New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2016.109

# Some properties of $K_{\preceq}$ set

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Received: 31 August 2016, Accepted: 30 October 2016
Published online: 13 November 2016.

**Abstract:** In this paper, an order $\preceq$ which is subset of the natural order $\leq$ of $[0,1]$ is considered. A set denoted by $K_{\preceq}$ containing some elements which are comparable with respect to $\leq$ but incomparable with respect to $\preceq$ is defined by using order $\preceq$. Some useful properties of $K_{\preceq}$ is investigated.

**Keywords:** Order, sub-order, partition

## 1 Introduction

In mathematics, a partially ordered set (poset) formalize ordering, sequencing or arrangment of the elements of a set. Simply, a poset is comprised of a set with a binary relation. The relation is called partially orders to express the fact that not every element precedes the other. Because such a construction is more general, partial order is very effective for algebra. One can easily argue that posets are very important and they plays fundamental roles in many particular fields of mathematics such as lattice theory [1,2,5], triangular norms [11], fuzzy logic and its applications [6,13].

Triangular norms (conorms) are binary operation, defined on $[0,1]$ unit real interval at first, satisfies properties of monotonicity, associativity, commutativity and neutral element. Therefore, $([0,1],\leq)$ is an useful poset for triangular norms. When viewed from this aspect and having importance for information science in mind, It is quite natural that many researchers have studied on t-norms and their properties [4,7,8,11,15]. On the other hand, the order $\preceq$ which is sub-order of $([0,1],\leq)$ is noticeable topic for both t-norms (t-conorms) and order-theory [3].

In this study, we worked $K_{\preceq}$ set and its some properties such as partition of $K_{\preceq}$ considering arbitrary sub-order of $([0,1],\leq)$.

This paper proceeds as follows: first, in Section 1 we give the basic definitions and notations. Secondly we investigate some properties of $K_{\preceq}$. If $K_{\preceq}$ is nonempty, we show that $K_{\preceq}$ is infinite. Again if $K_{\preceq}$ is nonempty, then for any $x \in K_x$, we proved that there exists a maximal interval contains $x$. After that, we show that every elements not comparable with the elements of $K_x$ according to $\preceq$ are also in $K_x$. With the help of these properties, we obtain a partition of $K_{\preceq}$.

## 2 Preliminaries

**Definition 1.**[1] *A partially ordered set or shortly poset $P$ is a set in which a binary relation $x \leq y$ is defined, which satisfies following conditions for $x,y,z$:*

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(i) *For all x*, $x \leq x$.
(ii) *If* $x \leq y$ *and* $y \leq x$, *then* $x = y$.
(iii) *If* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$.

*Furthermore, the binary relation* $\leq$ *which has the above properties is called an order on P. A poset P with respect to order* $\leq$ *is denoted by the pair of* $(P, \leq)$.

**Definition 2.** [1] *A poset which satisfies the following condition is said to be "simply" or "totally" ordered and is called a chain:*

$$\text{Given } x \text{ and } y, \text{ either } x \leq y \text{ or } y \leq x.$$

It is clear that every pair of elements $x,y$ of a poset $P$ may not provide $x \leq y$ or $y \leq x$. Such elements are called incomparable elements. Dually, if the pair of elements $x,y$ of a poset $P$ provides $x \leq y$ or $y \leq x$, such elements are called comparable elements. An upper bound of a subset $X$ of a poset $P$ is an element $a \in P$ containing every $x \in X$. The least upper bound is an upper bound contained in every other upper bound; it is denoted $l.u.b.X$ or $SupX$. By Definition 1, $SupX$ is unique if it exists. The notations of lower bound of $X$ and greatest lower bound ($g.l.b.X$ or $InfX$) of $X$ are defined dually. Again by Definition 1, $InfX$ is unique if it exists.

**Definition 3.** [1] *A lattice is a poset P any two elements have a* $g.l.b.$ *or "meet" denoted by* $x \wedge y$ *and,* $l.u.b.$ *or "join" denoted by* $x \vee y$. *A lattice L is complete when each of its subsets X has a* $l.u.b.$ *and a* $g.l.b.$ *in L.*

**Definition 4.** *Let P be a poset with* $\leq$. *If an order* $\preceq$ *provides the following condition, then* $\preceq$ *is called a subset of* $\leq$:

$$\forall x,y \in P, \quad x \preceq y \Rightarrow x \leq y.$$

Let $P$ be a poset with $\leq$, $\preceq$ be a subset of $\leq$ and $X \subseteq P$. We write $\bigvee X$ and $\bigwedge X$ if we mean respectively $SupX$ and $InfX$ with respect to $\leq$ and we write $\bigvee_{\preceq} X$ and $\bigwedge_{\preceq} X$ if we mean respectively $SupX$ and $InfX$ with respect to $\preceq$ (if they exist).

Let $(L, \leq)$ be a lattice and $\preceq$ be a subset of $\leq$. We consider the following equality:

$$\bigvee_{\tau}(x \wedge_{\preceq} y_{\tau}) = x \wedge_{\preceq} (\bigvee_{\tau} y_{\tau})$$

for all $\{x, y_{\tau} | \quad \tau \in T\} \subseteq L$. We sign this property with $* - property$. Also, hereafter $\leq$ denotes natural order of $[0,1]$ and an order $\preceq$ denotes any subset of $\leq$ in this work. It's known that $([0,1], \leq)$ is a chain (also a lattice). If we assume $\preceq \neq \leq$, then at least there are two elements $x, y \in [0,1]$ which are incomparable with respect to the order $\preceq$. Hence, in this situation the following set should be nonempty:

$$\{x \in [0,1] | \quad \text{for some} \quad y \in [0,1], \quad [x \leq y \quad \text{implies} \quad x \npreceq y] \quad \text{or} \quad [y \leq x \quad \text{implies} \quad y \npreceq x]\}.$$

We will use $K_{\preceq}$ symbol to denote this set.

## 3 Some properties of $K_{\preceq}$ set

**Proposition 1.** $K_{\preceq}$ *is an empty set if and only if* $([0,1], \preceq)$ *is a chain.*

*Proof.* Let $K_{\preceq}$ is an empty set. Then, for any $x \in [0,1]$ we can't find $y \in [0,1]$ provides $[x \leq y \quad \text{implies} \quad x \npreceq y]$ or $[y \leq x \quad \text{implies} \quad y \npreceq x]$. So, for all $x, y \in [0,1]$ we have $x \preceq y$ or $y \preceq x$. Conversely, if $([0,1], \preceq)$ is a chain,then for all $x \in [0,1]$, there exists no $y \in [0,1]$ provides $K_{\preceq}$ conditions. So, $K_{\preceq}$ is an empty set.

*Remark.* If $K_{\preceq}$ is empty, note that $\preceq = \leq$ in Proposition 1.

**Proposition 2.** *If $K_{\preceq}$ is a nonempty set, then there exists a subinterval of $K_{\preceq}$ containing for any element $x \in K_{\preceq}$, and so $K_{\preceq}$ is infinite.*

*Proof.* Let $K_{\preceq}$ be non-empty. Then, $K_{\preceq}$ contains at least one member. Let $x$ denote such an element. By the definition of the set $K_{\preceq}$, there is an element $y_x \in [0,1]$ such that $x \leq y_x$ but $x \npreceq y_x$, or $y_x \leq x$ but $y_x \npreceq x$. Without loss of generality, let us assume that $x \leq y_x$ but $x \npreceq y_x$. Now, we shall show that $[x, y_x] \subseteq K_{\preceq}$. Suppose that $[x, y_x] \nsubseteq K_{\preceq}$. Then, there is an element $c \in [x, y_x]$ such that $c \notin K_{\preceq}$. So, it must be $x \preceq c$ and $c \preceq y_x$. By the transitivity of the order $\preceq$, it is obtained that $x \preceq y_x$, a contradiction. Then, it must be $[x, y_x] \subseteq K_{\preceq}$. That means that $K_{\preceq}$ is infinite and $[x, y_x]$ is a subinterval of $K_{\preceq}$ containing the element $x \in K_{\preceq}$.

**Theorem 1.** *Let $K_{\preceq}$ be nonempty set. For $x \in K_{\preceq}$, there exists a maximal subinterval (the greatest subinterval) $K_x$ of $K_{\preceq}$ such that $x \in K_x$. Moreover, the family*

$$M := \{K_{x_i} | \quad K_{x_i}, \quad i \in I \quad \textit{is a maximal subinterval of } K_{\preceq}\}$$

*is a partition of $K_{\preceq}$, where the index set $I$ is finite or countably infinite.*

*Proof.* Let $x \in K_{\preceq}$ be arbitrary and $\mathscr{A}_x$ be a set defined by

$$\mathscr{A}_x := \{K | \quad K \subseteq K_{\preceq} \quad \text{is an subinterval such that} \quad x \in K\}.$$

By Proposition 2, $\mathscr{A}_x$ is non-empty. Also, it is clear that $(\mathscr{A}_x, \subseteq)$ is a poset. Let $\{K_j | \quad j \in J\}$ be any chain of $(\mathscr{A}_x, \subseteq)$. Then $\bigcup_j K_j = K^*$ is a subinterval of $K_{\preceq}$ i.e. $K^* \in \mathscr{A}_x$. Thus, by Zorn2s Lemma, $A_x$ has a maximal element. Let us denote by $K_x$ such a maximal interval.

Let us show that $M$ is a partition of $K_{\preceq}$. Again, by Proposition 2, $M$ is nonempty. Let $K_{x_i} \neq K_{x_j}$ for any $i, j \in I$. In this case, it is clear that $K_{x_i} \cap K_{x_j} = \emptyset$. Also it is clear that $\bigcup_{i \in I} K_{x_i} = K_{\preceq}$.

$(K_{x_i})_{i \in I}$ are the intervals and each of these intervals is nonempty and therefore, contains some rational numbers, which can be used as an index set of the corresponding interval. Consequently, the cardinality of the resulting index set $I$ can not exceed the cardinality of all rational numbers (in $[0,1]$), i.e, $I$ must be a finite or countably infinite set.

**Lemma 1.** *Let $K_x \subseteq K_{\preceq}$ be the greatest subinterval of $K_{\preceq}$ containing the element $x$. Then, every elements incomparable with the elements of $K_x$ according to $\preceq$ are also in $K_x$. Also, for any $y \in K_x$, $K_x = K_y$.*

*Proof.* For any $y \in K_x$, it is clear that $x \in K_x \subseteq K_y$. Since $x \in K_y$, it is clear that $K_y \subseteq K_x$. Then $K_x = K_y$.
Let $y \in K_x$ and $k_y$ be an incomparable element with $y$ according to $\preceq$. By the definition, either $y < k_y$ but $y \npreceq k_y$ or $k_y < y$ but $k_y \npreceq y$. Let $y < k_y$ but $y \npreceq k_y$. Suppose that $k_y \notin K_x = K_y$. Then, there exists the greatest subinterval $K_{k_y} \subseteq K_{\preceq}$ such that $k_y \in K_{k_y}$. Therefore, $K_y \neq K_{k_y}$ and in this case, it is clear that $K_y \cap K_{k_y} = \emptyset$. Then, there exist at least one an element $t \notin K_{\preceq}$ such that $y \leq t \leq k_y$. By the definition of $K_{\preceq}$, we have $y \preceq t \preceq k_y$. This inequality implies $y \preceq k_y$, a contradiction. It means that $k_y \in K_x$. Moreover, for any $y \in K_x$, it is clear that $K_x = K_y$.

**Proposition 3.** *Let $([0,1], \leq)$ provide $* -$ property and $K_x \subseteq K_{\preceq}$ be the greatest interval containing the element $x \in K_x$. Then, $K_x$ is a lower half-open interval.*

*Proof.* Let $K_x$ be not lower half-open interval. Then there is an element $c \in K_x$ such that for any $y \in K_x$, $c \leq y$. Since $c \in K_x$, there exists an element $x_c$ not comparable with $c$. By Lemma 1, $x_c \in K_x$. Thus, $c \leq x_c$ and $c \npreceq x_c$. There exists a sequence $(x_n)_{n \in \mathbb{N}}$, such that $\{x_n | n \in \mathbb{N}\} \nsubseteq K_{\preceq}$ and $sup x_n = c$. Thus for every $n \in \mathbb{N}$, $x_n \preceq x_c$. Then for some $l_n \in [0,1], n \in \mathbb{N}$,

$$x_n = x_c \wedge_{\preceq} l_n$$

Since $([0,1],\leq)$ provides $*-$ *property*, we have that

$$c=\bigvee_n x_n=\bigvee_n (x_c\wedge_\preceq l_n)=x_c\wedge_\preceq(\bigvee_n l_n).$$

Then it is obtained that $c\preceq x_c$, a contradiction. So $K_x$ is a lower half-open.

**Proposition 4.** *Every element of $K_\preceq$ is a derived point of $K_\preceq$.*

*Proof.* Let $x\in K_\preceq$ be arbitrary and let $U$ be any neighborhood of $x$. Then for some $1/n>0$, $B(x,1/n)\subseteq U$. Since $x\in K_\preceq$, there exists $y_x\in[0,1]$ such that $x\leq y_x$ implies $x\npreceq y_x$ or $y_x\leq x$ implies $y_x\npreceq x$. Let $y_x<x$ and $\varepsilon^*:=min\{\varepsilon,1/n\}$ for $\varepsilon:=x-y_x>0$. It follows $B(x,\varepsilon^*)\setminus\{x\}\subseteq U\setminus\{x\}$ from $B(x,\varepsilon^*)\subseteq B(x,1/n)\subseteq U$. By the proof of Proposition 2, since $[y_x,x]\subseteq K_\preceq$, we obtain that $[x-\varepsilon^*,x]\subseteq[y_x,x]\subseteq K_\preceq$. On the other hand, it is clear that $[x-\varepsilon^*,x]\subseteq B(x,\varepsilon^*)$. Then, we obtain that

$$U\setminus\{x\}\cap K_\preceq\neq\emptyset.$$

Thus, $x$ is an derived point of $K_\preceq$.

**Theorem 2.** *Let $\{0,1\}\subseteq B\subseteq[0,1]$ be an arbitrary set. If there exists a family $((u_i,v_i))_{i\in I}$ of pairwise disjoint open sub-intervals of $[0,1]$ such that*

$$\bigcup_{i\in I}(u_i,v_i)\subseteq[0,1]\setminus B\subseteq\bigcup_{i\in I}(u_i,v_i],$$

*where $I$ is finite or countably infinite index set, then there is an order $\preceq$ which is subset of $\leq$ such that $B$ coincides with the set of all comparable elements of $[0,1]$ with respect to $\preceq$.*

*Proof.* Let $B$ be a subset of $[0,1]$ satisfying the given inequalities and $I$ be a finite or countably infinite index set. Let $((u_i,v_i))_{i\in I}$ be a family of pairwise disjoint open subinterval of $[0,1]$. Then, $[0,1]\setminus B$ can be represented as a union of a finite or countably infinite family of pairwise disjoint intervals $(B_i)_{i\in I}$, where for each $i\in I$, either $B_i=(a_i,b_i)$ or $B_i=(a_i,b_i]$ for suitable $a_i,b_i\in[0,1]$ and where $B_i\cup B_j$ is not an interval for $i\neq j$. Then, the function $*:[0,1]\times[0,1]\to[0,1]$ defined by

$$x*y=\begin{cases}a_i & (x,y)\in B_i\times B_i,\\ min(x,y) & \text{otherwise,}\end{cases}$$

is clearly a binary operation and the order defined by

$$x\preceq y:\Leftrightarrow \text{for some } \ell\in[0,1]: x=y*\ell$$

is clearly a subset of $\leq$ on $[0,1]$. Now, we will show that the set of incomparable elements of $[0,1]$ with respect to $\preceq$ coincides $[0,1]\setminus B$.

Let consider $K_\preceq$, we shall prove that $K_\preceq=[0,1]\setminus B$. Let $x\in[0,1]\setminus B$. Then, for some $i\in I$, $x\in B_i$. We claim that for any $y\in B_i$ with $x<y$, it must be $x\npreceq y$. Suppose that for some $y\in B_i$ with $x<y$, $x\preceq y$. Then, for some $\ell\in[0,1]$, $x=y*\ell$. If $\ell\in B_i$, it would be $x=y*\ell=a_i\notin B_i$, which is a contradiction. Since $\ell\notin B_i$, $x=y*\ell=min(y,\ell)$. Since $x\neq y$, $x=\ell$ contradicts that $x\in B_i$. So, for any $y\in B_i$ with $x<y$, it must be $x\npreceq y$. Then, it is obtained that $x\in K_\preceq$.

Conversely, let $x\in K_\preceq$. Then, there is an element $y\in[0,1]$ such that $x<y$ implies $x\npreceq y$ or $y<x$ implies $y\npreceq x$. Assume that $x<y$ but $x\npreceq y$. If for every $i\in I$, $x\notin B_i$, then $x*y=min(x,y)=x$ contradicts that $x\npreceq y$. Then, for some $i\in I$, $x\in B_i$. Thus, $x\in\bigcup_{i\in I}B_i=[0,1]\setminus B$. So, it is obtained that $K_\preceq=[0,1]\setminus B$. Since $B=[0,1]\setminus K_\preceq$, $B$ is the set of all comparable elements of $[0,1]$ with respect to $\preceq$.

**Theorem 3.** *Let $\{0,1\} \subseteq B \subseteq [0,1]$ be an arbitrary set. If $([0,1],\leq)$ provides $*-$ property and $B$ coincides with the set of all comparable elements of $[0,1]$ with respect to $\preceq$, then there exists a finite or countably infinite index set $I$ and a family $((u_i,v_i))_{i\in I}$ of pairwise disjoint open subintervals of $[0,1]$ such that*

$$\bigcup_{i\in I}(u_i,v_i) \subseteq [0,1]\setminus B \subseteq \bigcup_{i\in I}(u_i,v_i].$$

*Proof.* By Theorem 1, it is clear that there exists such an index set $I$. Let $([0,1],\leq)$ be a $\preceq$-supremum distributive lattice and $B$ coincides with the set of all comparable elements of $[0,1]$ with respect to $\preceq$. Thus, $[0,1]\setminus B = K_{\preceq}$. By Theorem 1, there exists a partition of $K_{\preceq}$ such that for any $x_i \in K_{\preceq}$

$$\{K_{x_i} | \quad K_{x_i}, \quad i \in I \quad \text{is a maximal subinterval of } K_{\preceq}\}.$$

Since $([0,1],\leq)$ provides $*-$ *property* by Proposition 3 for every $i \in I$, $K_{x_i}$ is a lower half-open interval. Thus, for $u_i, v_i \in [0,1]$, $i \in I$, $K_{x_i} = (u_i,v_i)$ or $K_{x_i} = (u_i,v_i]$. Therefore, for any $i \in I$

$$(u_i,v_i) \subseteq K_{\preceq} \quad \text{or} \quad (u_i,v_i] \subseteq K_{\preceq}.$$

Then, clearly

$$\bigcup_{i\in I}(u_i,v_i) \subseteq K_{\preceq} = [0,1]\setminus B \subseteq \bigcup_{i\in I}(u_i,v_i].$$

## 4 Conclusion

In this paper, the order $\preceq$ which is subset of natural order $\leq$ on $[0,1]$ is handled and $K_{\preceq}$ set is defined using the order $\preceq$. In addition, some properties of $K_{\preceq}$ are investigated, in this manner, some results on the relation between $\preceq$ and $\leq$ are examined. On the other hand, $K_x$ set which is greatest interval of $K_{\preceq}$ containing the element $x$ is defined and properties of $K_x$, relationship between $K_x$ and $K_{\preceq}$ are researched.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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