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A Taylor matrix-collocation method based on residual error for solving Lane-Emden type differential equations

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Abstract: In this work, a new Taylor matrix method based on the collocation points and residual correction method is developed to solve the linear Lane-Emden differential equations with initial conditions. Correction of the approximate solution obtained by Taylor collocation method is performed by means of the error differential equation with the Residual function. Details of the method are presented and some numerical results along with error estimation are given to clarify the efficiency and high accuracy of the method.

Keywords: Taylor collocation method; Residual correction method; Lane-Emden equation, Residual function.

1 Introduction

Lane-Emden equations are singular initial value problems relating to second-order ordinary differential equations (ODEs) which have been used to model several phenomena in mathematical physics and astrophysics such as thermal explosions [1], stellar structure [2], the thermal behavior of a spherical cloud of gas, isothermal gas spheres and, thermionic currents [3]. One of the equations describing this type is Lane-Emden equation defined by

$$L[y(x)] = y''(x) + \frac{\gamma}{x}y'(x) + s(x)y(x) = f(x),\ \gamma \geq 0,\ 0 \leq x \leq 1,\ f(x) \in C[0,1], \tag{1}$$

with the initial conditions

$$y(0) = \alpha \ \ y'(0) = \beta, \tag{2}$$

where and are real constants.

In this study, by using the Tau method with error estimation [8] and the residual correction method [5-6-9] we develop an efficient error estimation for the Taylor collocation method and also a technique to obtain the corrected solution (high accuracy solution) of the problem [1-2].

2 Taylor collocation method

Let us consider the Lane-Emden equation defined by (1)

$$L[y(x)] = f(x)$$

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and find the solution in the truncated Taylor series

$$y(x) \cong \sum_{n=0}^{N} y_n x^n, \;\; y_n = \frac{y^n(0)}{n!}, \;\; 0 \le x \le 1 \tag{3}$$

where y_n $(n = 0,1,...,N)$ are unknown Taylor coefficients to be determined and N is chosen as $N \ge 2$. The solution (3) can be expressed in the matrix form as

$$y(x) = \mathbf{X}(x)\mathbf{Y} \tag{4}$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 \; x \; x^2 \cdots \; x^N \end{bmatrix}, \;\; \mathbf{Y} = \begin{bmatrix} y_0 \; y_1 \; y_2 \cdots \; y_N \end{bmatrix}^T$$

On the other hand, it is clearly seen that the relation between the matrix $\mathbf{X}(x)$ and its derivative $\mathbf{X}'(x)$ is given [7,10] as

$$\mathbf{X}'(x) = \mathbf{X}(x)\mathbf{B}; \;\; \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{5}$$

From the matrix equations (4) and (5), it follows that

$$y^{(s)}(x) = \mathbf{X}(x)\mathbf{B}^s\mathbf{Y}, \;\; s = 0,1,... \tag{6}$$

To obtain a numerical approximation to Lane-Emden equation (1) under the given conditions (2), substituting the matrix relations (6) into Eq. (1), we obtain the matrix equation

$$\mathbf{X}(x)\mathbf{B}^2\mathbf{Y} + \frac{\gamma}{x}\mathbf{X}(x)\mathbf{B}\mathbf{Y} + s(x)\mathbf{X}(x)\mathbf{Y} = f(x) \text{ or} \tag{7}$$

$$x\mathbf{X}(x)\mathbf{B}^2\mathbf{Y} + \gamma\mathbf{X}(x)\mathbf{B}\mathbf{Y} + xs(x)\mathbf{X}(x)\mathbf{Y} = xf(x).$$

By substituting the collocation points defined by

$$x_i = \frac{i}{N}, \;\; i = 0,1,...,N$$

into Eq. (7), we obtain the system of the matrix equations

$$x_i\mathbf{X}(x_i)\mathbf{B}^2\mathbf{Y} + \gamma\mathbf{X}(x_i)\mathbf{B}\mathbf{Y} + x_i s(x_i)\mathbf{X}(x_i)\mathbf{Y} = x_i f(x_i).$$

or the compact notation

$$\mathbf{PXB}^2\mathbf{Y} + \gamma\mathbf{XBY} + \mathbf{SXY} = \mathbf{F} \tag{8}$$

The augmented matrix for Eq. (8) becomes briefly

$$\mathbf{WY} = \mathbf{F}; \;\; \mathbf{W} = [w_{ij}] = \mathbf{PXB}^2 + \gamma\mathbf{XB} + \mathbf{SX}, \;\; i,j = 0,1,...,N \tag{9}$$

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which corresponds to a system of $N+1$ nonlinear algebraic equations with the unknown Taylor coefficients y_n $(n = 0,1,...,N)$. The matrices in Eq. (8) are as follows:

$$\mathbf{F} = \begin{bmatrix} x_0 f(x_0) \\ x_1 f(x_1) \\ \vdots \\ x_N f(x_N) \end{bmatrix}, \mathbf{P} = \begin{bmatrix} P(x_0) & 0 & \cdots & 0 \\ 0 & P(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P(x_N) \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} x_0 s(x_0) & 0 & \cdots & 0 \\ 0 & x_1 s(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_N s(x_N) \end{bmatrix}, \mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix}$$

The augmented matrix for Eq. (9) becomes, briefly,

$$[\mathbf{W} : \mathbf{F}] \tag{10}$$

Next, by means of relations (4) and (6), we can obtain the matrix forms corresponding to the initial conditions (2) as

$$\mathbf{X}(0)\mathbf{Y} = \alpha \;\text{ or }\; \begin{bmatrix} 1\;0\;0\cdots 0\;;0\;0\cdots 0\;\alpha \end{bmatrix} \tag{11}$$

and

$$\mathbf{X}(0)\mathbf{B}\mathbf{Y} = \beta \;\text{ or }\; \begin{bmatrix} 0\;1\;0\cdots 0\;;0\;0\cdots 0\;\beta \end{bmatrix} \tag{12}$$

Consequently, to obtain the solution of Eq. (1) under the conditions (2), by replacing the row matrices (11) and (12) by the last two rows of the augmented matrix (10), we have the required augmented matrix

$$[\tilde{\mathbf{W}} : \tilde{\mathbf{F}}] \tag{13}$$

By solving the matrix equation (13), the unknown Taylor coefficients y_n $(n = 0,1,...,N)$ are determined. Thus we get the Taylor polynomial solution

$$y_N(x) = \sum_{n=0}^{N} y_n x^n.$$

3 Residual correction and error estimation

In this section, we will give an error estimation for the Taylor collocation method and the residual correction of the Taylor approximate solution. For our purpose, we can define the residual function of the Taylor collocation method as

$$\mathbf{R}_N(x) = \mathbf{L}[y_N(x)] - f(x) \tag{14}$$

where $y_N(x)$, which is the Taylor polynomial solution defined by (3), is the approximate solution of the problem (1)-(2), Hence $y_N(x)$ satisfies the problem

$$L[y_N(x)] = y_N''(x) + \frac{\gamma}{x} y_N'(x) + s(x)y_N(x) = f(x) + R_N(x); \tag{15}$$

$$y_N(0) = \alpha,\;\; y_N'(0) = \beta$$

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Also, the error function $e_N(x)$ can be defined as

$$e_N(x) = y(x) - y_N(x) \tag{16}$$

where $y(x)$ is the exact solution of the problem (1)-(2). Substituting (16) into (1)-(2) and using (14)-(15), we have the error differential equation

$$L[e_N(x)] = L[y(x)] - L[y_N(x)] = -R_N(x)$$

with the homogenous initial conditions $e_N(0) = 0,\ e'_N(0) = 0$ or clearly, the problem

$$e''_N(x) + \frac{\gamma}{x}e'_N(x) + s(x)e_N(x) = -R_N(x);\ \ e_N(0) = 0,\ e'_N(0) = 0 \tag{17}$$

Note that, from (16), the inhomogeneous initial conditions $y(0) = \alpha,\ y'(0) = \beta$ and $y_N(0) = \alpha,\ y'_N(0) = \beta$ are reduced to the homogeneous initial conditions $e_N(0) = 0,\ e'_N(0) = 0$. Solving the problem (17) in the same way as Section 2, we get the approximation $e_{N,M}(x)$ to $e_N(x)$, $(M \geq N)$ which is the error function based on the Residual function $R_N(x)$.
Consequently, by means of the polynomials $y_N(x)$ and $e_{N,M}(x)$, $(M \geq N)$ we obtain the corrected Taylor polynomial solution $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$. Also, we construct the Taylor error function $e_{N,M}(x) = y(x) - y_N(x)$, the corrected Taylor error function $E_{N,M}(x) = e_N(x) - e_{N,M}(x) = y(x) - y_{N,M}(x)$ and the estimated error function $e_{N,M}(x)$.

4 Numerical examples

In this section, we show the efficiency of the presented method by using the numerical results of some examples, selected through Lane-Emden equations. Also, we calculate the values of the exact solution $y(x)$, the Taylor approximate solution $y_N(x)$, the corrected Taylor solution $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$, the Taylor error $|e_{N,M}(x)|$ and the corrected Taylor error $|E_{N,M}(x)|$ and the estimated error at the selected points of the given interval with 25 digits of accuracy with Maple programming (see Table1).

Example 1. [4] Firstly, let us consider the linear Lane-Emden equation

$$y''(x) + \frac{2}{x}y'(x) + y(x) = 6 + 12x + x^2 + x^3$$

subject to initial conditions

$$y(0) = 0,$$

and

$$y'(0) = 0$$

Using the given procedure in Section 3, the fundamental matrix equations of the equation and conditions, respectively, are obtained as follows:

$$\{\mathbf{PXB}^2 + 2\mathbf{XB} + \mathbf{SX}\} = \mathbf{F} \text{ and } \mathbf{X}(0)\mathbf{Y} = 0,\ \mathbf{X}'(0)\mathbf{BY} = 0$$

By taking $N = 3$, the Taylor coefficients are found as $\{y_0 = 0, y_1 = 0, y_2 = 1, y_3 = 1\}$.
Therefore, these coefficients give the exact solution $y(x) = x^2 + x^3$.

Example 2. We finally close our analysis by studying the inhomogeneous linear Lane-Emden equation

$$y''(x) + \frac{2}{x}y'(x) = 2(2x^2 + 3)y(x),\ y(0) = 1,\ y'(0) = 0$$

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with the exact solution $y(x) = e^{x^2}$ [4].
By following the procedure in Section 3 and Section4, we calculate the values of the exact solution $y(x)$, the Taylor solution $y_N(x)$, the corrected Taylor solution $y_{N,M}(x)$, the corrected Taylor error $|E_{N,M}(x)|$ and the estimated error $|e_{N,M}(x)|$ at the selected points of the given interval (see Table1). Table 1 shows that the presented method provides more accurate predictions of $y(x)$.

Table 1. : Error Analysis of Example 2.

x_i	Exact Solution $y(x) = e^{x^2}$	Taylor app. $y_N(x)$	Corr.Tay. sol. $y_{N,M}(x)$	Est. err. $\|e_{N,M}(x)\|$	Corr.Tay err $\|E_{N,M}(x)\|$
$N = 10, M = 15$					
0.0	1.0000000000	1.0000000000	1.0000000000	0.00000	0.00000
0.2	1.0408107741	1.0408274196	1.0408107722	1.66e-6	8.83e-9
0.4	1.1735108709	1.1735338152	1.1735108685	2.29e-5	7.74e-9
0.6	1.4333294145	1.4333586702	1.4333294115	2.21e-5	4.58e-9
0.8	1.8964808793	1.8965200963	1.8964808752	2.92e-5	1.59e-9
1.0	2.7182818284	2.7183383116	2.7182818226	5.64e-5	8.24e-9
$N = 20, M = 25$					
0.0	1.0000000000	1.0000000000	1.0000000000	0.00000	0.00000
0.2	1.0408107741	1.0408107741	1.0408107741	8.63e-14	9.43e-18
0.4	1.1735108709	1.1735108709	1.1735108709	1.04e-14	1.64e-18
0.6	1.4333294145	1.4333294145	1.4333294145	1.29e-14	3.59e-18
0.8	1.8964808793	1.8964808793	1.8964808793	1.92e-14	1.42e-18
1.0	2.7182818284	2.7182818284	2.7182818284	2.44e-14	5.44e-18

5 Conclusion

In this work, we carefully employed the reliable modified Taylor collocation method based on the residual correction method for the solution of linear Lane-Emden equations. Lane-Emden equations, which are of crucial importance in mathematical physics and astrophysics, are also solved by this method very accurately. Furthermore, an efficiently error estimation can be done by this method. Hence, it can be approximated by the corrected Taylor polynomial solution to the exact solution. From given illustrative examples, it can be seen that the method can obtain very accurate and satisfactory results. An interesting feature of this method is that the exact solution is obtained as demonstrated in Example 1 when the exact solution is polynomials.

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