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Rational approximations for solving cauchy problems

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Abstract: In this letter, numerical solutions of Cauchy problems are considered by multivariate Padé approximations (MPA). Multivariate Padé approximations (MPA) were applied to power series solutions of Cauchy problems that solved by using He's variational iteration method (VIM). Then, numerical results obtained by using multivariate Padé approximations were compared with the exact solutions of Cauchy problems.

Keywords: Cauchy problem, inviscid Burger's equation, multivariate padé approximaton (MPA).

1 Introduction

In recent times, univariate and multivariate padé approximaton have been succesfully applied to various problems in physical and engineering sciences [1-5]. "*Padé approximant represents a function by the ratio of two polynomials. The coefficients of the powers occurring in the polynomials are, however, determined by the coefficients in the Taylor series expansion of the function* [14]". Multivariate Padé approximation is based on univariate Padé approximation [12] but calculation methods and most of the theorems are different from each other [12]. Cuyt and her co-workers have established the uniqueness, nonuniqueness and existence results for homogeneous and nonhomogeneous multivariate Padé approximations of formal power series of several variables [15-17].

In many branches of applied sciences, the solution of a given problem is often obtained as a power series expansion. The question is then trying to approximate the function from its series expansion. A possible answer is to construct a rational function whose series expansion matches the original one as far as possible. Such rational functions are called Padé approximants [18]. In this paper, power series solutions of Cauchy problems were converted into multivariate Padé series. That is, multivariate Padé approximations were applied to the first-order partial differential equation in the form [6].

$$u_t(x,t)+a(x,t)\,u_x(x,t)=\phi(x),\ \ x\in\Re,\ t>0 \tag{1}$$

$$u(x,0)=\psi(x),\ \ x\in\Re. \tag{2}$$

The details about the above equations can be seen in [6].

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2 He's variational iteration method

The basic concepts and principles of He's variational iteration method can be seen in [7-11]. Zhou and Yao [6] obtained the following iteration formula by using the basic concepts and principles of He's variational iteration method:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \frac{\partial u_n(x,\xi)}{\partial \xi} + a(x,\xi) + \frac{\partial u_n(x,\xi)}{\partial \xi} - \phi(x) \right\} d\xi. \tag{3}$$

3 Multivariate Padé approximation

Consider the bivariate function $f(x,y)$ with Taylor power series development

$$f(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j \tag{4}$$

around the origin [12]. The Padé approximation problem of order for $f(x,y)$ consists in finding polynomials

$$p(x,y) = \sum_{k=0}^{m} A_k(x,y) \tag{5}$$

$$q(x,y) = \sum_{k=0}^{n} B_k(x,y) \tag{6}$$

such that in the power series $(fq-p)(x,y)$ the coefficients of x^i and y^j by solving the following equation system;

$$\begin{cases} C_0(x,y)B_0(x,y) = A_0(x,y) \\ C_1(x,y)B_0(x,y) + C_0(x,y)B_1(x,y) = A_1(x,y) \\ \vdots \\ C_m(x,y)B_0(x,y) + \cdots + C_{m-n}(x,y)B_n(x,y) = A_m(x,y) \end{cases} \tag{7}$$

$$\begin{cases} C_{m+1}(x,y)B_0(x,y) + C_{m+1-n}(x,y)B_n(x,y) = 0 \\ \vdots \\ C_{m+n}(x,y)B_0(x,y) + \cdots + C_m(x,y)B_n(x,y) = 0 \end{cases} \tag{8}$$

where $C_k = 0$ if $k < 0$. İf the equations (8) and (9) are solved then the coefficients A_k $(k = 0,\dots,m)$ and B_k $(k = 0,\dots,n)$ are obtained. So polynomials (5) and (6) are found. polynomials $p(x,y)$ and $q(x,y)$ are called Padé equations[12]. So the multivariate Padé approximant of order (m,n) for $f(x,y)$ is defined as,

$$r_{m,n}(x,y) = \frac{p(x,y)}{q(x,y)}. \tag{9}$$

Theorem 1. *(Cuyt and Wuytack [12]). For every nonnegative m and n a unique Padé approximant of order (m,n) for f exists.*

4 Applications and results

In this section multivariate Padé series solutions of Cauchy problems shall be illustrated by two examples. All the results were calculated by using the software Maple. The full VIM solutions of examples can be seen in Zhou and Yao [6].

Example 1. Consider the nonlinear cauchy problem [6]

$$u_t(x,t) + xu_x(x,t) = 0, \ \ x \in \Re, \, t > 0 \tag{10}$$

$$u(x,0) = x^2, \ \ x \in \Re. \tag{11}$$

According to the iteration formula (3) Zhou and Yao [6] obtained following solution,

$$u_n(x,t) = x^2\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \frac{(2t)^5}{5!} + \cdots\right) \tag{12}$$

The exact solution of (12) is given as $u(x,t) = x^2e^{-2t}$ in [6]. If the multivariate Padé approximation is applied to equation (12) for $m = 4$ and $n = 2$, according to the equation system (7) and (8) the following Padé equations are obtained;

$$p(x,t) = \frac{4t^4(t^2 - 3t + 3)x^6}{9} \tag{13}$$

and

$$q(x,t) = \frac{4t^4(t^2 + 3t + 3)x^4}{9}. \tag{14}$$

So the multivariate Padé approximant of order $(4,2)$ for equation (12) is,

$$r_{4,2}(x,t) = \frac{(t^2 - 3t + 3)x^2}{(t^2 + 3t + 3)}. \tag{15}$$

If the multivariate Padé approximation is applied to equation (12) for $m = 5$ and $n = 2$, according to the equation system (7) and (8) the following Padé equations are obtained,

$$p(x,t) = \frac{4t^6(2t^3 - 9t^2 + 18t - 15)x^6}{135} \tag{16}$$

and

$$q(x,t) = \frac{4t^6(t^2 + 4t + 5)x^4}{45}. \tag{17}$$

So the multivariate Padé approximant of order $(5,2)$ for equation (12) is,

$$r_{5,2}(x,t) = \frac{(2t^3 - 9t^2 + 18t - 15)x^2}{3(t^2 + 4t + 5)}. \tag{18}$$

If the multivariate Padé approximation is applied to equation (12) for $m = 6$ and $n = 2$, according to the equation system (7) and (8) the following Padé equations are obtained,

$$p(x,t) = 0.001975308642t^8\left(2t^4 - 12t^3 + 36t^2 - 60t + 45\right)x^6 \tag{19}$$

and

$$q(x,t) = 0.005925925926t^8\left(15 + 10t + 2t^2\right)x^4. \tag{20}$$

So the multivariate Padé approximant of order $(6,2)$ for equation (12) is,

$$r_{6,2}(x,t) = 0.3333333333\frac{\left(2t^4 - 12t^3 + 36t^2 - 60t + 45\right)x^2}{15 + 10t + 2t^2}. \tag{21}$$

If the numerical results are compared, following table and figures are obtained (Table 1 and Figure 1, Figure 2, Figure 3, Figure 4.);

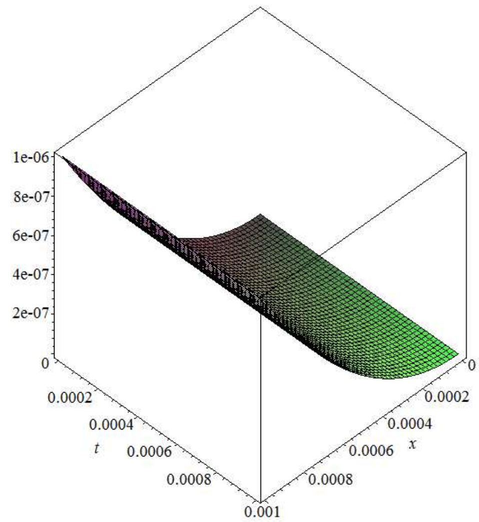

Fig. 1: Exact solution of equation (10) in Example 1.

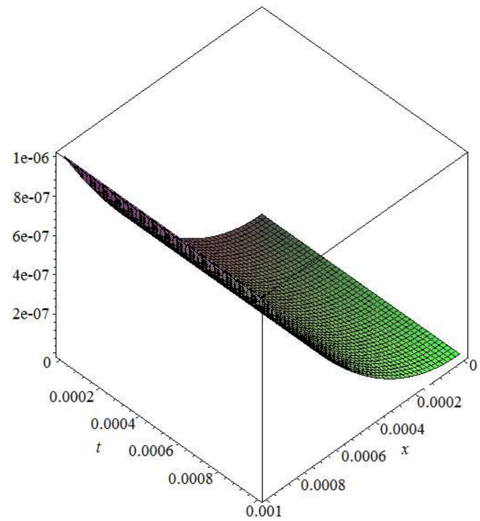

Fig. 2: $(r_{4,2}(x,t))$, Multivariate Padé approximant of order $(4,2)$ for equation (12).

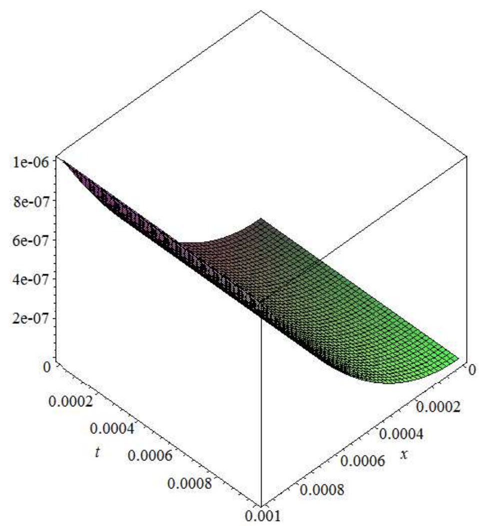

Fig. 3: $(r_{5,2}(x,t))$, Multivariate Padé approximant of order$(5,2)$ for equation (12).

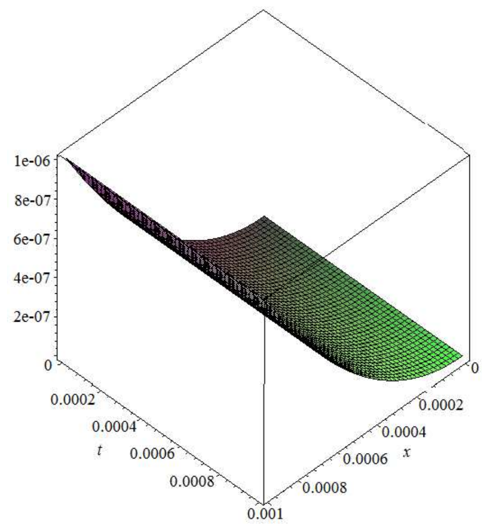

Fig. 4: $(r_{6,2}(x,t))$, Multivariate Padé approximant of order $(6,2)$for equation (12).

Table 1: Comparison of Exact solution of equation (10) and MPA solutions of equation (12).

x	t	Exact solution $u(x,t)=x^2e^{-2t}$	$r_{4,2}(x,t)$	$r_{5,2}(x,t)$	$r_{6,2}(x,t)$
0.001	0.001	$0.9980019987\times10^{-6}$	$0.9980019987\times10^{-6}$	$0.9980019987\times10^{-6}$	$0.9980019986\times10^{-6}$
0.002	0.002	$0.3984031957\times10^{-5}$	$0.3984031957\times10^{-5}$	$0.3984031957\times10^{-5}$	$0.3984031956\times10^{-5}$
0.003	0.003	$0.8946161677\times10^{-5}$	$0.8946161676\times10^{-5}$	$0.8946161683\times10^{-5}$	$0.8946161676\times10^{-5}$
0.004	0.004	0.00001587251064	0.00001587251064	0.00001587251063	0.00001587251064
0.005	0.005	0.00002475124584	0.00002475124584	0.00002475124585	0.00002475124583
0.006	0.006	0.00003557058166	0.00003557058166	0.00003557058167	0.00003557058166
0.007	0.007	0.00004831877967	0.00004831877967	0.00004831877967	0.00004831877966
0.008	0.008	0.00006298414849	0.00006298414848	0.00006298414850	0.00006298414846
0.009	0.009	0.00007955504362	0.00007955504362	0.00007955504363	0.00007955504363
0.01	0.01	0.00009801986733	0.00009801986733	0.00009801986733	0.00009801986732

Example 2. Consider the inviscid Burger's equation [6]

$$u_t(x,t)+u(x,t)u_x(x,t)=0,\ \ x\in\Re,\ t>0 \tag{22}$$

$$u(x,0)=x,\ \ x\in\Re. \tag{23}$$

According to the iteration formula (3) Zhou and Yao [6] obtained following solution,

$$\begin{aligned}u_4(x,t)&=x-tx+t^2x-t^3x+t^4x-\tfrac{13t^5x}{15}+\tfrac{2t^6x}{3}-\tfrac{t^7x}{29}+\tfrac{71t^8x}{252}-\tfrac{86t^9x}{567}\\&+\tfrac{22t^{10}x}{315}-\tfrac{5t^{11}x}{189}+\tfrac{t^{12}x}{126}-\tfrac{t^{13}x}{567}+\tfrac{t^{14}x}{3969}-\tfrac{t^{15}x}{59535}\end{aligned} \tag{24}$$

The exact solution of (22) is given as $u(x,t)=\frac{x}{1+t}$ in [13]. If the multivariate Padé approximation is applied to equation (24) for $m=9$ and $n=2$, according to the equation system (7) and (8) the following Padé equations are obtained;

$$\begin{aligned}&p(x,t)=t^{16}(197313169805t^8+194795648572t^7+161820668856t^6\\&-247699921980t^5+337516192020t^4-337516192020t^3+337516192020t^2\\&-307283385780t+673622025300)x^3/9084507566400\end{aligned} \tag{25}$$

and

$$q(x,t)=t^{16}(2482168t^2+30077064t+55305585)x^2/745854480. \tag{26}$$

So the multivariate Padé approximant of order $(9,2)$ for equation (24) is,

$$\begin{aligned}&r_{9,2}(x,t)=(197313169805t^8+194795648572t^7+161820668856t^6\\&-247699921980t^5+337516192020t^4-337516192020t^3+337516192020t^2\\&-307283385780t+673622025300)x/(12180(2482168t^2+30077064t+55305585))\end{aligned} \tag{27}$$

If the multivariate Padé approximation is applied to equation (24) for $m=11$ and $n=2$, according to the equation system (7) and (8) the following Padé equations are obtained;

$$\begin{aligned}&p(x,t)=t^{20}(3332274t^{10}+39785435t^9+280716390t^8\\&+240748200t^7-221629716t^5+323269380t^4-323269380t^3\\&+323269380t^2-195013980t+762297480)x^3/881040604500\end{aligned} \tag{28}$$

and

$$q(x,t) = t^{20}(390t^2 + 1725t + 2318)x^2/2679075. \tag{29}$$

So the multivariate Padé approximant of order $(11,2)$ for equation (24) is,

$$\begin{aligned} r_{11,2}(x,t) &= (3332274t^{10} + 39785435t^9 + 280716390t^8 \\ &+240748200t^7 - 221629716t^5 + 323269380t^4 \\ &-323269380t^3 + 323269380t^2 - 195013980t \\ &+762297480)x/(328860(390t^2 + 1725t + 2318)). \end{aligned} \tag{30}$$

If the multivariate Padé approximation is applied to equation (24) for $m = 13$ and $n = 2$, according to the equation system (7) and (8) the following Padé equations are obtained,

$$\begin{aligned} p(x,t) &= 0.3378040082.10^{-12}t^{24}(39150t^2 - 261812t^{11} \\ &+0.1010766.10^7t^{10} - 0.1330530.10^7t^9 + 0.15064245.10^8t^8 \\ &+0.9952740.10^7t^7 + 0.16706088.10^8t^6 - 0.23480604.10^8t^5 \\ &+0.29926260.10^8t^4 - 0.29926260.10^8t^3 + 0.29926260.10^8t^2 \\ &-0.26637660.10^8t + 0.48342420..10^8t)x^3 \end{aligned} \tag{31}$$

and

$$q(x,t) = 0.1110902261t^{24}(147 + 66t + 10t^2)x^2. \tag{32}$$

So the multivariate Padé approximant of order $(13,2)$ for equation (24) is,

$$\begin{aligned} r_{13,2}(x,t) &= 0.3040807638.10^{-5}(39150t^2 - 261812t^{11} \\ &+0.1010766.10^7t^{10} - 0.1330530.10^7t^9 + 0.15064245.10^8t^8 \\ &+0.9952740.10^7t^7 + 0.16706088.10^8t^6 - 0.23480604.10^8t^5 \\ &+0.29926260.10^8t^4 - 0.29926260.10^8t^3 + 0.29926260.10^8t^2 \\ &-0.26637660.10^8t + 0.48342420..10^8t)x/(147 + 66t + 10t^2). \end{aligned} \tag{33}$$

According to the numerical results following table and figures are obtained (Table 2 and Figure 5, Figure 6, Figure 7, Figure 8.),

Table 2: Comparison of Exact solution of equation (22) and MPA solutions of equation (24).

x	t	Exact solution $u(x,t) = \frac{x}{1+t}$	$r_{9,2}(x,t)$	$r_{11,2}(x,t)$	$r_{13,2}(x,t)$
0.001	0.001	0.0009990009990	0.0009990009992	0.0009990009989	0.0009990009992
0.002	0.002	0.001996007984	0.001996007984	0.001996007984	0.001996007984
0.003	0.003	0.002991026919	0.002991026920	0.002991026919	0.002991026918
0.004	0.004	0.003984063745	0.003984063745	0.003984063744	0.003984063744
0.005	0.005	0.004975124378	0.004975124379	0.004975124378	0.004975124375
0.006	0.006	0.005964214712	0.005964214713	0.005964214711	0.005964214711
0.007	0.007	0.006951340616	0.006951340616	0.006951340616	0.006951340616
0.008	0.008	0.007936507937	0.007936507936	0.007936507936	0.007936507936
0.009	0.009	0.008919722498	0.008919722496	0.008919722499	0.008919722499
0.01	0.01	0.009900990099	0.009900990099	0.009900990099	0.009900990099

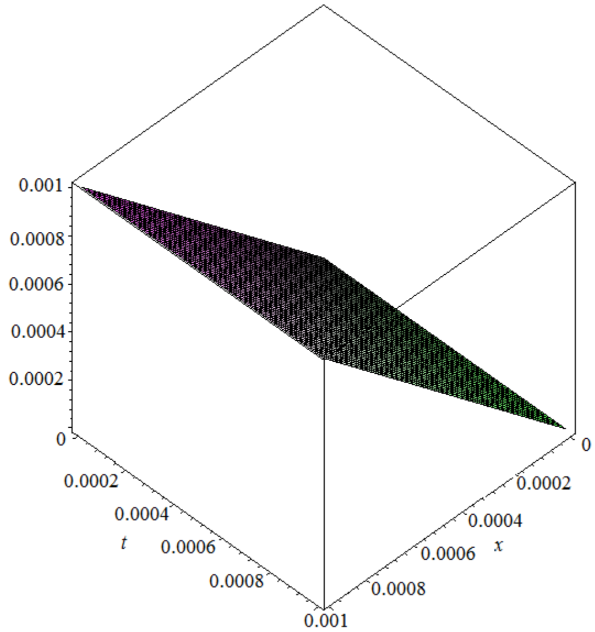

Fig. 5: Exact solution of equation (22) in Example 2.

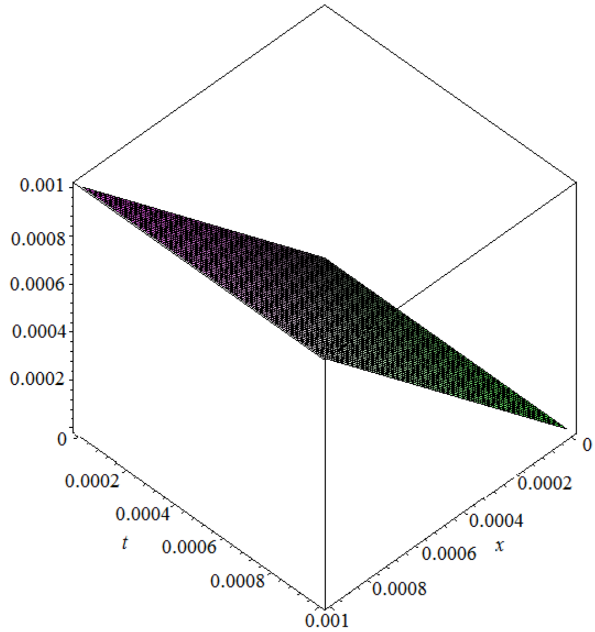

Fig. 6: $(r_{9,2}(x,t))$, Multivariate Padé approximant of order $(9,2)$for equation (24).

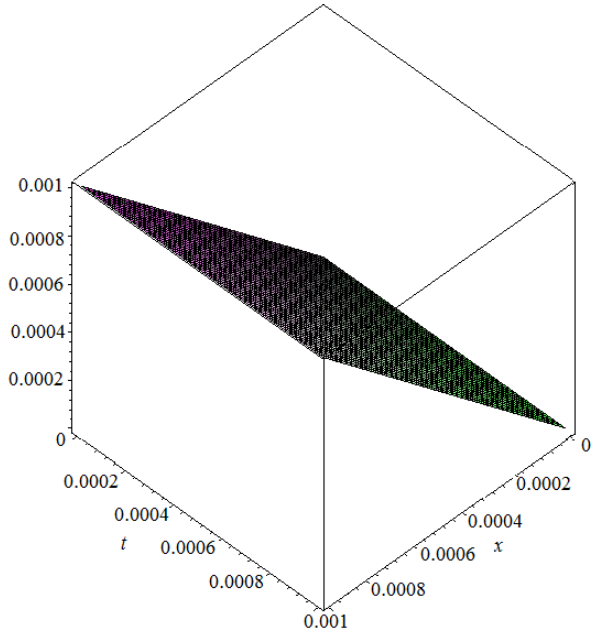

Fig. 7: $(r_{11,2}(x,t))$, Multivariate Padé approximant of order $(11,2)$ for equation (24).

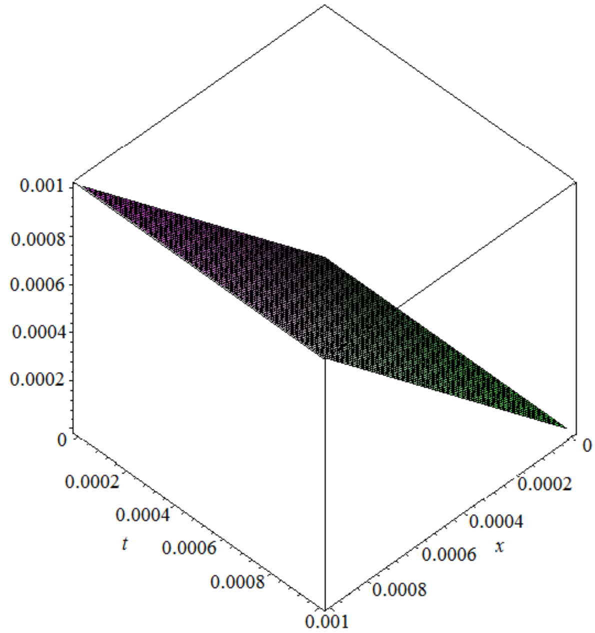

Fig. 8: $(r_{13,2}(x,t))$, Multivariate Padé approximant of order $(13,2)$ for equation (24).

5 Conclusion

In this paper, rational series solution of various kinds of Caucy problems were constructed by multivariare Padé approximation. The approximation is effective, easy to use and reliable and main benefit of the approximation is to offer rational approximation in a rapid convergent rational series form.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

6. References

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