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On the Mannheim surface offsets

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Abstract: In this paper, we study Mannheim surface offsets in dual space. By the aid of the E. Study Mapping, we consider ruled surfaces as dual unit spherical curves and define the Mannheim offsets of ruled surfaces by means of dual geodesic trihedron (dual Darboux frame). We obtain the relationships between the invariants of Mannheim ruled surfaces. Furthermore, we give the conditions for these surface offsets to be developable.

Keywords: Ruled surface; Mannheim offset; dual angle.

1 Introduction

Generally, an offset surface is offset a specified distance from the original along the parent surface's normal. Offsetting of curves and surfaces is one of the most important geometric operations in CAD/CAM due to its immediate applications in geometric modeling, NC machining, and robot navigation [4]. Especially, the offsets of ruled surfaces, which are the surfaces generated by continuously moving of a straight line, have an important role in (CAGD) [11,12]. These surfaces are used in different kinds of applications of Computer Aided Geometric Design (CAGD), moving geometry and kinematics. In [13], Ravani and Ku defined and studied the well-known offsets of ruled surfaces called Bertrand trajectory ruled surfaces. Then, Küçük and Gürsoy have introduced closed Bertrand trajectory ruled surfaces in dual space in terms of their integral invariants [6]. They have obtained the relations between the pitches and angle of pitches of closed Bertrand trajectory ruled surfaces. Also, they have given some characterizations including the relationships between the area of projections of spherical images and integral invariants of Bertrand trajectory ruled surfaces.

Recently, a new offset of ruled surfaces has been defined by Orbay, Kasap and Aydemir [7]. They have called this new offset as Mannheim offset and studied the developable Mannheim offset surfaces. Later, Önder and Uğurlu have defined and studied Mannheim offsets of ruled surfaces in the Minkowski 3-space E_1^3 [8,9]. Furthermore, Mannheim offsets of closed ruled surfaces in dual space have been studied according to Blaschke frame in [10] and the characterizations of Mannheim offsets of ruled surfaces in terms of integral invariants and areas of projection have been given.

In this paper, by considering E. Study mapping, we define the Mannheim offsets of the ruled surfaces according to dual geodesic trihedron and we give some theorems and new results for Mannheim offsets of ruled surfaces.

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2 Dual representation of ruled surfaces

W.K. Clifford (1845-1879) had been introduced dual numbers such as a dual number is a double in the form $\bar{a} = (a, a^*) = a + \varepsilon a^*$ where a and a^* are real numbers and $\varepsilon = (0, 1)$ is called dual unit with the property that $\varepsilon^2 = 0$ [1]. Let $\bar{a} = (a, a^*) = a + \varepsilon a^*$ and $\bar{b} = (b, b^*) = b + \varepsilon b^*$ be two dual numbers. The product of these numbers is defined by

$$\bar{a}\bar{b} = (a, a^*)(b, b^*) = (ab, ab^* + a^*b) = ab + \varepsilon(ab^* + a^*b). \tag{1}$$

Then, the set of dual numbers is denoted by D,

$$D = \left\{\bar{a} = a + \varepsilon a^* : \ a, a^* \in \mathrm{R}, \ \varepsilon^2 = 0\right\}. \tag{2}$$

Dual differentiable function of a dual variable has been studied by Dimentberg [2]. He derived the following general expression for a dual (differentiable) function

$$f(\bar{x}) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x), \tag{3}$$

where $f'(x)$ shows the derivative of $f(x)$ with respect to x. From this definition, we can give the following dual expressions for some well-known functions,

$$\begin{cases} \cos(\bar{x}) = \cos(x + \varepsilon x^*) = \cos(x) - \varepsilon x^* \sin(x), \\ \sin(\bar{x}) = \sin(x + \varepsilon x^*) = \sin(x) + \varepsilon x^* \cos(x), \\ \sqrt{\bar{x}} = \sqrt{x + \varepsilon x^*} = \sqrt{x} + \varepsilon \frac{x^*}{2\sqrt{x}}, \ (x > 0). \end{cases} \tag{4}$$

Let consider the set $D^3 = D \times D \times D$ of triples of dual numbers. Then we write

$$D^3 = \{\tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) : \quad \bar{a}_i \in D, \ i = 1, 2, 3\}. \tag{5}$$

which is called dual space and the triples $\tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3)$ are called dual vectors. Similar to the dual numbers, a dual vector $\tilde{a}$ has the form $\tilde{a} = \mathbf{a} + \varepsilon \mathbf{a}^* = (\mathbf{a}, \mathbf{a}^*)$, where $\mathbf{a}$ and $\mathbf{a}^*$ are the vectors of R^3. Then scalar product and cross product of dual vectors $\tilde{a} = \mathbf{a} + \varepsilon \mathbf{a}^*$ and $\tilde{b} = \mathbf{b} + \varepsilon \mathbf{b}^*$ in D^3 have given by,

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \langle \mathbf{a}, \mathbf{b} \rangle + \varepsilon \left(\langle \mathbf{a}, \mathbf{b}^* \rangle + \langle \mathbf{a}^*, \mathbf{b} \rangle \right), \tag{6}$$

and

$$\tilde{a} \times \tilde{b} = \mathbf{a} \times \mathbf{b} + \varepsilon \left(\mathbf{a} \times \mathbf{b}^* + \mathbf{a}^* \times \mathbf{b} \right), \tag{7}$$

respectively, where $\langle \mathbf{a}, \mathbf{b} \rangle$ and $\mathbf{a} \times \mathbf{b}$ are scalar product and cross product of the vectors $\mathbf{a}$ and $\mathbf{b}$ in R^3, respectively.

The norm of a dual vector $\tilde{a}$ is given by

$$\|\tilde{a}\| = \sqrt{\langle \tilde{a}, \tilde{a} \rangle} = \|\mathbf{a}\| + \varepsilon \frac{\langle \mathbf{a}, \mathbf{a}^* \rangle}{\|\mathbf{a}\|}. \tag{8}$$

Then if $\tilde{a}$ has the norm $1 + \varepsilon 0$, it is called dual unit vector and the set of dual unit vectors is called dual unit sphere and defined by

$$\tilde{S}^2 = \left\{\tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) \in D^3 : \ \langle \tilde{a}, \tilde{a} \rangle = 1 + \varepsilon 0\right\}, \tag{9}$$

(See [1,3]).

In 3-dimensional space $\mathbb{R}^3$, it is enough to have a point $p \in L$ and a unit vector $\mathbf{a}$ to determine an oriented line L. Then, the vector $\mathbf{a}^* = \mathbf{p} \times \mathbf{a}$ is called moment vector which does not depend on the points p. Then the pair $(\mathbf{a}, \mathbf{a}^*)$ represents the oriented line L. Conversely, if a pair $(\mathbf{a}, \mathbf{a}^*)$ is given, the line L can be obtained as $L = \{(\mathbf{a} \times \mathbf{a}^*) + \lambda \mathbf{a} : \ \mathbf{a}, \mathbf{a}^* \in \mathbb{R}^3, \ \lambda \in \mathbb{R}\}$. From the above discussion, we have that

$$\langle \mathbf{a}, \mathbf{a} \rangle = 1, \ \langle \mathbf{a}, \mathbf{a}^* \rangle = 0. \tag{10}$$

The components $a_i, \ a_i^* \quad (1 \le i \le 3)$ of the vectors $\mathbf{a}$ and $\mathbf{a}^*$ are called the normalized Plucker coordinates of the line L. From (6), (9) and (10), we see that the dual unit vector $\tilde{a} = \mathbf{a} + \varepsilon \mathbf{a}^*$ corresponds to the line L and this correspondence is called E. Study Mapping [1,3]. This correspondence has an important role to derive the properties of spatial motion of a line and consequently differential geometry of ruled surfaces. Hence, we study the geometry of ruled surfaces by considering dual curves lying fully on $\tilde{S}^2$.

The angle $\bar{\theta} = \theta + \varepsilon \theta^*$ between two dual unit vectors $\tilde{a}, \ \tilde{b}$ is called dual angle and defined by

$$\langle \tilde{a}, \tilde{b} \rangle = \cos \bar{\theta} = \cos \theta - \varepsilon \theta^* \sin \theta. \tag{11}$$

The geometric interpretation of dual angle is that θ is the real angle between the lines $L_1, \ L_2$ corresponding to the dual unit vectors $\tilde{a}, \ \tilde{b}$, respectively, and θ^* is the shortest distance between those lines [3].

In [14], Veldkamp introduced the dual geodesic trihedron of a ruled surface. Then, we speak to his procedure briefly as follows:

Let dual unit vector $\tilde{e}(u) = \mathbf{e}(u) + \varepsilon \mathbf{e}^*(u)$ represents a dual curve $(\tilde{k})$. The spherical curve drawn by a unit vector $\mathbf{e}$ on the real unit sphere S^2 is called the (real) indicatrix of $(\tilde{k})$ and supposed that it is not a single point. If we consider the parameter u as the arc-length parameter s of the real indicatrix, then we have $\langle \mathbf{e}', \mathbf{e}' \rangle = 1$ where $\mathbf{e}' = \mathbf{t}$ is unit tangent vector of the indicatrix. Let consider the equation $\mathbf{e}^*(s) = \mathbf{p}(s) \times \mathbf{e}(s)$ which has infinity of solutions for the function $\mathbf{p}(s)$. Taking $\mathbf{p}_o(s)$ as a solution, the set of all solutions is given by $\mathbf{p}(s) = \mathbf{p}_o(s) + \lambda(s)\mathbf{e}(s)$, where λ is a real scalar function of s. Then, we obtain $\langle \mathbf{p}', \mathbf{e}' \rangle = \langle \mathbf{p}_o', \mathbf{e}' \rangle + \lambda$. If we take $\lambda = \lambda_o = -\langle \mathbf{p}_o', \mathbf{e}' \rangle$, we have that $\mathbf{p}_o(s) + \lambda_o(s)\mathbf{e}(s) = \mathbf{c}(s)$ is the unique solution for $\mathbf{p}(s)$ with $\langle \mathbf{c}', \mathbf{e}' \rangle = 0$. Then, the dual curve $(\tilde{k})$ corresponding to the ruled surface

$$\varphi_e = \mathbf{c}(s) + v\mathbf{e}(s), \tag{12}$$

may be represented by dual unit vector

$$\tilde{e}(s) = \mathbf{e} + \varepsilon \mathbf{c} \times \mathbf{e}, \tag{13}$$

where

$$\langle \mathbf{e}, \mathbf{e} \rangle = 1, \quad \langle \mathbf{e}', \mathbf{e}' \rangle = 1, \quad \langle \mathbf{c}', \mathbf{e}' \rangle = 0. \tag{14}$$

Then we get

$$\left\| \tilde{e}' \right\| = \mathbf{t} + \varepsilon \det(\mathbf{c}', \mathbf{e}, \mathbf{t}) = 1 + \varepsilon \Delta, \tag{15}$$

where $\Delta = \det(\mathbf{c}', \mathbf{e}, \mathbf{t})$. The dual arc-length $\bar{s}$ of dual curve $(\tilde{k})$ is given by

$$\bar{s} = \int_0^s \left\| \tilde{e}'(u) \right\| du = \int_0^s (1+\varepsilon\Delta) du = s + \varepsilon \int_0^s \Delta du. \tag{16}$$

Then $\bar{s}' = 1 + \varepsilon\Delta$. Therefore, the dual unit tangent to the curve $\tilde{e}(s)$ is

$$\frac{d\tilde{e}}{d\bar{s}} = \frac{\tilde{e}'}{\bar{s}'} = \frac{\tilde{e}'}{1+\varepsilon\Delta} = \tilde{t} = \mathbf{t} + \varepsilon(\mathbf{c}\times\mathbf{t}). \tag{17}$$

Finally, defining dual unit vector $\tilde{g} = \mathbf{g} + \varepsilon\mathbf{c}\times\mathbf{g}$ by $\tilde{g} = \tilde{e}\times\tilde{t}$, we have dual frame $\{\tilde{e}, \tilde{t}, \tilde{g}\}$ which is called dual Darboux frame (dual geodesic trihedron) of φ_e(or $(\tilde{e})$). Moreover, the real orthonormal frame $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ along the striction curve of ruled surface φ_e is called the Frenet frame of φ_e and has the derivative formulae

$$\mathbf{e}' = \mathbf{t}, \ \ \mathbf{t}' = \gamma\mathbf{g} - \mathbf{e}, \ \ \mathbf{g}' = -\gamma\mathbf{t}, \tag{18}$$

where γ is called the conical curvature [5]. Corresponding dual form of the formulae given in (18) for the dual frame $\{\tilde{e}, \tilde{t}, \tilde{g}\}$ can be introduced as follows

$$\frac{d\tilde{e}}{d\bar{s}} = \tilde{t}, \ \ \frac{d\tilde{t}}{d\bar{s}} = \bar{\gamma}\tilde{g} - \tilde{e}, \ \ \frac{d\tilde{g}}{d\bar{s}} = -\bar{\gamma}\tilde{t}, \tag{19}$$

where

$$\bar{\gamma} = \gamma + \varepsilon(\delta - \gamma\Delta), \quad \delta = \left\langle \mathbf{c}', \mathbf{e} \right\rangle, \tag{20}$$

and the dual darboux vector of the frame is $\tilde{d} = \bar{\gamma}\tilde{e} + \tilde{g}$. From the definition of Δ and (20), we also have

$$\mathbf{c}' = \delta\mathbf{e} + \Delta\mathbf{g}. \tag{21}$$

The dual curvature of dual curve (ruled surface) $\tilde{e}(s)$ is

$$\bar{R} = \frac{1}{\sqrt{1+\bar{\gamma}^2}}. \tag{22}$$

The unit vector $\tilde{d}_o$ of Darboux vector $\tilde{d} = \bar{\gamma}\tilde{e} + \tilde{g}$ is given by

$$\tilde{d}_o = \frac{\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}\tilde{e} + \frac{1}{\sqrt{1+\bar{\gamma}^2}}\tilde{g}. \tag{23}$$

Then, if $\bar{\rho}$ is the dual angle between dual unit vectors $\tilde{d}_o$ and $\tilde{e}$, we have

$$\cos\bar{\rho} = \frac{\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}, \ \ \sin\bar{\rho} = \frac{1}{\sqrt{1+\bar{\gamma}^2}}, \tag{24}$$

where $\bar{\rho}$ is the dual spherical radius of curvature and so, $\bar{R} = \sin\bar{\rho}, \ \bar{\gamma} = \cot\bar{\rho}$ (For details see [14]).

3 Characterizations for Mannheim surface offsets

Mannheim offsets of ruled surfaces have been defined by Orbay and et al as follows:

Definition 3.1.([7]) Assume that φ and φ^* be two ruled surfaces in R^3 with the parametrizations

$$\varphi(s,v) = \mathbf{c}(s) + v\mathbf{q}(s), \quad \|\mathbf{q}(s)\| = 1,$$
$$\varphi^*(s,v) = \mathbf{c}^*(s) + v\mathbf{q}^*(s), \quad \|\mathbf{q}^*(s)\| = 1,$$

respectively, where $(\mathbf{c})$ (resp. $(\mathbf{c}^*)$) is the striction curve of ruled surfaces φ (resp. φ^*). Let the Frenet frames of ruled surfaces φ and φ^* be $\{\mathbf{q},\mathbf{h},\mathbf{a}\}$ and $\{\mathbf{q}^*,\mathbf{h}^*,\mathbf{a}^*\}$, respectively. The ruled surface φ^* is said to be Mannheim offset of the ruled surface φ if there exists a one to one correspondence between their rulings such that the asymptotic normal vector $\mathbf{a}$ of φ is the central normal vector $\mathbf{h}^*$ of φ^*. In this case, (φ,φ^*) is called a pair of Mannheim ruled surfaces.

Then, the dual version of Definition 3.1 can be given according to Darboux frame as follows:

Definition 3.2. Let consider the ruled surfaces φ_e and φ_{e_1} generated by dual unit vectors $\tilde{e}$ and $\tilde{e}_1$ and let $\{\tilde{e}(\bar{s}),\, \tilde{t}(\bar{s}),\, \tilde{g}(\bar{s})\}$ and $\{\tilde{e}_1(\bar{s}_1),\, \tilde{t}_1(\bar{s}_1),\, \tilde{g}_1(\bar{s}_1)\}$ be the dual Darboux frames of φ_e and φ_{e_1}, respectively. Then, φ_e and φ_{e_1} are called Mannheim surface offsets, if

$$\tilde{g}(\bar{s}) = \tilde{t}_1(\bar{s}_1), \tag{25}$$

holds along the striction lines of the surfaces, where $\bar{s}$ and $\bar{s}_1$ are the dual arc-lengths of φ_e and φ_{e_1}, respectively.

From definition 3.2, the relationship between trihedrons of ruled surfaces φ_e and φ_{e_1} is

$$\begin{pmatrix} \tilde{e}_1 \\ \tilde{t}_1 \\ \tilde{g}_1 \end{pmatrix} = \begin{pmatrix} \cos\bar{\theta} & \sin\bar{\theta} & 0 \\ 0 & 0 & 1 \\ \sin\bar{\theta} & -\cos\bar{\theta} & 0 \end{pmatrix} \begin{pmatrix} \tilde{e} \\ \tilde{t} \\ \tilde{g} \end{pmatrix}, \tag{26}$$

where $\bar{\theta} = \theta + \varepsilon\theta^*$, $(0 \le \theta \le \pi,\ \theta^* \in \mathrm{R})$ is dual angle between the generators $\tilde{e}$ and $\tilde{e}_1$ of Mannheim ruled surface φ_e and φ_{e_1}. The real angle θ is called the offset angle and the real number θ^* is called the offset distance. Then, $\bar{\theta} = \theta + \varepsilon\theta^*$ is called dual offset angle of the Mannheim ruled surface φ_e and φ_{e_1}. If $\theta = 0$ and $\theta = \pi/2$ then the Mannheim offsets are called oriented offsets and right offsets, respectively.

Theorem 3.1. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then the relations between offset angle θ, offset distance θ^* and arc length s are given by*

$$\theta = -s + c, \ \theta^* = -\int_0^s \Delta du + c^*, \tag{27}$$

respectively, where c and c^* are real constants.

Proof. Suppose that ruled surface φ_{e_1} is a Mannheim offset of ruled surface φ_e. Then (26) gives us

$$\tilde{e}_1 = \cos\bar{\theta}\tilde{e} + \sin\bar{\theta}\tilde{t}. \tag{28}$$

Differentiating (28) with respect to $\bar{s}$ gives

$$\frac{d\tilde{e}_1}{d\bar{s}} = -\sin\bar{\theta}\left(1 + \frac{d\bar{\theta}}{d\bar{s}}\right)\tilde{e} + \cos\bar{\theta}\left(1 + \frac{d\bar{\theta}}{d\bar{s}}\right)\tilde{t} + \bar{\gamma}\sin\bar{\theta}\tilde{g}. \tag{29}$$

Since $\frac{d\tilde{e}_1}{d\bar{s}}$ and $\tilde{g}$ are linearly dependent, from (29) we get $\frac{d\bar{\theta}}{d\bar{s}} = -1$. Then for the dual constant $\bar{c} = c + \varepsilon c^*$ we write

$$d\bar{\theta} = -d\bar{s},$$
$$\bar{\theta} = -\bar{s} + \bar{c},$$
$$\theta + \varepsilon\theta^* = -s - \varepsilon s^* + c + \varepsilon c^*,$$

and from (16) we have

$$\theta = -s + c,\ \ \theta^* = -\int_0^s \Delta du + c^*,$$

where c and c^* are real constants.

Corollary 3.1. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then φ_e is developable if and only if* $\theta^* = c^* = \text{constant}$.

Proof. Since φ_e and φ_{e_1} form a Mannheim surface offset, we have Theorem 4.1. Thus from (27) we see that φ_e is developable i.e. $\Delta = 0$ if and only if $\theta^* = c^* = \text{constant}$.

Theorem 3.2. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then the relationship between the dual arc-length parameters of φ_e and φ_{e_1} is given by*

$$\frac{d\bar{s}_1}{d\bar{s}} = \bar{\gamma}\sin\bar{\theta}. \tag{30}$$

Proof. Suppose that φ_e and φ_{e_1} form a Mannheim offset. Considering Theorem 3.1, we get

$$\frac{d\tilde{e}_1}{d\bar{s}_1} = \tilde{t}_1 = \bar{\gamma}\sin\bar{\theta}\frac{d\bar{s}}{d\bar{s}_1}\tilde{g}. \tag{31}$$

From (26) we have $\tilde{t}_1 = \tilde{g}$. Then (31) gives us

$$\bar{\gamma}\sin\bar{\theta}\frac{d\bar{s}}{d\bar{s}_1} = 1, \tag{32}$$

and from (32) we get (30).

Theorem 3.3. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then there are the following relationships between the real arc-length parameters and invariants of φ_e and φ_{e_1}*

$$\begin{cases} \frac{ds_1}{ds} = \gamma\sin\theta, \\ \Delta_1 = \theta^*\cot\theta + \frac{\delta}{\gamma}. \end{cases} \tag{33}$$

Proof. Let φ_e and φ_{e_1} form a Mannheim surface offset. Then from Theorem 3.2, (30) holds. By considering (20), the real and dual parts of (30) are

$$\frac{ds_1}{ds} = \gamma\sin\theta,\ \ \frac{ds\,ds_1^* - ds^*ds_1}{ds^2} = \theta^*\gamma\cos\theta + (\delta - \gamma\Delta)\sin\theta, \tag{34}$$

respectively. Furthermore from (16) we have

$$ds^* = \Delta ds,\ \ ds_1^* = \Delta_1 ds_1. \tag{35}$$

Writing the equalities (35) in (34), we have (33).

Corollary 3.2. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then, the Mannheim offset φ_{e_1} is developable if and only if*

$$\theta^* = -\frac{\delta}{\gamma}\tan\theta, \tag{36}$$

holds.

Proof. Assume that φ_e and φ_{e_1} form a Mannheim surface offset. Then Theorem 3.3 holds. So, we have that φ_{e_1} is developable i.e, $\Delta_1 = 0$ if and only if

$$\theta^* = -\frac{\delta}{\gamma}\tan\theta. \tag{37}$$

holds which finishes the proof.

Theorem 3.4. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then*

$$\delta_1 = \frac{\delta}{\gamma}\cot\theta - \theta^* \tag{38}$$

holds.

Proof. Let the striction lines of φ_e and φ_{e_1} be $c(s)$ and $c_1(s_1)$, respectively and let φ_e and φ_{e_1} form a Mannheim surface offset. Then, we can write

$$\mathbf{c}_1 = \mathbf{c} + \theta^*\mathbf{g}. \tag{39}$$

Differentiating (39) with respect to s_1 we have

$$\frac{d\mathbf{c}_1}{ds_1} = \left(\frac{d\mathbf{c}}{ds} - \theta^*\gamma\mathbf{t} + \frac{d\theta^*}{ds}\mathbf{g}\right)\frac{ds}{ds_1}. \tag{40}$$

From (20) we know that $\delta_1 = \langle d\mathbf{c}_1/ds_1, e_1\rangle$. Then from (26) and (40) we obtain

$$\delta_1 = (\cos\theta\,\langle d\mathbf{c}/ds,\, \mathbf{e}\rangle + \sin\theta\,\langle d\mathbf{c}/ds,\, \mathbf{t}\rangle - \theta^*\gamma\sin\theta\,\langle \mathbf{t},\, \mathbf{t}\rangle)\,\frac{ds}{ds_1}. \tag{41}$$

Since $\delta = \langle d\mathbf{c}/ds, e\rangle$ and $\langle d\mathbf{c}/ds,\, \mathbf{t}\rangle = 0$, from (41) we write

$$\delta_1 = (\delta\cos\theta - \theta^*\gamma\sin\theta)\,\frac{ds}{ds_1}. \tag{42}$$

Furthermore, from (33) we have

$$\frac{ds}{ds_1} = \frac{1}{\gamma\sin\theta}, \tag{43}$$

and substituting (43) in (42) we obtain

$$\delta_1 = \frac{\delta}{\gamma}\cot\theta - \theta^*.$$

Theorem 3.5. *Let φ_e and φ_{e_1} form a Mannheim surface offset. The conical curvature of φ_{e_1} is obtained as follows,*

$$\gamma_1 = \cot\theta. \tag{44}$$

Proof. From (18) and (26) we have

$$\begin{aligned}\gamma_1 &= -\langle \mathbf{g}_1', \mathbf{t}_1\rangle \\ &= -\left\langle \tfrac{d}{ds_1}(\sin\theta\mathbf{e} - \cos\theta\mathbf{t}),\ \mathbf{g}\right\rangle\end{aligned} \tag{45}$$

which gives

$$\gamma_1 = \gamma\cos\theta\frac{ds}{ds_1}. \tag{46}$$

From the first equality of (33) and (46) we have (44).

Theorem 3.6. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then, there exits the following relation between dual curvature $\bar{R}_1$ of φ_{e_1} and dual offset angle $\bar{\theta}$,*

$$\bar{R}_1 = \sin\bar{\theta}. \tag{47}$$

Proof. From (22) and (44), Eq. (47) is obtained immediately.

From Eq. (23), (24) and Theorem 3.6, we have the following corollaries.

Corollary 3.3. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then, dual unit Darboux vector $\tilde{d}_{0_1}$ of φ_{e_1} is given by*

$$\tilde{d}_{0_1} = \cos\bar{\theta}\tilde{e}_1 + \sin\bar{\theta}\tilde{g}_1, \tag{48}$$

and from (26) it means that the ruled surface generated by dual unit Darboux vector $\tilde{d}_{0_1}$ of φ_{e_1} is a Mannheim offset of φ_{e_1}.

Corollary 3.4. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then, the relation between dual offset angle and dual conical curvature $\bar{\gamma}_1$ of φ_{e_1} is given by*

$$\cos\bar{\theta} = \frac{\bar{\gamma}_1}{\sqrt{1+\bar{\gamma}_1^2}},\quad \sin\bar{\theta} = \frac{1}{\sqrt{1+\bar{\gamma}_1^2}}. \tag{49}$$

Moreover, from Eq. (26) and Corollary 3.4, we have the following corollary.

Corollary 3.5. *Let φ_e and φ_{e_1} form a Mannheim surface offset. Then, the relation between dual Darboux frames of the surfaces is given by*

$$\begin{pmatrix}\tilde{e}_1\\ \tilde{t}_1\\ \tilde{g}_1\end{pmatrix} = \begin{pmatrix}\frac{\bar{\gamma}_1}{\sqrt{1+\bar{\gamma}_1^2}} & \frac{1}{\sqrt{1+\bar{\gamma}_1^2}} & 0\\ 0 & 0 & 1\\ \frac{1}{\sqrt{1+\bar{\gamma}_1^2}} & -\frac{\bar{\gamma}_1}{\sqrt{1+\bar{\gamma}_1^2}} & 0\end{pmatrix}\begin{pmatrix}\tilde{e}\\ \tilde{t}\\ \tilde{g}\end{pmatrix}. \tag{50}$$

Example 3.1. Let consider the hyperbolic paraboloid surface φ_e given by the parametrization

$$\varphi_e(s,v) = \left(\frac{1}{2}s,\ \frac{1}{2}s,\ 0\right) + v\left(\frac{1}{2},\ -\frac{1}{2},\ s\right), \tag{51}$$

and rendered in Fig. 1.

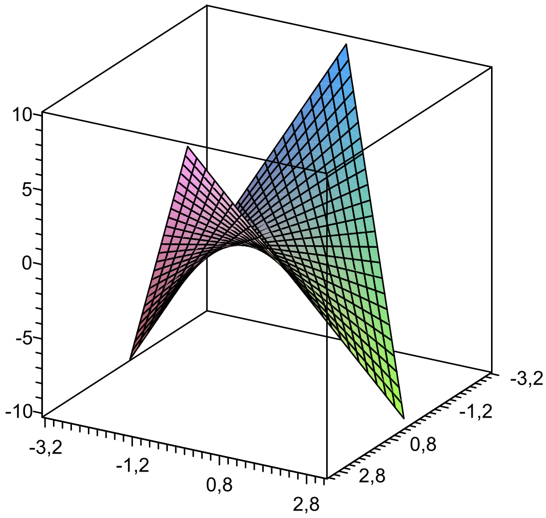

Fig. 1: Hyperbolic paraboloid surface φ_e.

From the E. Study Mapping, the dual spherical curve representing (51) is

$$\tilde{e}(s) = \frac{\sqrt{2}}{\sqrt{1+2s^2}}\left[\left(\frac{1}{2},-\frac{1}{2},s\right) + \varepsilon\left(\frac{1}{2}s^2,-\frac{1}{2}s^2,-\frac{1}{2}s\right)\right]. \tag{52}$$

Then, the dual Darboux frame of φ_e is obtain as follows

$$\begin{cases} \tilde{e}(s) = \frac{\sqrt{2}}{\sqrt{1+2s^2}}\left[\left(\frac{1}{2},-\frac{1}{2},s\right)+\varepsilon\left(\frac{1}{2}s^2,-\frac{1}{2}s^2,-\frac{1}{2}s\right)\right] \\ \tilde{t}(s) = \frac{1}{\sqrt{1+8\tan^2(\sqrt{2}s)}}\left[\left(-2\tan(\sqrt{2}s),2\tan(\sqrt{2}s),1\right)\right. \\ \qquad \left. +\varepsilon\left(\tan(\sqrt{2}s),-\tan(\sqrt{2}s),4\tan^2(\sqrt{2}s)\right)\right] \\ \tilde{g}(s) = \left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0\right) \end{cases}$$

The general equation of the Mannheim offset surface of φ_e is

$$\begin{aligned} \varphi_{e_1}(s,v) &= \left(\tfrac{1}{2}s-\theta^*\tfrac{\sqrt{2}}{2},\ \tfrac{1}{2}s-\theta^*\tfrac{\sqrt{2}}{2},\ 0\right) \\ &\quad + v\left(\frac{\sqrt{2}}{\sqrt{1+2s^2}}\cos\theta\left(\tfrac{1}{2},\ -\tfrac{1}{2},\ s\right)+\sin\theta\left(-\tfrac{\sqrt{2}}{2},-\tfrac{\sqrt{2}}{2},0\right)\right). \end{aligned} \tag{53}$$

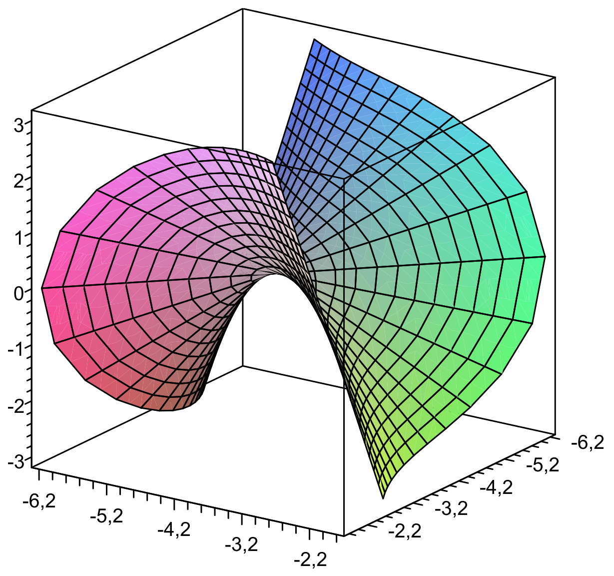

Fig. 2: Mannheim offset φ_{e_1} with dual offset angle $\bar{\theta} = 0 + \varepsilon 4\sqrt{2}$.

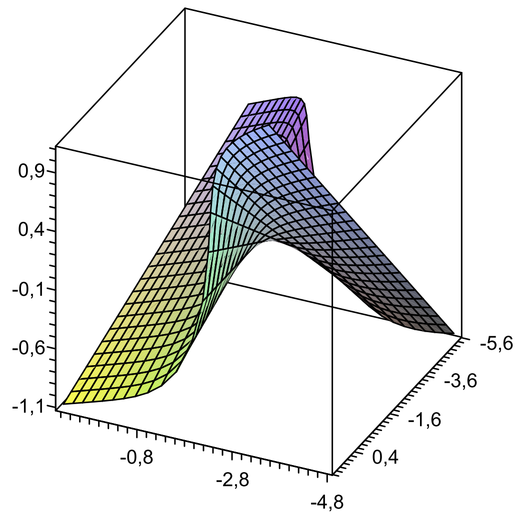

Fig. 3: Mannheim offset φ_{e_1} with dual offset angle $\bar{\theta} = \pi/4 + \varepsilon 2\sqrt{2}$.

From (53) we can give the following special cases:

i) The Mannheim offset φ_{e_1} with dual offset angle $\bar{\theta} = 0 + \varepsilon 4\sqrt{2}$ is

$$\varphi_{e_1}(s,v) = \left(\frac{1}{2}s - 4,\ \frac{1}{2}s - 4,\ 0\right) + v\left(\frac{\sqrt{2}}{\sqrt{1+2s^2}}, -\frac{\sqrt{2}}{\sqrt{1+2s^2}}, \frac{\sqrt{2}s}{\sqrt{1+2s^2}}\right)$$

which is an oriented offset of φ_e(Fig. 2).

ii) The Mannheim offset φ_{e_1} with dual offset angle $\bar{\theta} = \pi/4 + \varepsilon 2\sqrt{2}$ is

$$\varphi_{e_1}(s,v) = \left(\frac{1}{2}s-2,\ \frac{1}{2}s-2,\ 0\right) + v\left(\frac{1}{2\sqrt{1+2s^2}} - \frac{1}{2}, \frac{-1}{2\sqrt{1+2s^2}} - \frac{1}{2}, \frac{s}{\sqrt{1+2s^2}}\right)$$

4 Conclusions

In the surface theory, offset surfaces have an important role and large applications in many areas. Especially, the ruled surface offsets are interesting since these surfaces can be generated by a continuous moving of a straight line. In this paper, some new results including the characterizations of Mannheim surface offsets have been obtained in dual space. Furthermore, the relationships for Mannheim surface offsets to be developable have been introduced.

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