row matrix (19), we reduce (1) under conditions (2) to the following linear system of algebraic equations

$$\tilde{W}A=\tilde{G} \tag{20}$$

where

$$\tilde{W}=\begin{bmatrix} w_{01} & w_{02} & \cdots & w_{0N} & ; & g(t_0)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(t_0)\\ \vdots & \vdots & \ddots & \vdots & ; & \vdots\\ w_{(N-i)0} & w_{(N-i)1} & \cdots & w_{(N-i)N} & ; & g(t_{N-i})\\ u_{00} & u_{01} & \cdots & u_{0N} & ; & c_0\\ u_{10} & u_{11} & \cdots & u_{1N} & ; & c_1\\ \vdots & \vdots & \ddots & \vdots & ; & \vdots\\ u_{(n-1)0} & u_{(n-1)1} & \cdots & u_{(n-1)N} & ; & c_{n-1}\end{bmatrix}. \tag{21}$$

If $rank\tilde{W} = rank\left[\tilde{W} : \tilde{G}\right] = N+1$, the linear system (20) has a unique solution and the matrix A, which is represented Hermite coefficients, is determined by $A = \left(\tilde{W}\right)^{-1}\tilde{G}$. On the other hand, if $\det(\tilde{W}) = 0$ and $rank\tilde{W} = rank\left[\tilde{W} : \tilde{G}\right] < N+1$, then we may obtain the particular solutions. Otherwise, if $rank\tilde{W} \neq rank\left[\tilde{W} : \tilde{G}\right]$, then there is no solution.

5 Numerical examples

In this section, some numerical examples are given to illustrate the applicability, accuracy, and effectiveness of the proposed method. The obtained results denote that this method can be considered as an alternative to the other methods in the literature in terms of the purpose of solving linear NFDEs in general.

Example 1. Firstly, let us consider the following NFDE with proportional delay [15,17,21,22,23,24,37]:

$$\begin{cases} y''(t) = \frac{3}{4}y(t) + y(\frac{t}{2}) + y'(\frac{t}{2}) + \frac{1}{2}y''(\frac{t}{2}) - t^2 - t + 1 \\ y(0) = y'(0) = 0. \end{cases} \tag{22}$$

By applying the present method to obtain the approximate solution $y_N(t)$ for $N = 3$, we seek the approximate solution in the form

$$y_3(t) = \sum_{j=0}^{3} a_j H_j(t) \tag{23}$$

where $\lambda(t) = 3/4, \beta_0(t) = \beta_1(t) = 1,\ \beta_2(t) = 1/2, q_0 = q_1 = q_2 = 1/2, g(t) = -t^2 - t + 1$. Using the collocation points for N=3, which are calculated as $\{t_0 = 0, t_1 = 1/3,\ t_2 = 2/3, t_3 = 1\}$, and using (15) the matrix equation of the (22) is

$$\left\{H_1\left(M^T\right)^2 - \lambda H_1 - \beta_0 H_{1/2} - \beta_1 H_{1/2}\, M^T - \beta_2 H_{1/2}\left(M^T\right)^2\right\} A = G \tag{24}$$

where

$$\lambda = \begin{bmatrix} 3/4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 \\ 0 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 3/4 \end{bmatrix}, \beta_0 = \beta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \beta_2 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, M^T = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 2/3 & -14/9 & -100/27 \\ 1 & 4/3 & -2/9 & -152/27 \\ 1 & 2 & 2 & -4 \end{bmatrix}, H_{1/2} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 1/3 & -17/9 & -53/27 \\ 1 & 2/3 & -14/9 & -100/27 \\ 1 & 1 & -1 & -5 \end{bmatrix}.$$

The augmented matrix for (22) is

$$[W;G] = \begin{bmatrix} -7/4 & -2 & 15/2 & 12 & ; & 1 \\ -7/4 & -17/6 & 103/18 & 758/27 & ; & 5/9 \\ -7/4 & -11/3 & 55/18 & 1114/27 & ; & -1/9 \\ -7/4 & -9/2 & -1/2 & 50 & ; & -1 \end{bmatrix}.$$

From (19), the matrix forms for initial conditions are

$$[U_0; c_0] = [1 \quad 0 \quad -2 \quad 0],\ \ [U_1; c_1] = [0 \quad 2 \quad 0 \quad -12].$$

From system (21), the new augmented matrix can be obtained as follows

$$[\tilde{W};\tilde{G}] = \begin{bmatrix} -7/4 & -2 & 15/2 & 12 & ; & 1 \\ -7/4 & -17/6 & 103/18 & 758/27 & ; & 5/9 \\ 1 & 0 & -2 & 0 & ; & 0 \\ 0 & 2 & 0 & -12 & ; & 0. \end{bmatrix}.$$

Solving this system, the unknown Hermite coefficients vector is found as

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} & 0 \end{bmatrix}^T.$$

Hence, the solution of (22) for N=3 is obtained $y_3(t) = t^2$ which is the exact solution.

Example 2. Now, let us consider the NFDE with proportional delay [16,17,21]

$$\begin{cases} y'(t) = -y(t) + 0.1y(0.8t) + 0.5y'(0.8t) + (0.32t - 0.5)e^{-0.8t} + e^{-t} \\ y(0) = 0 \end{cases} \tag{25}$$

where $\lambda(t) = -1, \beta_0(t) = 0.1, \beta_1(t) = 0.5,\ q_0 = q_1 = 0.8, g(t) = (0.32t - 0.5)e^{-0.8t} + e^{-t}$. From (15), the matrix equation of the (25) is

$$\{H_1 M^T - \lambda H_1 - \beta_0 H_{0.8} - \beta_1 H_{0.8} M^T\} A = G. \tag{26}$$

By applying the HCM for different values of N=3 and N=5, we obtain the approximate solutions. Fig. 1 shows exact solution and the approximate solutions for N=3 and N=5 are compared. Also, in Table 1, the absolute error functions $E_n(t) = |y(t) - y_N(t)|$ at the selected points of the given interval are compared with other methods.

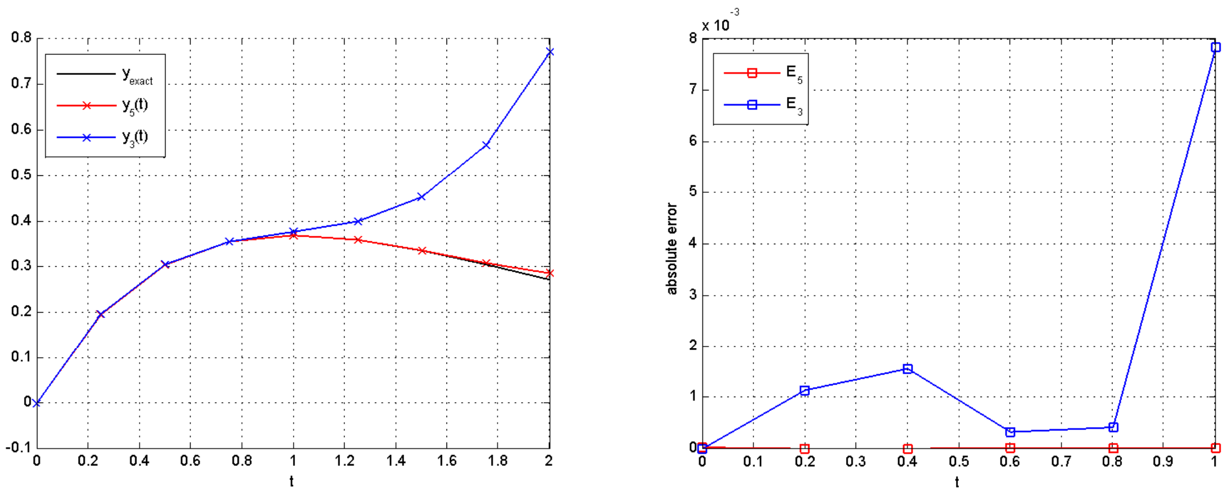

Fig. 1: First figure shows that comparison of exact and the approximate solutions for N=3 and N=5. Second figure shows that the absolute error functions $E_n(t)$ for N=5 and N=9.

Example 3. Let us consider the NFDE with proportional delay [15,21]

$$\begin{cases} y'''(t) = y(t) + y'\left(\frac{t}{2}\right) + y''\left(\frac{t}{3}\right) + \frac{1}{2}y'''\left(\frac{t}{4}\right) \\ y(0) = y'(0) = y''(0) = 0. \end{cases} \tag{27}$$

Table 1: Comparison of the absolute errors corresponding to different methods for (25).

t_i	HCM for N=5	RKHSM [21]	One-leg θ method [22,23]	RKTM [24]	VIM [17]
0.2	1.28e-6	1.17e-4	1.45e-2	1.49e-3	2.14e-3
0.4	3.44e-6	7.59e-4	3.60e-2	2.16e-3	2.84e-3
0.6	2.41e-5	4.73e-4	5.03e-2	2.31e-3	2.67e-3
0.8	1.17e-5	2.75e-4	5.47e-2	2.17e-3	2.04e-3
1.0	1.14e-5	1.43e-4	5.03e-2	1.86e-3	1.22e-3.

Using (15), the matrix equation of the (27) is

$$\left\{H_1\left(M^T\right)^3-\lambda H_1-\beta_1 H_{1/2}\,M^T-\beta_2 H_{1/3}\left(M^T\right)^2-\beta_3 H_{1/4}\left(M^T\right)^3\right\}A=G \tag{28}$$

where $\lambda(t)=1,\ \beta_0(t)=0,\ \beta_1(t)=1,\ \beta_2(t)=1,\ \beta_3(t)=1/2, q_1=1/2, q_2=1/3, q_3=1/4, g(t)=0$. By applying the HCM for N=4, the new augmented matrix in (21) can be obtained as follows

$$[\tilde{W};\tilde{G}]=\begin{bmatrix}-1 & -2 & -6 & 36 & 84 & ; & 0\\ -1 & -5/2 & -29/4 & 69/2 & 8711/48 & ; & 3959/768\\ 1 & 0 & -2 & 0 & 12 & ; & 0\\ 0 & 2 & 0 & -12 & 0 & ; & 0\\ 0 & 0 & 8 & 0 & -96 & ; & 0\end{bmatrix}.$$

Solving this system, the unknown Hermite coefficients vector is

$$A=\begin{bmatrix}\frac{3}{4} & 0 & \frac{3}{4} & 0 & \frac{1}{16}\end{bmatrix}^T.$$

Therefore, the solution of (27) for N=4 is obtained $y_4(t)=t^4$ which is the exact solution.

Example 4. Finaly, let us consider the NFDE with variable coefficients[21,22]:

$$\begin{cases} y''(t)=y'(0.5t)-0.5ty''(0.5t)+2\\ y(0)=1, y'(0)=0\end{cases} \tag{29}$$

From (15), the matrix equation of the (29) is:

$$\left\{H_1\left(M^T\right)^2-\beta_1 H_{0.5}-\beta_2 H_{0.5}\left(M^T\right)^2\right\}A=G \tag{30}$$

where $\beta_1(t)=1,\ \beta_2(t)=0.5t,\ q_1=q_2=0.5,\ g(t)=2$. By applying the HCM for N=3, the new augmented matrix in (21) can be obtained as follows:

$$[\tilde{W};\tilde{G}]=\begin{bmatrix}0 & -2 & 8 & 12 & ; 2\\ 0 & -2 & 8 & 86/3 & ; 2\\ 1 & 0 & -2 & 0 & ; 1\\ 0 & 2 & 0 & -12 & ; 0\end{bmatrix}.$$

Solving this system, the unknown Hermite coefficients vector is

$$A=\begin{bmatrix}\frac{3}{2} & 0 & \frac{1}{4} & 0\end{bmatrix}^T.$$

Hence, the solution of (29) for N=3 is obtained $y_3(t)=t^2+1$ which is the exact solution.

6 Conclusion

The fundamental aim of this paper is to improve the Hermite collocation method (HCM) to numerically solve the NFDEs with proportional delays. The comparison of the results shows that this approach can solve the NFDEs effectively and this method is consistent with the existing results in the literature. The validity and accuracy of this method is based on the assumption that it converges by increasing the number of collocation points. We conclude that the HCM can be considered as an accurate and reliable method for NFDEs with proportional delays.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] G.A. Adomian,Review of the decomposition method in applied mathematics,J. Math. Anal. Appl. 135,501–544 (1988).
[2] M. Bayram,Computer in Biology and Medicine,Automatic analysis of the control of metabolic networks,26(5),401–408 (1996).
[3] N. Guzel and M. Bayram,Numerical solution of differential-algebraic equations with index-2,Appl. Math. Comput.,174(2),1279–1289 (2006).
[4] J.K. Hale, Theory of Functional Differential Equations,Spring-Verlag, New York, 1993.
[5] J.H. He,Approximate analytical solution for seepage flow with fractional derivatives in porous media,Comput. Methods Appl. Mech. Engrg., 167 ,27–68 (1998).
[6] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations,Elsevier, New York, 2006.
[7] S.J. Liao,Proposed homotopy analysis technique for the solution of nonlinear problems,Ph.D. Dissertation, Shanghai Jiao Tong University, 1992.
[8] A.M. Wazwaz, Partial Differential Equations: Methods and Applications,Balkema, The Netherlands, 2002.
[9] J.K. Zhou, Differential Transformation and its Application for Electrical Circuits,Huazhong University Press, Wuhan, China, 1986.
[10] W.G. Ajello, H.I. Freedman and J. Wu,A model of stage structured population growth with density depended time delay,SIAM J.Appl.Math.,52,855–869 (1992).
[11] O. Arino and O. Sidki,An abstract neutral functional-differential equation arising from a cell population model,J. Math. Anal. Appl.,235(2) ,435–453 (1999).
[12] A. Bellen and M. Zennaro,Numerical Methods for Delay Differential Equations , Numerical Mathematics and Scientific Computation, The Clarendon Press Oxford University Press, New York, USA, 2003.
[13] L. Fox, D.F. Mayers, J.A. Ockendon, and A.B. Tayler,On a functional differential equation,J. Ins. Math. Appl. ,8 ,271–307 (1971).
[14] J. R. Ockendon and A. B. Tayler,Dynamics of a current collection system for an electric locomotive,Proc. Royal Soc. London A, 332,447–468(1971).
[15] A.H. Bhrawy, L.M. Assas, E. Tohidi and M.A. Alghamdi,A Legendre-Gauss collocation method for neutral functional-differential equations with proportional delays,Adv. Difference Equ., DOI:10.1186/1687-1847-2013-63, (2013).
[16] J. Biazar and B. Ghanbari,The homotopy perturbation method for solving neutral functional differential equations with proportional delays,Journal of King Saud University-Science, 24,33–37(2010).
[17] X. Chen and L. Wang, The variational iteration method for solving a neutral functional differential equation with proportional delays,Comput. Math. Appl., 59,2696–2702 (2010).
[18] L. F. Cordero and R. Escalante,Segmented tau approximation for test neutral functional differential equations,Appl. Math. Comput. , 187(2) ,725–740 (2007).
[19] Z. Jackiewicz and E. Lo,The numerical solution of neutral functional differential equations by Adams predictor-corrector methods,Appl. Numer. Math., 8 ,477–491 (1991).

[20] Z. Jackiewicz and E. Lo,Numerical solution of neutral functional differential equations by Adams methods in divided difference form,J. Comput. Appl. Math., 189(1-2) ,592–605 (2006).
[21] X. Lv and Y. Gao,The RKHSM for solving neutral functional-differential equations with proportional delays,Math. Methods Appl. Sci., 36(6) ,642–649 (2012).
[22] W.S. Wang and S.F. Li, On the one-leg δ -methods for solving nonlinear neutral functional differential equations,Appl. Math. Comput. ,193(1) ,285–301 (2007).
[23] W.S. Wang, T. Qin and S.F. Li, Stability of one-leg ? -methods for nonlinear neutral differential equations with proportional delay,Appl. Math. Comput., 213(1) ,177–183 (2009).
[24] W.S. Wang, Y. Zhang and S.F. Li,Stability of continuous Runge-Kutta-type methods for nonlinear neutral delay-differential equations,Appl. Math. Modell., 33(8) ,3319–3329 (2009).
[25] S. Wu and C. Huang,Convergence analysis of waveform relaxation methods for neutral differential-functional systems,J. Comput. Appl. Math., 223(1) ,263–277 (2009).
[26] A.H. Bhrawy and A.S. Alofi,A Jacobi-Gauss collocation method for solving nonlinear Lane-Emden type equations,Commun. Nonlinear Sci., 17(1) ,62–70 (2012).
[27] A.A. Dascioglu,A Chebyshev polynomial approach for linear Fredholm-Volterra integro-differential equations in the most general form,Appl. Math. Comput., 181(1) ,103–112 (2006).
[28] E.H. Doha, A.H. Bhrawy and S.S. Ezz-Eldien,A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order,Comput. Math. Appl., 62(5) ,2364–2373 (2011).
[29] S. Esmaeili, M. Shamsi and Y. Luchko,Numerical solution of fractional differential equations with a collocation method based on MÃ$\frac{1}{4}$ntz polynomials,Comput. Math. Appl., 62(3) ,918–929 (2011).
[30] M. Gulsu, M. Sezer and Z. Guney,Approximate solution of general high-order linear nonhomogeneous difference equations by means of Taylor collocation method,Appl. Math. Comput., 173(2) ,683–693 (2006).
[31] M. Gulsu, Y. Ozturk and M. Sezer,Approximate solution of general high-order linear nonhomogeneous difference equations by means of Taylor collocation method,Appl. Math. Comput., 216(7) ,2183–2198 (2010).
[32] M. Gulsu, B. Gurbuz, Y. Ozturk and M. Sezer,Laguerre polynomial approach for solving linear delay difference equations,Appl. Math. Comput., 217(15) ,6765–6776 (2011).
[33] M. M. Khader,N. H. Sweilam and A. M. S. Mahdy,Numerical Study for the Fractional Differential Equations Generated by Optimization Problem Using Chebyshev Collocation Method and FDM,Appl. Math. Inf. Sci.,7(5),2011–2018 (2013).
[34] K. Maleknejad, B. Basirat and E. Hashemizadeh, A Bernstein operational matrix approach for solving a system of high order linear Volterra-Fredholm integro-differential equations,Math. Comput. Model., 55(3-4),1363–1372 (2012).
[35] A. Saadatmandi and M. Dehghan,A new operational matrix for solving fractional-order differential equations,Comput. Math. Appl., 59(3) ,1326–1336 (2010).
[36] S. Yalcinbas, M. Aynigul and M. Sezer,A collocation method using Hermite polynomials for approximate solution of pantograph equations,J. Frankl. Ins., 348(6) ,1128–1139 (2011).
[37] S. Yuzbasi,An efficient algorithm for solving multi-pantograph equation systems,Comput. Math. Appl.,64(4) ,589–603 (2012).
[38] E. Tohidi, A.H. Bhrawy and K. Erfani,A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation,Appl. Math. Modell., 37(6),4283–4294 (2013).