Journal of Abstract and Computational Mathematics
http://www.ntmsci.com/jacm

Solving systems of ordinary differential equations in unbounded domains by exponential Chebyshev collocation method

Mohamed A. Ramadan[1] *and Mohamed. A. Abd El Salam*[2]

[1]Mathematics Department, Faculty of Science, Menoufia University, Shebein El-Koom, Egypt
[2]Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, 11884, Cairo, Egypt

Received: 23 June 2016, Accepted: 29 September 2016
Published online: 1 October 2016.

Abstract: The purpose of this paper is to investigate the use of exponential Chebyshev collocation method for solving systems of linear ordinary differential equations with variable coefficients in unbounded domains, with most general form of conditions. The definition of the exponential Chebyshev (EC) functions allows us to deal with systems of differential equations defined in the whole domain and with infinite boundaries without singularities or divergence. The method transforms the system of differential equations and the given conditions to block matrix equation with unknown EC coefficients. By means of the obtained matrix equations, a new system of equations which corresponds to the system of linear algebraic equations is gained. Numerical examples are included to illustrate the validity and applicability of the method.

Keywords: Exponential Chebyshev functions, system of differential equations, collocation method, unbounded domains.

1 Introduction

In recent years, systems of high order ordinary differential equations have been solved intensively by using approximate iterative methods such as variational iteration method [1], differential transformation method [2], Adomian decomposition method [3], differential transform method [4], Homotopy analysis method [5]. In addition to these methods, the spectral methods are also used to solving systems of linear differential equations. Chebyshev collocation method [6] and Taylor collocation method [7] are also applied to solve these systems of differential equations. The well-known Chebyshev polynomials $T_n(t)$ are orthogonal polynomials on the interval [-1, 1], see [8]. These polynomials have many applications in numerical analysis and spectral methods. One of the applications of Chebyshev polynomials is the solution of systems of differential equations with mixed conditions, with collocation points [6]. Therefore, this limitation of the Chebyshev approach fails in the problems that are naturally defined on all domains, especially including infinity. Under a transformation that maps the interval [-1, 1] into a semi-infinite domain [0,∞), Boyd [9, 10], Parand and Razzaghi [11, 12], Sezer et al. [13, 14], and Ramadan et al. [15-19] successfully applied different spectral methods to solve problems on semi-infinite domain. Recently, the authors of [20] have proposed modified form of Chebyshev polynomials as an alternative to the solutions of the problems given in all domains. In their studies, the basis functions called exponential Chebyshev functions (EC) $E_n(t)$that are orthogonal in $(-\infty,\infty)$. This kind of extension tackles the problems over the whole real domain. The EC functions are defined as

$$E_n(t) = T_n\left(\frac{e^t-1}{e^t+1}\right), \tag{1}$$

* Corresponding author e-mail: [1]ramadanmohamed13@yahoo.com, [2]mohamed_salam1985@yahoo.com

where, the corresponding recurrence relation is

$$E_0(t)=1,\ E_1(t)=\frac{e^t-1}{e^t+1},\ E_{n+1}(t)=2\left(\frac{e^t-1}{e^t+1}\right)E_n(t)-E_{n-1}(t).\ n\geq 1 \tag{2}$$

Now, in this paper we will use the EC collocation method for solving systems of linear ordinary differential equations with variable coefficients in unbounded domains, with most general form of conditions. The paper is organized as follows. In section 2, preliminaries introduced while in section 3 properties of the exponential Chebyshev (EC) functions are presented. In section 4, we seek the form of the fundamental matrix relation based on collocation points. In section 5, method of solution is presented. Finally, section 6 contains numerical illustrations and results that are compared with the exact solutions to demonstrate the applicability of the present method.

2 Preliminaries

The system of high-order linear ordinary differential equations system considered here is a set of k linear differential equations with variable coefficients of the mth order in the form [6]

$$\sum_{n=0}^{m}\sum_{j=1}^{k} p_{ij}^{n}(t)\,y_j^{(n)}(t)=f_i(t),\, i=1,2,\ldots,k \tag{3}$$

This system can be written in compact matrix notation as

$$\sum_{i=0}^{m}\mathbf{P}_i(t)\,\mathbf{y}^{(i)}(t)=\mathbf{f}(t), \tag{4}$$

where the $p_{ij}^{n}(t)$ and $f_i(t)$ are well defined functions on the interval $(-\infty,\infty)$, where the matrices $P_i(t)$, $y^{(i)}(t)$ and $f(t)$ are of the form

$$\mathbf{P}_i(t)=\begin{bmatrix} p_{11}^i & p_{12}^i & \cdots & p_{1k}^i \\ p_{21}^i & p_{22}^i & \cdots & p_{2k}^i \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ p_{k1}^i & p_{k2}^i & \cdots & p_{kk}^i \end{bmatrix},\ \mathbf{y}^{(i)}(t)=\begin{bmatrix} y_1^{(i)}(t) \\ y_2^{(i)}(t) \\ . \\ . \\ . \\ y_k^{(i)}(t)\end{bmatrix},\ \mathbf{f}(t)=\begin{bmatrix} f_1(t) \\ f_2(t) \\ . \\ . \\ . \\ f_k(t)\end{bmatrix}.$$

We consider the above system under the most general form of conditions defined as

$$\sum_{i=0}^{m-1} a_i y^{(i)}(a)+b_i y^{(i)}(b)+c_i y^{(i)}(c)=\lambda,-\infty<a\leq c\leq b<\infty \tag{5}$$

where a_i, b_i, c_i and λ are real valued vectors, and a, b maybe tends to the boundaries that is $a,\ b\to\pm\infty$.

3 Properties of exponential Chebyshev (EC) functions

In this section we list some of the properties of the EC functions.

3.1 Orthogonality of EC functions

The weight function $w(t)$ corresponding to EC functions [20], such that they are orthogonal in the interval $(-\infty,\infty)$ is given by $\sqrt{e^t}/\left(e^t+1\right)$, with the orthogonality condition

$$\int_{-\infty}^{\infty} E_n(t)E_m(t)w(t)dt = \frac{c_m\pi}{2}\delta_{nm}, \tag{6}$$

where

$$c_m = \begin{cases} 2, & m=0 \\ 1, & m\neq 0 \end{cases}$$

and δ_{nm} is the Kronecker delta function. Also the product relation of the EC functions is

$$E_n(t)E_m(t) = \frac{1}{2}[E_{n+m}(t)+E_{|n-m|}(t)], \tag{7}$$

which used in the derivative relations.

3.2 Function expansion in terms of EC functions

A function $f(t)$ that well-defined over the interval $(-\infty,\infty)$, may be expanded as

$$f(t) = \sum_{n=0}^{\infty} a_n E_n(t), \tag{8}$$

where

$$a_n = \frac{2}{c_n\pi}\int_{-\infty}^{\infty} E_n(t)f(t)w(t)dt.$$

If $f(t)$ in expression (8) is a truncated to $N<\infty$ in terms of the EC functions takes the form

$$f(t) \cong \sum_{n=0}^{N} a_n E_n(t), \tag{9}$$

also, the (k)th-order derivative of $f(t)$ can be written as

$$f^{(k)}(t) \cong \sum_{n=0}^{N} a_n \left(E_n(t)\right)^{(k)}, \tag{10}$$

where $\left(E_n(t)\right)^{(0)} = E_n(t)$.

3.3 The derivatives relations

In the next proposition the operational matrix of the derivative of EC functions that introduced for first time in [20] is presented and proved.

Proposition 1. *The relation between the vector $E(x)$ and its (r)th-order derivative is given as*

$$E^{(r)}(t) \cong E(t)(D^T)^r, \tag{11}$$

where, D is the $(N+1)\times(N+1)$operational matrix for the derivative, and the general form of the matrix D is a tridiagonal matrix which is obtained from

$$D{=}diag\left(\frac{i}{4},\ 0,\ \frac{i}{4}\right),\qquad i=0,\ 1,\ ...,\ N \tag{12}$$

Proof. Derivatives of the EC functions can be found by differentiating relation (2), and by the help of (8) we get

$$(E_0(t))' = 0, \tag{13}$$

$$(E_1(t))' = \frac{2e^t}{(1+e^t)^2} = \frac{1}{4}E_0(t) - \frac{1}{4}E_2(t), \tag{14}$$

and

$$\begin{aligned}(E_{n+1}(t))' &= \tfrac{d}{dx}[2E_1(t)E_n(x) - E_{n-1}(t)]\\ &= \tfrac{d}{dx}[2\,(E_1(t))^{(0)}\,(E_n(t))^{(0)} - (E_{n-1}(t))^{(0)}]\\ &= [2\,(E_1(t))^{(1)}\,(E_n(t))^{(0)} + 2\,(E_1(t))^{(0)}\,(E_n(t))^{(1)} - (E_{n-1}(t))^{(1)}],\end{aligned}$$

that can be written as

$$(E_{n+1}(t))' = 2\{(E_1(t)E_n(t))'\} - (E_{n-1}(t))'. \tag{15}$$

By using the relations (13)-(15) and by the help of product relation (7) for $n=0,\,1,\,...,\,N$ then we get

$$\begin{cases} (E_0(t))' = 0,\\ (E_1(t))' = \frac{1}{4}E_0(t) - \frac{1}{4}E_2(t),\\ (E_2(t))' = \frac{1}{2}E_1(t) - \frac{1}{2}E_3(t),\\ \quad\vdots\\ (E_n(t))' = \frac{n}{4}E_{n-1}(t) - \frac{n}{4}E_{n+1}(t), \qquad n>1. \end{cases} \tag{16}$$

The above equalities (16) form $(N+1)\times(N+2)$ rectangular matrix. Then a truncation to the last column gives square operational matrix D given in (12), then to obtain the matrix $E^{(r)}(t)$ we can use (16) as

$$E'(t) \cong E(t)D^T,$$

$$E''(t) \cong E'(t)D^T = (E(t)D^T)D^T = E(t)(D^T)^2,$$

$$E^{(3)}(t) \cong E''(t)D^T = E(t)(D^T)^3,$$

$$\vdots$$

then by induction we can write

$$E^{(r)}(t) \cong E(t)(D^T)^r. \tag{17}$$

Proposition and its proof derived a regular scheme with truncation in the matrix D, for the relation between the vector $E(t)$ and its (r)th-order derivative.

Now, we turn to the improved scheme without any truncation for the relation of $E(t)$ and its (r)th-order derivative that leads us to get equality sign in (17), that introduced by us in [21] in next proposition.

Proposition 2. *The derivatives of the vector $E(t){=}[E_0(t)\ \ E_1(t)\ \ ....\ \ E_N(t)]$, can be expressed with equality sign if we added the truncated last column in the following form as*

$$E'(t) = E(t)D^T + B(t), \tag{18}$$

where D is given in (2), and $B(t)$ is $1\times(N+1)$ row vector which was truncated in (16) and will be the actual term to get the equality sign of (17). The row vector $B(t)$ is deduced as shown next

$$B(t)=[0 \quad 0 \quad .... \quad 0 \quad \frac{-N}{4}E_{N+1}(t)]. \tag{19}$$

Consequently, to obtain the matrix $E^{(r)}(t)$, we can use (18) as

$$\begin{aligned} &E'(t)=E(t)D^{T}+B(t),\\ &(E(t))''=E'(t)D^{T}+B'(t)=\left(E(t)D^{T}+B(t)\right)D^{T}+B'(t), \end{aligned}$$

then by induction we find

$$E^{(r)}(t)=E(t)(D^{T})^{r}+\sum_{i=0}^{r-1}B^{(i)}(t)\,(D^{T})^{r-i-1}\,,\quad r\geq 1 \tag{20}$$

where

$$B^{(i)}(t)=\left[0\;0\cdots 0\;\tfrac{-N}{4}E_{N+1}^{(i)}(t)\right].$$

If N = 4 the form of D and B(t) is

$$D=\begin{bmatrix}0&0&0&0&0\\ \frac{1}{4}&0&\frac{-1}{4}&0&0\\ 0&\frac{1}{2}&0&\frac{-1}{2}&0\\ 0&0&\frac{3}{4}&0&\frac{-3}{4}\\ 0&0&0&1&0\end{bmatrix},\qquad B(t)=[0\;\;0\;\;...\;\;0\;\;-E_{5}(t)].$$

4 Fundamental matrix relation based on collocation points

In this section we will provide the fundamental matrix relation based on collocation points of the solution of (3), with mixed conditions (5), by the representation of the derivative of EC functions given in equation (20).

Now, we define the collocation points [20] and [21], so that $-\infty<t_s<\infty$, as

$$t_s=Ln\left[\frac{1+\cos\left(\frac{s\pi}{N}\right)}{1-\cos\left(\frac{s\pi}{N}\right)}\right],\qquad s=1,\,...,\,N-1 \tag{21}$$

and at the boundaries i.e. $(s=0,\;\; s=N)$ $t_0\to\infty$, $t_N\to-\infty$, since the EC functions are convergent at both boundaries $\pm\infty$, i.e their values are ±1. Then, the appearance of infinity in the collocation points does not cause a problem in the method or cause any divergence.

Now, assume that the solutions $y_i(t)$ of (3) can be expressed in the form (9), which is a truncated Chebyshev series in terms of EC functions. Then $y_i(t)$ and its derivatives $y_i^{(j)}(t)$ can be written in the matrix form as

$$y_i(t)=E(t)A_i, \tag{22}$$

and

$$y_i^{(j)}(t)=E^{(j)}(t)A_i,\;\; i=1,2,\ldots,k,\, j=0,1,\ldots,m \tag{23}$$

where

$$A_i=\left[a_{i0}\;a_{i1}\;\cdots\;a_{iN}\right]^{T}.$$

Then, we substitute the collocation points (21) into (4) to obtain the system

$$\sum_{i=0}^{m} \tilde{P}_i Y^{(i)} = F, \tag{24}$$

where

$$\tilde{\mathbf{P}}_i = \begin{bmatrix} \mathbf{P}_i(\mathrm{t}_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_i(\mathrm{t}_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_i(\mathrm{t}_N) \end{bmatrix}, \mathbf{Y}^{(i)} = \begin{bmatrix} \mathbf{y}^{(i)}(\mathrm{t}_0) \\ \mathbf{y}^{(i)}(\mathrm{t}_1) \\ \vdots \\ \mathbf{y}^{(i)}(\mathrm{t}_N) \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \mathbf{f}(\mathrm{t}_0) \\ \mathbf{f}(\mathrm{t}_1) \\ \vdots \\ \mathbf{f}(\mathrm{t}_N) \end{bmatrix},$$

Now, substituting relation (23) into (20), we get

$$y_i^{(j)}(t) = \left(E(t)(D^T)^k + \sum_{i=0}^{k-1} B^{(i)}(t)\ (D^T)^{k-i-1} \right) A_i. \quad j = 0, 1, \ldots, m \tag{25}$$

Hence, the matrix $\mathbf{y}^{(i)}(t)$ defined as a column matrix that is formed of i^{th} derivatives of unknown functions, can be expressed by

$$\mathbf{y}^{(i)}(t) = \left(\mathbf{E}(t)(\mathbf{D}^T)^k + \sum_{i=0}^{k-1} \mathbf{B}^{(i)}(t)\ (\mathbf{D}^T)^{k-i-1} \right) \mathbf{A}, \tag{26}$$

where

$$\mathbf{E}(t) = \begin{bmatrix} E(t) & 0 & \cdots & 0 \\ 0 & E(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E(t) \end{bmatrix}_{k\times k}, \mathbf{D}^T = \begin{bmatrix} D^T & 0 & \cdots & 0 \\ 0 & D^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^T \end{bmatrix}_{k\times k},$$

$$\mathbf{A} = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_k \end{bmatrix}_{k\times 1}, \quad \mathbf{B}^{(i)}(t) = \begin{bmatrix} B^{(i)}(t) & 0 & \cdots & 0 \\ 0 & B^{(i)}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{(i)}(t) \end{bmatrix}_{k\times k}.$$

Putting the collocation points t_s, in relation (26) we have the matrix system

$$\mathbf{y}^{(i)}(t_s) = \left(\mathbf{E}(t_s)(\mathbf{D}^T)^k + \sum_{i=0}^{k-1} \mathbf{B}^{(i)}(t_s)\ (\mathbf{D}^T)^{k-i-1} \right) \mathbf{A}, \tag{27}$$

this system can be written as

$$\mathbf{Y}^{(\mathrm{i})} = \left(\tilde{\mathbf{E}}\,(\mathbf{D}^{\mathrm{T}})^{\mathrm{k}} + \sum_{\mathrm{i}=0}^{\mathrm{k}-1} \tilde{\mathbf{B}}^{(\mathrm{i})}\ (\mathbf{D}^{\mathrm{T}})^{\mathrm{k}-\mathrm{i}-1} \right) \mathbf{A},$$

where

$$\tilde{\mathbf{E}} = \begin{bmatrix} \mathbf{E}(\mathrm{t}_0) \\ \mathbf{E}(\mathrm{t}_1) \\ \vdots \\ \mathbf{E}(\mathrm{t}_N) \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}(\mathrm{t}_0) \\ \mathbf{B}(\mathrm{t}_1) \\ \vdots \\ \mathbf{B}(\mathrm{t}_N) \end{bmatrix}.$$

with the aid of this equation, expression (4) becomes

$$\sum_{i=0}^{m}\tilde{\mathbf{P}}_{\mathrm{i}}\left(\tilde{\mathbf{E}}\left(\mathbf{D}^{\mathrm{T}}\right)^{\mathrm{k}}+\sum_{\mathrm{i}=0}^{\mathrm{k}-1}\tilde{\mathbf{B}}^{(\mathrm{i})}\left(\mathbf{D}^{\mathrm{T}}\right)^{\mathrm{k-i-1}}\right)\mathbf{A}=\mathbf{F}. \tag{28}$$

Similarly, we form the matrix representations of the mixed conditions.

Substituting the matrices $\mathbf{y}^{(i)}(a),\mathbf{y}^{(i)}(b)$ and $\mathbf{y}^{(i)}(c)$ which depend on the exponential Chebyshev coefficients matrix $\mathbf{A}$ into (5) and simplifying the result we obtain

$$\begin{array}{l}\sum_{i=0}^{m-1}a_i\left(\mathbf{E}(a)(\mathbf{D}^T)^k+\sum_{i=0}^{k-1}\mathbf{B}^{(i)}(a)\,(\mathbf{D}^T)^{k-i-1}\right)\mathbf{A}\\ +b_i\left(\mathbf{E}(b)(\mathbf{D}^T)^k+\sum_{i=0}^{k-1}\mathbf{B}^{(i)}(b)\,(\mathbf{D}^T)^{k-i-1}\right)\mathbf{A}\\ \quad +\mathrm{c_i}\left(\mathbf{E}(c)(\mathbf{D}^T)^k+\sum_{i=0}^{k-1}\mathbf{B}^{(i)}(c)\,(\mathbf{D}^T)^{k-i-1}\right)\mathbf{A}=\lambda\end{array}. \tag{29}$$

5 The collocation method

The fundamental matrix equation (28) for (4) corresponds to system of $k(N+1)$ algebraic equations for $k(N+1)$ unknown coefficients $a_{i0},a_{i1},\ldots,a_{iN}$, $i=1,\ 2,\ldots,k$.

Now, we can write equation (28) in short form as:

$$\boldsymbol{WA}\text{=}\boldsymbol{F}\quad or\quad [\boldsymbol{W};\,\boldsymbol{F}], \tag{30}$$

We can obtain the matrix form for the conditions (5), by means of equations (29) in a short form as

$$UA=[\lambda_i], \tag{31}$$

And, the definition of $\boldsymbol{W}$ and U is obtained as:

$$\mathrm{W}=[\mathrm{w_{pq}}]=\sum_{i=0}^{m}\tilde{\mathbf{P}}_{\mathrm{i}}\left(\tilde{\mathbf{E}}\left(\mathbf{D}^{\mathrm{T}}\right)^{\mathrm{k}}+\sum_{\mathrm{i}=0}^{\mathrm{k}-1}\tilde{\mathbf{B}}^{(\mathrm{i})}\left(\mathbf{D}^{\mathrm{T}}\right)^{\mathrm{k-i-1}}\right),p,q=1,2,\ldots,\ k(N+1)$$

similarly, the elements of U formed by equation (29).

Now, the solution of (3), under the conditions (5) can then be obtained by replacing the rows of matrices (31) by some rows of the matrix (30), we get the required augmented matrix

$$\tilde{\mathbf{W}}\mathbf{A}=\tilde{\mathbf{F}}\qquad or\qquad \left[\tilde{\mathbf{W}};\tilde{\mathbf{F}}\right]. \tag{32}$$

Hence, EC coefficients can be simply computed and the approximate solution of system (3) under the mixed conditions (5) can be obtained.

6 Numerical test examples

In this section, six numerical test examples are given to illustrate the accuracy and effectiveness of our method. All examples are performed on the computer using software programs written in MATHEMATICA 7.0.

Example 1. First, we consider the simple system of constant coefficients

$$\begin{array}{l} x' - y' + y = \tanh\left(\frac{t}{2}\right) \\ x' + y' + x = \frac{2+\sinh(t)}{1+\cosh(t)}, \qquad -\infty < t < \infty \end{array} \tag{33}$$

with the conditions $x(0) = 0$, *and* $y(t) = 1$ at $t \to \infty$, where the exact solutions $x(t) = y(t) = \tanh\left(\frac{t}{2}\right)$.For this example we have,

$$\begin{array}{l} k = 2,\ m = 1,\ f_1(t) = \tanh\left(\frac{t}{2}\right),\ f_2(t) = \frac{2+\sinh(t)}{1+\cosh(t)},\ p^0_{11}(t) = 0, \\ p^0_{12}(t) = 1,\ p^0_{21}(t) = 1,\ p^0_{22}(t) = 0,\ p^1_{11}(t) = 1,\ p^1_{12}(t) = -1, \\ p^1_{21}(t) = 1,\ p^1_{22}(t) = 1. \end{array}$$

Then, for $N = 2$, the collocation points are $t_0 \to \infty$, $\quad t_1 = 0$, $\quad t_2 \to -\infty$, and the fundamental matrix of the problem using our proposed method is

$$\{\tilde{\mathbf{P}}_0 \tilde{\mathbf{E}}\,(\mathbf{D}^{\mathrm{T}}) + \tilde{\mathbf{P}}_1 \tilde{\mathbf{E}}\,(\mathbf{D}^{\mathrm{T}} + \tilde{\mathbf{B}})\}\mathbf{A} = \mathbf{F},$$

where $\tilde{\mathbf{P}}_0$, $\tilde{\mathbf{P}}_1$, $\tilde{\mathbf{E}}$, $\mathbf{D}$, $\tilde{\mathbf{B}}$ are matrices of order (6×6) given as:

$$\tilde{\mathbf{P}}_0 = \begin{bmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \end{bmatrix}, \quad \tilde{\mathbf{E}} = \begin{bmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 1&0&-1&0&0&0 \\ 0&0&0&1&0&-1 \\ 1&-1&1&0&0&0 \\ 0&0&0&1&-1&1 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 0&0&0&0&0&0 \\ \frac{1}{4}&0&-\frac{1}{4}&0&0&0 \\ 0&\frac{1}{2}&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&\frac{1}{4}&0&-\frac{1}{4} \\ 0&0&0&0&\frac{1}{2}&0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0&0&-\frac{1}{2}&0&0&0 \\ 0&0&0&0&0&-\frac{1}{2} \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&\frac{1}{2}&0&0&0 \\ 0&0&0&0&0&\frac{1}{2} \end{bmatrix},$$

$$\tilde{\mathbf{P}}_1 = \begin{bmatrix} 1&-1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&1&-1&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&-1 \\ 0&0&0&0&1&1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1&1&0&1&-1&-1 \end{bmatrix}^T,$$

the augmented matrix for the given conditions with $N = 2$ is

$$\begin{bmatrix} 1\ 0\ -1\ 0\ 0\ 0\ ;\ 0 \end{bmatrix},$$

for the first condition $x(0) = 0$, and for the other condition $y = 1$ at $t \to \infty$ is

$$\begin{bmatrix} 0\ 0\ 0\ 1\ 1\ 1\ ;\ 1 \end{bmatrix}.$$

Table 1: Numeric results of approximate and exact solution in example 2.

t	Exact solution	$u(x), N = 8$	$u(x), N = 16$	$u(x), N = 24$
0.0	-1.000000	-1.00116	-1.000000	-1.000000
0.5	-0.886819	-0.887187	-0.886818	-0.886819
1.0	-0.648054	-0.645356	-0.648055	-0.648054
1.6	-0.387978	-0.385007	-0.387977	-0.387978
1.8	-0.321805	-0.319594	-0.321805	-0.321805
2.0	-0.265802	-0.264341	-0.265803	-0.265802
2.5	-0.163071	-0.162688	-0.163071	-0.163071
3.0	-0.0993279	-0.0991032	-0.0993274	-0.0993279

Table 2: Error norms of example 2.

	L_2	L_∞ (max error)
$N = 8$	6.6718×10^{-5}	0.00353909
$N = 16$	7.40886×10^{-12}	1.04052×10^{-6}
$N = 24$	4.18177×10^{-18}	7.72064×10^{-10}

After the augmented matrices of the system and conditions are computed, we obtain the coefficients solution as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

Therefore, we find the solutions as

$$x(t) = 0*E_0 + 1*E_1 + 0*E_2 \quad \text{and} \quad y(t) = 0*E_0 + 1*E_1 + 0*E_2$$

or in the form $x(t) = y(t) = \frac{e^t-1}{e^t+1} = \tanh\left(\frac{t}{2}\right)$, which represent the exact solution of this problem.

Example 2. It is clear that if $k = 1$ in (3) the proposed system reduced to be high-order ordinary differential equations and that will be special case of our method. Boyd in his paper [22] and his book [10] list some examples are naturally defined in the infinite interval we apply our method to the transformed associated Legender equation [22] in the following form

$$u'' + 2\sec h^2(x)\,u = -\sec h(x) \tag{34}$$

equation (34) has exact solution $P_1^1[\tanh(x)]$ where $P_n^m[x]$ is the associated Legender polynomials and the transformation which produced equation (34) made by $x \to \tanh(x)$.

Where the subjected conditions of (34) are $u(x) = 0$ where $|x| \to \infty$. Some numeric results found in Table.1 of approximate and exact solution with different N and Table.2 represent the error norms L_2, L_∞, where

$$L_2 = \sqrt{h\sum_{i=0}^{I}\left(y^i_{Exact} - y^i_{Approximat}\right)^2}, \quad L_\infty = Max\left|y^i_{Exact} - y^i_{Approximat}\right|.$$

where the Figure.2 obtained the comparison of absolute errors at N=8,16 and 24 and Figure.1 comparing the results of exact and approximate solutions at N=8,16 and 24 where $x \in [-5, 5]$.

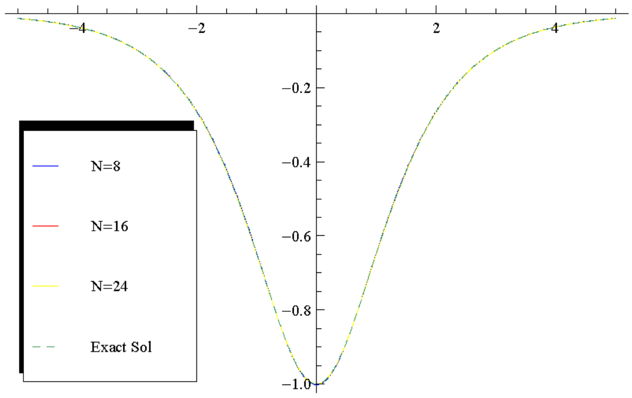

Fig. 1: Exact and approximate solutions at different N.

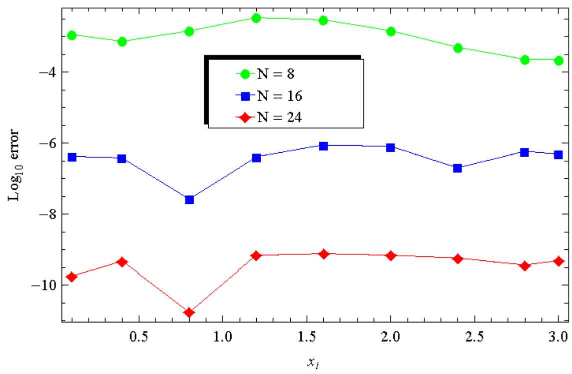

Fig. 2: Comparison of absolute errors.

Example 3. Consider now the following linear system of first order with variable coefficients and the subjected condition tends to infinity as

$$\begin{aligned} &x' + y' + x + y = \tfrac{1+3e^t}{(1+e^t)^3} \\ &x' + y' + \tfrac{1}{1+e^t}x + \tfrac{1}{1+e^t}y = \tfrac{1-2e^{2t}}{(1+e^t)^3}, \qquad -\infty < t < \infty \end{aligned} \tag{35}$$

with the conditions $x = 0$, $y = 0$ at $t \to \infty$,where the exact solutions are $x(t) = \frac{1}{1+e^t}$ and $y(t) = \frac{e^t}{(1+e^t)^2}$. Then, for N =4 and k=2, the fundamental matrix is

$$\{\tilde{\mathbf{P}}_0\tilde{\mathbf{E}}(\mathbf{D}^{\mathrm{T}}) + \tilde{\mathbf{P}}_1\tilde{\mathbf{E}}(\mathbf{D}^{\mathrm{T}} + \tilde{\mathbf{B}})\}\mathbf{A} = \mathbf{F},$$

and the formed system become (10×10), finally we obtain the coefficients as

$$\mathbf{A} = \left[\tfrac{1}{2} \ -\tfrac{1}{2} \ 0 \ 0 \ 0 \ \tfrac{1}{8} \ 0 \ -\tfrac{1}{8} \quad 0 \ 0\right]^T.$$

Therefore, we find the solution $x(t) = \frac{1}{2}E_0 - \frac{1}{2}E_1$, and $y(t) = \frac{1}{8}E_0 - \frac{1}{8}E_2$, or in the form

$$x(t) = \frac{1}{2} - \frac{1}{2}\left(\frac{e^t - 1}{e^t + 1}\right) = \frac{1}{e^t + 1},$$

and

$$y(t) = \frac{1}{8} - \frac{1}{8}\left[2\left(\frac{e^t - 1}{e^t + 1}\right)^2 - 1\right] = \frac{e^t}{(1 + e^t)^2},$$

which is the exact solution of example 3.

Example 4. Consider the system of the form

$$\begin{aligned} &x' + x + y = -\operatorname{sec}h(t)(-2 + \tanh(t)), \\ &x' - y' + x - 2y = -\operatorname{sec}h(t), \qquad -\infty < t < \infty \end{aligned} \tag{36}$$

and the given conditions found as $x(t) = 0$, $y(t) = 0$ at $t \to \infty$, and exact solutions is

$$x(t) = y(t) = \operatorname{sec}h(t).$$

Table 3: Error norms of example 2.

	L_2 for $x(t)$	L_2 for $y(t)$	L_∞ for $x(t)$	L_∞ for $y(t)$
$N=6$	0.00678295	0.0139659	0.0404235	0.0871772
$N=10$	6.22781×10^{-5}	8.60452×10^{-6}	0.00451133	0.00283974
$N=16$	9.56755×10^{-10}	6.55061×10^{-10}	2.3491×10^{-5}	2.184×10^{-5}

Table 4: Numeric results approximate and exact solution in example 4.

t	Exact	$x(t), N=10$	$y(t), N=10$	$x(t), N=16$	$y(t), N=16$
-3.0	0.0993279	0.0948166	0.0964882	0.0993514	0.0993061
-2.5	0.163071	0.158355	0.163204	0.163048	0.163061
-2.0	0.265802	0.26457	0.26707	0.265806	0.265800
-1.8	0.321805	0.322034	0.322959	0.321813	0.321802
-1.6	0.387978	0.389091	0.388816	0.387980	0.387975
-1.0	0.648054	0.648187	0.648181	0.648054	0.648054
-0.5	0.886819	0.886735	0.886999	0.886817	0.886820
0.0	1.000000	1.000000	0.999783	1.000000	0.999999
0.5	0.886819	0.886968	0.887074	0.886819	0.886820
1.0	0.648054	0.648059	0.647989	0.648054	0.648054
1.6	0.387978	0.387927	0.387924	0.387978	0.387978
1.8	0.321805	0.321803	0.321835	0.321805	0.321804
2.0	0.265802	0.265833	0.265873	0.265802	0.265802
2.5	0.163071	0.163100	0.163102	0.163071	0.163071
3.0	0.0993279	0.0993268	0.0993085	0.0993279	0.0993278

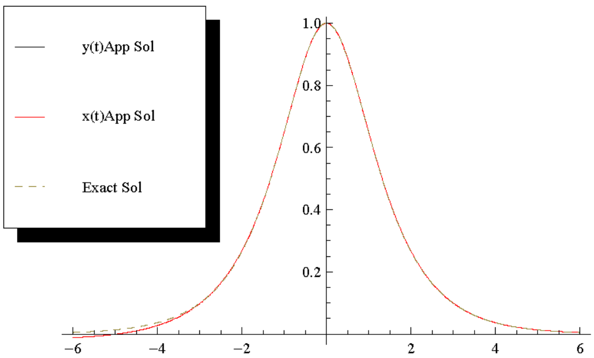

Fig. 3: Exact and approximate, $N=10$.

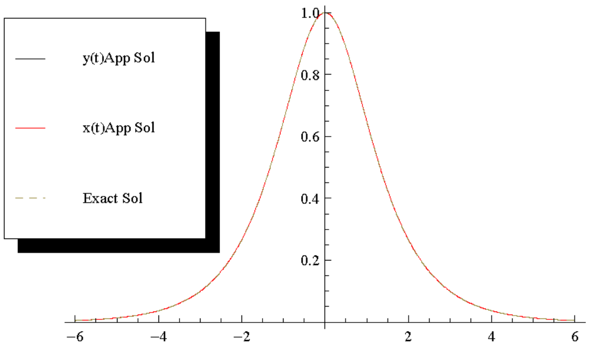

Fig. 4: Exact and approximate, $N=16$.

The numerical solutions obtained using the proposed method for N = 6, N = 10 and N = 16 are compared with the exact solution in Table.4 where Table.3 shows the error norms L_2, L_∞, to estimate the errors at the different N. Figure.3 show the exact and approximate solutions at different N , $t \in [-6, 6]$. In Figure.4 the comparison of the absolute errors of $x(t)$, $y(t)$for the three cases N=6, 10, and 16 are given, and show that the greater N give good accuracy.

Example 5. Consider the second order system of two equations as

$$\begin{aligned} x''+y''-x &= -\tfrac{1+3e^t}{(1+e^t)^3} \\ y''+2x' &= -\tfrac{1}{1+\cosh(t)} \qquad -\infty < t < \infty \end{aligned} \tag{37}$$

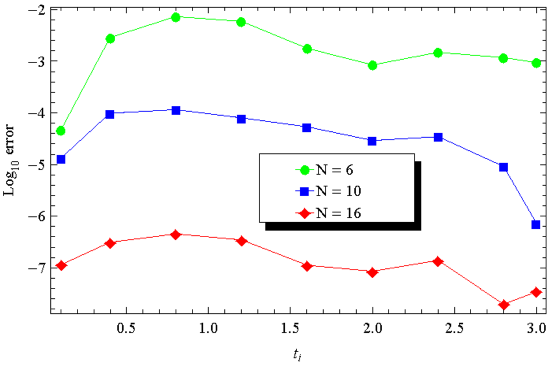

Fig. 5: Comparison of absolute errors of $x(t)$.

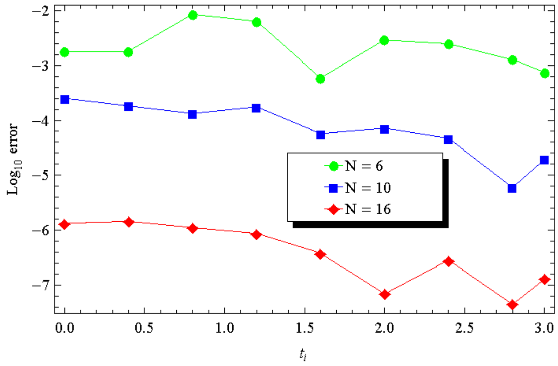

Fig. 6: Comparison of absolute errors of $y(t)$.

with conditions $x(0)=\frac{1}{2}$, $y(0)=1$, $x=0$ at $t\to\infty$, and $y=1$ at $t\to\infty$,

where the exact solutions taken as $x(t)=\frac{1}{1+e^t}$ and $y(t)=1$.

In this example we have, $k=2,\ \ m=2$

The fundamental matrix of this problem using our proposed method is

$$\left\{\tilde{\mathbf{P}}_0\tilde{\mathbf{E}}\left(\mathbf{D}^{\mathrm{T}}\right)+\tilde{\mathbf{P}}_1\left[\tilde{\mathbf{E}}\left(\mathbf{D}^{\mathrm{T}}\right)+\tilde{\mathbf{B}}\right]+\tilde{\mathbf{P}}_2\left[\tilde{\mathbf{E}}\left(\mathbf{D}^{\mathrm{T}}\right)^2+\tilde{\mathbf{B}}\mathrm{D}^{\mathrm{T}}+\tilde{\mathbf{B}}'\right]\right\}\mathbf{A}=\mathbf{F}.$$

For N=4, we have the coefficient solution as

$$\mathbf{A}=\left[\tfrac{1}{2}\ -\tfrac{1}{2}\ 0\ 0\ 0\ 1\ 0\ 0\ \ 0\ 0\right]^T,$$

Therefore, we find the solution

$$x(t)=\frac{1}{2}E_0(t)-\frac{1}{2}E_1(t)\qquad and\quad y(t)=E_0(t),$$

or in the form

$$x(t)=\frac{1}{2}-\frac{1}{2}\left(\frac{e^t-1}{e^t+1}\right)=\frac{1}{e^t+1},$$

and

$$y(t)=1,$$

which is the exact solution of the problem.

Example 6. Consider the linear system of three equations

$$\begin{aligned}
&y_1'-y_2'-y_3'+y_3=\tfrac{e^t(-2+e^t)}{(1+e^t)^4}\\
&y_1'+y_2'-y_1+2y_3=-\tfrac{1+3e^{2t}}{(1+e^t)^3}\\
&3y_2'+y_3'-y_1-y_2=-\tfrac{1+7e^{2t}+5e^{3t}}{(1+e^t)^4}\qquad -\infty<t<\infty
\end{aligned}\tag{38}$$

with the conditions $y_1(t)=y_2(t)=y_3(t)=0$ at $t\to\infty$, where the exact solutions are $y_1(t)=\frac{1}{1+e^t}$, $y_2(t)=\frac{e^t}{(1+e^t)^2}$, $y_3(t)=\frac{e^t}{(1+e^t)^3}$.

In this example we have, $k = 3$, $m = 1$, for N=4, we have the solution

$$\mathbf{A} = [0.5\ -0.5\ 0\ 0\ 0\ 0.125\ 0\ -0.125\ 0\ 0\ 0.0625\ -0.03125\ -0.0625\ 0.03125\ 0\ ]^T.$$

Therefore, we find the solution

$$y_1(t) = 0.5E_0 - 0.5E_1$$

$$y_2(t) = 0.125E_0 - 0.125E_2$$

$$y_3(t) = 0.0625E_0 - 0.03125E_1 - 0.0625E_2 + 0.03125E_3$$

After simplifying we get the exact solution of the problem (37).

7 Conclusions

Systems of high-order linear differential equations are usually difficult to solve analytically especially with variable coefficients under mixed conditions. In many cases, obtaining the approximate solutions is necessary. For this reason, the exponential Chebyshev collocation method can be proposed to obtain approximate solution of high-order linear systems in infinite domain. The definition of the EC functions allow us solve systems of high-order differential equations in unbounded domains. The systems and the subjected conditions were transformed to matrix equation with unknown EC coefficients. On the other hand, the EC functions approach deals directly with infinite boundaries without divergence. This variant for our method gave us freedom to deal with the systems of differential equations with boundary conditions tends to infinity. Illustrative examples are used to demonstrate the applicability of the proposed technique. Future work:
Recently, our research group examine a new operational matrix of derivatives of EC functions for solving ODEs in unbounded domains [23] that may be applied for systems. In addition, we introduced a form of exponential Chebyshev for the second kind (reported on line [24], [25]) for ordinary and partial differential equations that also can solve systems which is still under revision but some modifications are required.

References

[1] M. Tatari and M. Denghan, *Improvement of He's variational iteration method for solving systems of differential equations*, Computational Mathematics and Applications, 58, 2160-2166, 2009.

[2] A. Hassan, *application to differential transformation method for solving systems of differential equations*, Applied Mathematical Modeling, 32, 2552-2559, 2008.

[3] H. Jafari and V. Daftardar-Gejji, *Revised Adomian decomposition method for solving systems of ordinary and fractional differential equations*, Applied Mathematics and Computations, 181, 598-608, 2006.

[4] M. Thangmoon and S. Pusjuso, *The numerical solutions of differential transform method and Laplace transform method for a system of differential equations*, Nonlinear Analysis: Hybrid systems, 4, 425-431, 2010.

[5] M. Zurigat, S. Momani and Z. Odibat, *The homotopy analysis method for handling systems of fractional differential equations*, Applied Mathematical Modeling, 34, 24-35, 2010.

[6] A. Akyüz and M. Sezer, *Chebyshev polynomial solutions of systems of high-order linear differential equations with variable coefficients*, Applied Mathematics and Computation,144 , 237-247, 2003.

[7] E. GÃ¶kmen and M. Sezer, *Taylor collocation method for systems of high-order linear differential–difference equations with variable coefficients*, Ain Shams Engineering Journal, 4, 117-125, 2013.

[8] J. C. Mason and D. C. Handscomb, Chebyshev polynomials, CRC Press, Boca Raton, 2003.

[9] J. P. Boyd, *Orthogonal rational functions on a semi-infinite interval*, Journal of Computational Physics, 70, 63-88, 1987.

[10] J. P. Boyd, Chebyshev and Fourier spectral methods, Second Edition, DOVER Publications, Mineola, 2000.

[11] K. Parand and M. Razzaghi, *Rational Chebyshev tau method for solving higher-order ordinary differential equations*, International Journal of Computer Mathematics, 81, 73-80, 2004.

[12] K. Parand and M. Razzaghi, *Rational Chebyshev tau method for solving Volterra population model*, Applied mathematics and computation 149, 893-900, 2004.

[13] M. Sezer, M. Gulsu and B. Tanay, *Rational Chebyshev collocation method for solving higher-order linear ordinary differential equations*, Wiley Online Library, DOI 10.1002/num.20573, 2010.

[14] S. Yalcinbas, N. Ozsoy and M. Sezer, *Approximate solution of higher-order linear differential equations by means of a new rational Chebyshev collocation method*, Mathematical and Computational Applications, 15: 1, 45-56, 2010.

[15] M. A. Ramadan, K. R. Raslan and M. A. Nassar, *An approximate analytical solution of higher-order linear differential equations with variable coefficients using improved rational Chebyshev collocation method*, Applied and Computational Mathematics, 3: 6, 315-322, 2014.

[16] M. A. Ramadan, K. R. Raslan and M. A. Nassar, *Numerical solution of system of higher order linear ordinary differential equations with variable coefficients using two proposed schemes for rational Chebyshev functions*, Global Journal of Mathematics, 3: 2, 322-327, 2015.

[17] M. A. Ramadan, K. R. Raslan and M. A. Nassar, *An approximate solution of systems of high-order linear differential equations with variable coefficients by means of a rational Chebyshev collocation method*, Electronic Journal of Mathematical Analysis and Applications, 4: 1, 86-98, 2016.

[18] M. A. Ramadan, K. R. Raslan, A. R. Hadhoud , and M. A. Nassar, *Rational Chebyshev functions with new collocation points in semi-infinite domains for solving higher-order linear ordinary differential equations*, Journal of advances in mathematics, 7, 5403-5410, 2015.

[19] M. A. Ramadan, K. R. Raslan, A. R. Hadhoud and M. A. Nassar, *Numerical solution of high-order linear integro-differential equations with variable coefficients using two proposed schemes for rational Chebyshev functions*, New trends in mathematical sciences, 4: 3, 22-35, 2016.

[20] A. Betul Koc and A. Kurnaz, *A new kind of double Chebyshev polynomial approximation on unbounded domains*, Boundary Value Problems, 2013: 1, 1687-2770, 2013.

[21] M. A. Ramadan, K. R. Raslan T. S. El Danaf and M. A. Abd El salam, *On the exponential Chebyshev approximation in unbounded domains: A comparison study for solving high-order ordinary differential equations*, International Journal of Pure and Applied Mathematics, 105: 3, 399-413, 2015.

[22] J. P. Boyd, *Spectral methods using rational basis functions on an infinite interval*, Journal of Computational Physics 69:1, 112-142, 1987.

[23] M. A. Ramadan, K. R. Raslan T. S. El Danaf and M. A. Abd El salam, *A new exponential Chebyshev operational matrix of derivatives for solving high-order ordinary differential equations in unbounded domains*, Journal of Modern Methods in Numerical Mathematics 7:1 (2016), 19-30.

[24] M. A. Ramadan, K. R. Raslan T. S. El Danaf and M. A. Abd El salam, *An exponential Chebyshev second kind approximation for solving high-order ordinary differential equations in unbounded domains, with application to Dawson's integral*, Journal of the Egyptian Mathematical Seocity, in press (2016).

[25] M. A. Ramadan, K. R. Raslan T. S. El Danaf and M. A. Abd El salam, *Solving high-order partial differential equations in unbounded domains by means of double exponential second kind Chebyshev approximation*, Computational Methods for Differential Equations, in press (2016).