NTMSCI **4**, No. 3, 296-306 (2016) BISKA 296

New Trends in Mathematical Sciences

http://dx.doi.org/10.20852/ntmsci.2016320385

The Generalized difference of $\int \chi^{2I}$ of fuzzy real numbers over $p-$ metric spaces defined by Musielak Orlicz function

Deepmala Rai[1], *N. Subramanian*[2] *and Vishnu Narayan Mishra*[3]

[1] SQC and OR Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700 108, West Bengal, India
[2] Department of Mathematics, SASTRA University, Thanjavur-613 401, India
[3] Applied Mathematics and Humanities Department,Sardar Vallabhbhai National Institute of Technology,Ichchhanth Mahadev Dumas Road, Surat 395 007, India

Received: 6 February 2016, Accepted: 28 April 2016
Published online: 13 August 2016.

Abstract: In this article we introduce the sequence spaces

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \quad \text{and} \quad \left[\Lambda^{2q}_{f\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)},$$

associated with the integrated sequence space defined by Musielak. We study some basic topological and algebraic properties of these spaces. We also investigate some inclusion relations related to these spaces.

Keywords: Analytic sequence, double sequences, χ^2 space, difference sequence space, Musielak - Orlicz function, $p-$ metric space, Ideal, ideal convergent, fuzzy number; multiplier space.

1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on, they were investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy [6], Turkmenoglu [7], and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \; for\, some\, l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

* Corresponding author e-mail: dmrai23@gmail.com

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m,n \in \mathbb{N}$; $\mathscr{M}_u(t), \mathscr{C}_p(t), \mathscr{C}_{0p}(t), \mathscr{L}_u(t), \mathscr{C}_{bp}(t)$ and $\mathscr{C}_{0bp}(t)$ reduce to the sets $\mathscr{M}_u, \mathscr{C}_p, \mathscr{C}_{0p}, \mathscr{L}_u, \mathscr{C}_{bp}$ and $\mathscr{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that $\mathscr{M}_u(t)$ and $\mathscr{C}_p(t), \mathscr{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathscr{M}_u(t)$ and $\mathscr{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [10] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [11] and Tripathy have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [12] have defined the spaces $\mathscr{BS}, \mathscr{BS}(t), \mathscr{CS}_p, \mathscr{CS}_{bp}, \mathscr{CS}_r$ and $\mathscr{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathscr{M}_u, \mathscr{M}_u(t), \mathscr{C}_p, \mathscr{C}_{bp}, \mathscr{C}_r$ and $\mathscr{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathscr{BS}, \mathscr{BV}, \mathscr{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathscr{CS}_{bp}$ and $\mathscr{CS}_r$ of double series. Basar and Sever [13] have introduced the Banach space $\mathscr{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathscr{L}_q$. Quite recently Subramanian and Misra [14] have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [15] as an extension of the definition of strongly Cesàro summable sequences. Connor [16] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [17] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [18]-[19], and [20] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,n \in \mathbb{N})$.

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m,n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn}) (m,n \in \mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary Orlicz functions. Then, we have:
(i) For all $u,y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality)[See[21]] \tag{2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{4}$$

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \Sigma_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \Sigma_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of Orlicz function is called a Musielak-Orlicz function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - (f_{mn})(u) : u \geq 0\}, m,n = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak Orlicz function f, the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m,n \to \infty \right\},$$

where M_f is a convex modular defined by

$$M_f(x) = \Sigma_{m=1}^{\infty} \Sigma_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn}\left\{ inf\left(\Sigma_{m=1}^{\infty}\Sigma_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leq 1 \right\}$$

If X is a sequence space, we give the following definitions:

(i) X' = the continuous dual of X;

(ii) $X^{\alpha} = \left\{ a = (a_{mn}) : \Sigma_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty,\ for\, each\, x \in X \right\};$

(iii) $X^{\beta} = \left\{ a = (a_{mn}) : \Sigma_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent,\ foreach\, x \in X \right\};$

(iv) $X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn\geq 1}\left|\Sigma_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, foreachx \in X \right\};$

(v) $let\, X\, be\, an\, FK - space \supset \phi; then\, X^f = \left\{ f(\Im_{mn}) : f \in X' \right\};$

(vi) $X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn}|a_{mn}x_{mn}|^{1/m+n} < \infty,\ foreach\, x \in X \right\};$

$X^{\alpha}.X^{\beta},X^{\gamma}$ are called $\alpha-(or \quad K\ddot{o}the-Toeplitz)$ dual of $X,\beta-(or \quad generalized-K\ddot{o}the-Toeplitz)$ dual of $X,\gamma-dual$ of $X,\delta-dual$ of X respectively. X^{α} is defined by Gupta and Kamptan . It is clear that $X^{\alpha}\subset X^{\beta}$ and $X^{\alpha}\subset X^{\gamma}$, but $X^{\beta}\subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\}$$

for $Z=c,c_0$ and ℓ_{∞}, where $\Delta x_k=x_k-x_{k+1}$ for all $k\in\mathbb{N}$.
Here c,c_0 and ℓ_{∞} denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1\leq p\leq\infty$ by Başar and Altay and in the case $0<p<1$ by Altay and Başar. The spaces $c(\Delta),c_0(\Delta),\ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\|=|x_1|+sup_{k\geq 1}|\Delta x_k| \text{ and } \|x\|_{bv_p}=\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p},(1\leq p<\infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta)=\left\{x=(x_{mn})\in w^2:(\Delta x_{mn})\in Z\right\}$$

where $Z=\Lambda^2,\chi^2$ and $\Delta x_{mn}=(x_{mn}-x_{mn+1})-(x_{m+1n}-x_{m+1n+1})=x_{mn}-x_{mn+1}-x_{m+1n}+x_{m+1n+1}$ for all $m,n\in\mathbb{N}$.

2 Definition and preliminaries

Let $n\in\mathbb{N}$ and X be a real vector space of dimension w, where $n\leq m$. A real valued function $d_p(x_1,\ldots,x_n)=\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p=0$ if and and only if $d_1(x_1,0),\ldots,d_n(x_n,0)$ are linearly dependent,
(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1,0),\ldots,d_n(x_n,0))\|_p=|\alpha|\,\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p,\alpha\in\mathbb{R}$
(iv) $d_p((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n))=(d_X(x_1,x_2,\cdots x_n)^p+d_Y(y_1,y_2,\cdots y_n)^p)^{1/p}\ for 1\leq p<\infty$; (or)
(v) $d((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)):=\sup\{d_X(x_1,x_2,\cdots x_n),d_Y(y_1,y_2,\cdots y_n)\}$,
for $x_1,x_2,\cdots x_n\in X,y_1,y_2,\cdots y_n\in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X=\mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E=sup\left(|det(d_{mn}(x_{mn},0))|\right)=sup\left(\begin{vmatrix} d_{11}(x_{11,},0) & d_{12}(x_{12},0) & \ldots d_{1n}(x_{1n},0)\\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots d_{2n}(x_{1n},0)\\ . & & \\ . & & \\ . & & \\ d_{n1}(x_{n1},0) & d_{n2},0(x_{n2},0) & \ldots d_{nn}(x_{nn},0)\end{vmatrix}\right)$$

where $x_i=(x_{i1},\cdots x_{in})\in\mathbb{R}^n$ for each $i=1,2,\cdots n$.

If every Cauchy sequence in X converges to some $L\in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Definition 1. *Let X be a linear metric space. A function $\rho : X \to \mathbb{R}$ is called paranorm, if*
(1) $\rho(x) \geq 0$, *for all* $x \in X$;
(2) $\rho(-x) = \rho(x)$, *for all* $x \in X$;
(3) $\rho(x+y) \leq \rho(x) + \rho(y)$, *for all* $x, y \in X$;
(4) *If* (σ_{mn}) *is a sequence of scalars with* $\sigma_{mn} \to \sigma$ *as* $m, n \to \infty$ *and* (x_{mn}) *is a sequence of vectors with* $\rho(x_{mn} - x) \to 0$ *as* $m, n \to \infty$, *then* $\rho(\sigma_{mn} x_{mn} - \sigma x) \to 0$ *as* $m, n \to \infty$.

A paranorm w for which $\rho(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [23], Theorem 10.4.2, p.183).

The notion of deal convergence was introduced first by Kostyrko et al.[24] as a generalization of statistical convergence which was further studied in toplogical spaces by Kumar et al.[25,26] and also more applications of ideals can be deals with various authors by B.Hazarika [27-39] and B.C.Tripathy and B. Hazarika [40-43].

Definition 2. *A family $I \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if*
(1) $\phi \in I$
(2) $A, B \in I$ *imply* $A \bigcup B \in I$
(3) $A \in I, B \subset A$ *imply* $B \in I$.
While an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}\times\mathbb{N}$. A sequence $(x_{mn})_{m,n\in\mathbb{N}\times\mathbb{N}}$ in X is said to be $I-$ convergent to $0 \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \left\{m, n \in \mathbb{N}\times\mathbb{N} : \|(d_1(x_1), \ldots, d_n(x_n)) - 0\|_p \geq \varepsilon\right\}$ belongs to I.

Definition 3. *A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if*
(1) $\phi \in F$
(2) *for each* $A, B \in F$, *we have imply* $A \bigcap B \in F$
(3) *each* $A \in F$ *and each* $A \subset B$, *we have* $B \in F$.

Definition 4. *An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.*

Definition 5. *A non-trivial ideal $I \subset 2^X$ is called*
(i) *admissible if and only if* $\{\{x\} : x \in X\} \subset I$.
(ii) *maximal if there cannot exists any non-trivial ideal* $J \neq I$ *containing* I *as a subset.*

If we take $I = I_f = \{A \subseteq \mathbb{N}\times\mathbb{N} : A \text{ is a finite subset }\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N}\times\mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asyptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}\times\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}\times\mathbb{N}$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$, $d(X,Y) = max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$.

Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space. Also the relation "$\leq$" is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}\times\mathbb{R}$ i.e. a mapping $X : R \to J(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

Definition 6. *A fuzzy number X is said to be*
(i) *convex if* $X(t) \geq X(s) \wedge X(r) = min\{X(s), X(r)\}$, *where* $s < t < r$.

(ii) *normal if there exists* $t_0 \in \mathbb{R} \times \mathbb{R}$ *such that* $X(t_0) = 1$.
(iii) *upper semi-continuous if for each* $\varepsilon > 0, X^{-1}([0, a+\varepsilon])$ *for all* $a \in [0,1]$ *is open in the usual topology of* $\mathbb{R} \times \mathbb{R}$.

Let $R(J)$ *denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, i.e. if* $X \in \mathbb{R}(J) \times \mathbb{R}(J)$ *the for any* $\alpha \in [0,1]$, $[X]^{\alpha}$ *is compact, where*

$$[X]^{\alpha} = \{t \in \mathbb{R} \times \mathbb{R} : X(t) \geq \alpha, \quad if \quad \alpha \in [0,1]\}, [X]^0 = closure \ \ of \quad (\{t \in \mathbb{R} \times \mathbb{R} : X(t) > \alpha, \quad if \quad \alpha = 0\}).$$

The set $\mathbb{R}$ *of real numbers can be embedded* $\mathbb{R}(J)$ *if we define* $\bar{r} \in \mathbb{R}(J) \times \mathbb{R}(J)$ *by*

$$\bar{r}(t) = \begin{cases} 1, & if\, t = r: \\ 0, & if\, t \neq r \end{cases}$$

The absolute value, $|X|$ *of* $X \in \mathbb{R}(J)$ *is defined by*

$$|X|(t) = \begin{cases} max\{X(t), X(-t)\}, & if\, t \geq 0; \\ 0, & if\, t < 0 \end{cases}$$

Define a mapping $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}^+ \cup \{0\}$*by*

$$\bar{d}(X,Y) = sup_{0 \leq \alpha \leq 1} d\left([X]^{\alpha}, [Y]^{\alpha}\right).$$

It is known that $\left(\mathbb{R}(J), \bar{d}\right)$ *is a complete metric space.*

Definition 7. *A metric on* $\mathbb{R}(J)$ *is said to be translation invariant if* $\bar{d}(X+Z, Y+Z) = \bar{d}(X,Y)$*, for* $X,Y,Z \in \mathbb{R}(J)$.

Definition 8. *A sequence* $X = (X_{mn})$ *of fuzzy numbers is said to be*
(i) *convergent to a fuzzy number* X_0 *if for every* $\varepsilon > 0$*, there exists a positive integer* n_0 *such that* $\bar{d}(X_{mn}, X_0) < \varepsilon$ *for all* $n \geq n_0$.
(ii) *bounded if the set* $\{X_{mn} : m,n \in \mathbb{N}\}$ *of fuzzy numbers is bounded.*

Definition 9. *A sequence* $X = (X_{mn})$ *of fuzzy numbers is said to be*
(i) *I-convergent to a fuzzy number* X_0 *if for each* $\varepsilon > 0$ *such that*

$$A = \left\{m,n \in \mathbb{N} : \bar{d}(X_{mn}, X_0) \geq \varepsilon\right\} \in I.$$

The fuzzy number X_0 *is called I-limit of the sequence* (X_{mn}) *of fuzzy numbers and we write* $I - lim X_{mn} = X_0$.
(ii) *I-bounded if there exists* $M > 0$ *such that*

$$\left\{m,n \in \mathbb{N} : d\left(X_{mn}, \bar{0}\right) > M\right\} \in I.$$

Definition 10. *A sequence space* E_F *of fuzzy numbers is said to be*
(i) *solid (or normal) if* $(Y_{mn}) \in E_F$ *whenever* $(X_{mn}) \in E_F$ *and* $\bar{d}\left(Y_{mn}, \bar{0}\right) \leq \bar{d}\left(X_{mn}, \bar{0}\right)$ *for all* $m,n \in \mathbb{N}$.
(ii) *symmetric if* $(X_{mn}) \in E_F$ *implies* $\left(X_{\pi(mn)}\right) \in E_F$ *where* π *is a permutation of* $\mathbb{N} \times \mathbb{N}$.

Let $K = \{k_1 < k_2 < \ldots\} \subseteq \mathbb{N}$ *and* E *be a sequence space. A K-step space of* E *is a sequence space*

$$\lambda_{mn}^{E} = \left\{\left(X_{m_p n_p}\right) \in w^2 : (m_p n_p) \in E\right\}.$$

A canonical preimage of a sequence $\left\{\left(x_{m_p n_p}\right)\right\} \in \lambda_K^E$ *is a sequence* $\{y_{mn}\} \in w^2$ *defined as*

$$y_{mn} = \begin{cases} x_{mn}, & if\, m,n \in E \\ 0, & otherwise. \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

Definition 11. *A sequence space E_F is said to be monotone if E_F contains the canonical pre-images of all its step spaces.*

The following well-known inequality will be used throughout the article. Let $p = (p_{mn})$ be any sequence of positive real numbers with $0 \le p_{mn} \le sup_{mn} p_{mn} = G, D = max\{1, 2G-1\}$ then

$$|a_{mn} + b_{mn}|^{p_{mn}} \le D\left(|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}}\right) \text{ for all } m,n \in \mathbb{N} \text{ and } a_{mn}, b_{mn} \in \mathbb{C}.$$

Also $|a_{mn}|^{p_{mn}} \le max\left\{1, |a|^G\right\}$ for all $a \in \mathbb{C}$.

First we procure some known results; those will help in establishing the results of this article.

Lemma 1. *A sequence space E_F is normal implies E_F is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [44], page 53).*

Lemma 2. *(Kostyrko et al., [24], Lemma 5.1). If $I \subset 2^{\mathbb{N}}$ is a maximal ideal, then for each $A \subset \mathbb{N}$ we have either $A \in I$ or $\mathbb{N} - A \in I$.*

3 Some new integrated sequence spaces of fuzzy numbers

The main aim of this article to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let $p = (p_{mn})$ be a sequence of positive real numbers for all $m,n \in \mathbb{N}$. $f = (f_{mn})$ be a Musielak-Orlicz function, $\left(X, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)$ be a $p-$metric space, and $\left(\lambda_{mn}^{-1}\right)$ be a sequence of non-zero scalars and $\mu_{mn}(X) = \bar{d}\left(\frac{\Delta^m X_{mn}}{\lambda_{mn}}, \bar{0}\right)$ be a sequence of fuzzy numbers, we define the following sequence spaces as follows:

$$\left[\chi_{f\mu}^{2q}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} =$$
$$\left\{(X_{mn}) : \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \ge \varepsilon\right\} \in I\right\},$$

$$\left[\Lambda_{f\mu}^{2q}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} =$$
$$\left\{K > 0 : \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \ge K\right\} \in I\right\}.$$

Theorem 1. *Let $f = (f_{mn})$ be a Musielak-Orlicz function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence spaces $\left[\chi_{f\mu}^{2q}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ and $\left[\Lambda_{f\mu}^{2q}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ are linear spaces.*

Proof. We prove the result only for the space $\left[\chi_{f\mu}^{2q}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$. The other spaces can be treated, similarly. Let $X = (X_{mn})$ and $Y = (Y_{mn})$ be two elements $\left[\chi_{f\mu}^{2q}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$. We have

$$A_{\frac{\varepsilon}{2}} = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \ge \frac{\varepsilon}{2}\right\} \in I$$

and

$$B_{\frac{\varepsilon}{2}} = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \ge \frac{\varepsilon}{2}\right\} \in I.$$

Let α and β be two scalars. By the Musielak continuity of the function $f = (f_{mn})$ the following inequality holds:

$$\left[f_{mn}\left(\left\|\frac{\mu_{mn}(\alpha x+\beta y)}{|\alpha|+|\beta|},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \leq$$
$$D\left[\frac{|\alpha|}{|\alpha|+|\beta|}f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}$$
$$+D\left[\frac{|\beta|}{|\alpha|+|\beta|}f_{mn}\left(\|\mu_{mn}(y),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \leq$$
$$D\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}$$
$$+D\left[f_{mn}\left(\|\mu_{mn}(y),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}.$$

From the above relation we obtain the following:

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\left[f_{mn}\left(\left\|\frac{\mu_{mn}(\alpha x+\beta y)}{|\alpha|+|\beta|},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\geq\varepsilon\right\}\subseteq$$
$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:DK\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\geq\frac{\varepsilon}{2}\right\}\cup$$
$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:DK\left[f_{mn}\left(\|\mu_{mn}(y),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\geq\frac{\varepsilon}{2}\right\}\in I.$$

This completes the proof.

Remark. It is easy to verify $\left[\Lambda^{2q}_{f\mu},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]^{I(F)}$ is a linear space.

Theorem 2. *The classes of sequences*

$$\left[\chi^{2q}_{f\mu},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]^{F}$$

and

$$\left[\Lambda^{2q}_{f\mu},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]^{F}$$

are paranormed spaces paranormed by g, defined by

$$g(X)=inf\left\{\frac{q_{mn}}{H}:sup_{mn}f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\leq 1\right\}$$

where $H=max\{1,sup_{mn}q_{mn}\}$.

Proof. Clearly $g(X)\geq 0, g(-X)=g(X)$ and $g(X+Y)\leq g(X)+g(Y)$. Next we show the continuity of the product. Let α be fixed and $g(X)\to 0$. Then it is obvious that $g(\alpha X)\to 0$. Next let $\alpha\to 0$ and X be fixed. Since f_{mn} are continuous, we have $f_{mn}\left(\alpha\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\to 0$, as $\alpha\to 0$. Thus we have

$$inf\left\{\frac{q_{mn}}{H}:sup_{mn}f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\leq 1\right\}\to 0, \text{ as } \alpha\to 0.$$

Hence $g(\alpha X)\to 0$ as $\alpha\to 0$. Therefore g is a paranorm.

Proposition 1. $\left[\chi^{2q}_{f\mu},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]^{I(F)}\subset\left[\Lambda^{2q}_{f\mu},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]^{I(F)}$ *and the inclusion is proper.*

Proof. Let $I(F)=I$, $f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)=(-1)^{m+n}$, $\frac{1}{\lambda_{mn}}=q_{mn}=m=1$ then $\mu(x)=\left[\Lambda^{2q}_{f\mu},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]^{I(F)}$ but $(x_{mn})\notin\left[\chi^{2q}_{f\mu},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]^{I(F)}$.

Theorem 3. *The spaces*

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \quad \text{and} \quad \left[\Lambda^{2q}_{f\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$$

are neither solid nor monotone in general.

Proof. Let (x_{mn}) be a given sequence and (α_{mn}) be a sequence of scalars such that $|\alpha_{mn}| \le 1$, for all $m,n \in \mathbb{N}$. Then, for all $m,n \in \mathbb{N}$, we have

$$\left[f_{mn}\left(\|\mu_{mn}(\alpha x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \le \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}.$$

If $\Delta_{mn} = 1$ then solidness follows above inequality. The monotonicity follows by lemma 2.12. The first part of the proof follows from the following example.

Example 1. Let $I(F) = I$,

$$\begin{aligned}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} &= \left[f\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \\ &= \left[\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}},\end{aligned}$$

$m = 1, \frac{1}{\lambda_{mn}} = 1$ for all $m,n \in \mathbb{N}, q_{mn} = 1$ for m,n odd, $q_{mn} = 3$ for m,n even, $(m+n)!(x_{mn}) = (mn)^{m+n}$ for all $m,n \in \mathbb{N}$ belongs to $\left[\Lambda^{2q}_{\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I}$. For E, a sequence space, consider its step space E_J defined by $(y_{mn}) \in E_J$ implies $y_{mn} = 0$ for all m,n odd and $y_{mn} = x_{mn}$ for m,n even. Then $(y_{mn}) \in \left[\Lambda^{2q}_{\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I}_{J}$. Hence the spaces are not monotone. Hence are not solid.

Theorem 4. *The spaces*

$$\left[\chi^{2q}_{\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I} \quad \text{and} \quad \left[\Lambda^{2q}_{\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I}$$

are not convergence free.

Example 2. Let $I(F) = I$,

$$\begin{aligned}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} &= \left[f\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \\ &= \left[\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}},\end{aligned}$$

$m = 1, \frac{1}{\lambda_{mn}} = 1$ for all $m,n \in \mathbb{N}, q_{mn} = 1$ for m,n odd, $q_{mn} = 2$ for m,n even, consider the sequence $(m+n)!(x_{mn}) = (mn)^{-(m+n)}$ for all $m,n \in \mathbb{N}$ belongs to each of $\left[\chi^{2q}_{\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I}$ and $\left[\Lambda^{2q}_{\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I}$. Consider the sequence (y_{mn}) defined by $((m+n)!y_{mn})^{1/m+n} = m^2n^2$, for all $m,n \in \mathbb{N}$. Then (y_{mn}) neither belongs to $\left[\chi^{2q}_{\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I}$ nor $\left[\Lambda^{2q}_{\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I}$. Hence the spaces are not convergence free.

Acknowledgement

The authors wish to thank the unknown referees who patiently went through the article and whose suggestions considerably improved its presentation and readability. The research of the first author Deepmala is supported by the Science and Engineering Research Board (SERB), Department of Science and Technology (DST), Government of India under SERB National Post-Doctoral fellowship scheme File Number: PDF/2015/000799.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] T.J.I'A. Bromwich, An introduction to the theory of infinite series, Macmillan and Co.Ltd. ,New York, (1965).
[2] G.H. Hardy, On the convergence of certain multiple series, Proc. Camb. Phil. Soc., 19 (1917), 86-95.
[3] F. Moricz, Extentions of the spaces c and c_0 from single to double sequences, Acta. Math. Hung., 57(1-2), (1991), 129-136.
[4] F. Moricz and B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Camb. Phil. Soc., 104, (1988), 283-294.
[5] M. Basarir and O. Solancan, On some double sequence spaces, J. Indian Acad. Math., 21(2) (1999), 193-200.
[6] B.C. Tripathy, On statistically convergent double sequences, Tamkang J. Math., 34(3), (2003), 231-237.
[7] A. Turkmenoglu, Matrix transformation between some classes of double sequences, J. Inst. Math. Comp. Sci. Math. Ser., 12(1), (1999), 23-31.
[8] A. G$\ddot{o}$khan and R. Çolak, The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$, Appl. Math. Comput., 157(2), (2004), 491-501.
[9] A. G$\ddot{o}$khan and R. Çolak, Double sequence spaces ℓ_2^∞, ibid., 160(1), (2005), 147-153.
[10] M. Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.
[11] M. Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl., 288(1), (2003), 223-231.
[12] B. Altay and F. Baąar, Some new spaces of double sequences, J. Math. Anal. Appl., 309(1), (2005), 70-90.
[13] F. Başar and Y. Sever, The space $\mathscr{L}_p$ of double sequences, Math. J. Okayama Univ, 51, (2009), 149-157.
[14] N. Subramanian and U.K. Misra, The semi normed space defined by a double gai sequence of modulus function, Fasciculi Math., 46, (2010).
[15] I.J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Cambridge Philos. Soc, 100(1) (1986), 161-166.
[16] J. Cannor, On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull., 32(2), (1989), 194-198.
[17] A. Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, Math. Ann., 53, (1900), 289-321.
[18] H.J. Hamilton, Transformations of multiple sequences, Duke Math. J., 2, (1936), 29-60.
[19] H.J. Hamilton, A Generalization of multiple sequences transformation, Duke Math. J., 4, (1938), 343-358.
[20] H.J. Hamilton, Preservation of partial Limits in Multiple sequence transformations, Duke Math. J., 4, (1939), 293-297.
[21] P.K. Kamthan and M. Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, 65 Marcel Dekker, In c., New York , 1981.
[22] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math., 10 (1971), 379-390.
[23] A. Wilansky, Summability through Functional Analysis, North-Holland Mathematical Studies, North-Holland Publishing, Amsterdam, Vol.85(1984).

[24] P. Kostyrko, T. Salat and W. Wilczynski, $I-$ convergence, *Real Anal. Exchange*, **26(2)** (2000-2001), 669-686, MR 2002e:54002.
[25] V. Kumar and K. Kumar, On the ideal convergence of sequences of fuzzy numbers, *Inform. Sci.*, **178(24)** (2008), 4670-4678.
[26] V. Kumar, On I and I^*- convergence of double sequences, *Mathematical communications*, **12** (2007), 171-181.
[27] B. Hazarika, On Fuzzy Real Valued $I-$ Convergent Double Sequence Spaces, *The Journal of Nonlinear Sciences and its Applications* **(in press)**.
[28] B. Hazarika, On Fuzzy Real Valued $I-$ Convergent Double Sequence Spaces defined by Musielak-Orlicz function, *J. Intell. Fuzzy Systems*, **25(1)** (2013), 9-15, DOI: 10.3233/IFS-2012-0609.
[29] B. Hazarika, Lacunary difference ideal convergent sequence spaces of fuzzy numbers, *J. Intell. Fuzzy Systems*, **25(1)** (2013), 157-166, DOI: 10.3233/IFS-2012-0622.
[30] B. Hazarika, On $\sigma-$ uniform density and ideal convergent sequences of fuzzy real numbers, *J. Intell. Fuzzy Systems*, DOI: 10.3233/IFS-130769.
[31] B. Hazarika, Fuzzy real valued lacunary $I-$ convergent sequences, *Applied Math. Letters*, **25(3)** (2012), 466-470.
[32] B. Hazarika, Lacunary $I-$ convergent sequence of fuzzy real numbers, *The Pacific J. Sci. Techno.*, **10(2)** (2009), 203-206.
[33] B. Hazarika, On generalized difference ideal convergence in random 2-normed spaces, *Filomat*, **26(6)** (2012), 1265-1274.
[34] B. Hazarika, Some classes of ideal convergent difference sequence spaces of fuzzy numbers defined by Orlicz function, *Fasciculi Mathematici*, **52** (2014)(Accepted).
[35] B. Hazarika, $I-$ convergence and Summability in Topological Group, *J. Informa. Math. Sci.*, **4(3)** (2012), 269-283.
[36] B. Hazarika, Classes of generalized difference ideal convergent sequence of fuzzy numbers, *Annals of Fuzzy Math. and Inform.*, (in press).
[37] B. Hazarika, On ideal convergent sequences in fuzzy normed linear spaces, *Afrika Matematika*, DOI: 10.1007/s13370-013-0168-0.
[38] B. Hazarika and E. Savas, Some $I-$ convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz functions, *Math. Comp. Modell.*, **54(11-12)** (2011), 2986-2998.
[39] B. Hazarika,K. Tamang and B.K. Singh, Zweier Ideal Convergent Sequence Spaces Defined by Orlicz Function, *The J. Math. and Computer Sci.*, (Accepted).
[40] B. Hazarika and V.Kumar, Fuzzy real valued $I-$ convergent double sequences in fuzzy normed spaces, *J. Intell. Fuzzy Systems*, (accepted).
[41] B.C. Tripathy and B. Hazarika, $I-$ convergent sequence spaces associated with multiplier sequences, *Math. Ineq. Appl.*, **11(3)** (2008), 543-548.
[42] B.C. Tripathy and B. Hazarika, Paranorm $I-$ convergent sequence spaces, *Math. Slovaca*, **59(4)** (2009), 485-494.
[43] B.C. Tripathy and B. Hazarika, Some $I-$ convergent sequence spaces defined by Orlicz functions, *Acta Math. Appl. Sinica*, **27(1)** (2011), 149-154.
[44] B. Hazarika and A. Esi, On ideal convergent sequence spaces of fuzzy real numbers associated with multiplier sequences defined by sequence of Orlicz functions, *Annals of Fuzzy Mathematics and Informatics*, (in press).