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Hermite-Hadamard-Fejér type inequalities for harmonically convex functions via fractional integrals

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Abstract: In this paper, firstly, Hermite-Hadamard-Fejér type inequality for harmonically convex functions in fractional integral forms have been established. Secondly, an integral identity and some Hermite-Hadamard-Fejér type integral inequalities for harmonically convex functions in fractional integral forms have been obtained. The some results presented here would provide extensions of those given in earlier works.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard-Fejér inequality, Riemann-Liouville fractional integral, harmonically convex function.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature as Hermite-Hadamard's inequality [5].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [4], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1).

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be convex function. Then, the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{2}$$

holds, where $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve and extend inequalities (1) and (2) see [1,6,7,15,17].

We recall the following inequality and special functions which are known as Beta and hypergeometric function

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respectively

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x,y>0,$$

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\ c>b>0,|z|<1\,(\text{see [12]}).$$

Lemma 1. *For $0<\alpha\leq 1$ and $0\leq a<b$ we have*

$$|a^{\alpha}-b^{\alpha}|\leq(b-a)^{\alpha}.$$

(see [14,19]).

We will now give definitions of the right-hand side and left-hand side Riemann-Liouville fractional integrals which are used throughout this paper.

Definition 1. *Let $f\in L[a,b]$. The right-hand side and left-hand side Riemann-Liouville fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $b>a\geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\ x>a \text{ and}$$
$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\ x<b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int\limits_0^{\infty}e^{-t}t^{\alpha-1}dt$ (see [12]).

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [3,8,9,16,18,19].

In [11], İşcan gave definition of harmonically convex functions and established following Hermite-Hadamard type inequality for harmonically convex functions as follows.

Definition 2. *Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq tf(y)+(1-t)f(x) \tag{3}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (3) is reversed, then f is said to be harmonically concave.

Theorem 2. *Let $f:I\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities holds:*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}. \tag{4}$$

(see [11]).

In [10], İşcan and Wu presented Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral forms as follows.

Theorem 3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a,b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{1/a-}(f\circ h)(1/b)+J^{\alpha}_{1/b+}(f\circ h)(1/a)\right] \leq \frac{f(a)+f(b)}{2} \tag{5}$$

with $\alpha > 0$ and $h(x) = 1/x$.

In [13] Latif et. al. gave the following definition.

Definition 3. *A function $g : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $2ab/a+b$ if*

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for all $x \in [a,b]$.

In [2] Chan and Wu presented Hermite-Hadamard-Fejér inequality for harmonically convex functions as follows.

Theorem 4. *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$ and $g : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx &\leq \int_a^b \frac{f(x)g(x)}{x^2}dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx. \end{aligned} \tag{6}$$

In this paper, we firstly presented Hermite-Hadamard-Fejér inequality for harmonically convex function in fractional integral forms which is the weighted generalization of Hermite-Hadamard inequality for harmonically convex functions (5). Secondly, we obtained some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for harmonically convex function in fractional integrals.

2 Main results

Throughout this section, let $\|g\|_\infty = \sup_{t\in[a,b]} |g(t)|$, for the continuous function $g : [a,b] \to \mathbb{R}$.

Lemma 2. *If $g : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is integrable and harmonically symmetric with respect to $2ab/a+b$ with $a < b$, then*

$$J^{\alpha}_{1/b+}(g\circ h)(1/a) = J^{\alpha}_{1/a-}(g\circ h)(1/b) = \frac{1}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]$$

with $\alpha > 0$ and $h(x) = 1/x$, $x \in \left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. Since g is harmonically symmetric with respect to $2ab/a+b$, from Definition 3 we have $g\left(\frac{1}{x}\right) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)$, for all $x \in \left[\frac{1}{b},\frac{1}{a}\right]$. Setting $t = \frac{1}{a}+\frac{1}{b}-x$ and $dt = -dx$ gives

$$\begin{aligned} J^{\alpha}_{1/b+}(g\circ h)(1/a) &= \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1} g\left(\frac{1}{t}\right)dt = \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1} g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx \\ &= \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1} g\left(\frac{1}{x}\right)dx = J^{\alpha}_{1/a-}(g\circ h)(1/b). \end{aligned}$$

This completes the proof.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a harmonically convex function with $a<b$ and $f\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a+b$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{2ab}{a+b}\right)\begin{bmatrix} J^{\alpha}_{1/b+}(g\circ h)(1/a) \\ +J^{\alpha}_{1/a-}(g\circ h)(1/b)\end{bmatrix} \le \begin{bmatrix} J^{\alpha}_{1/b+}(fg\circ h)(1/a) \\ +J^{\alpha}_{1/a-}(fg\circ h)(1/b)\end{bmatrix} \le \frac{f(a)+f(b)}{2}\begin{bmatrix} J^{\alpha}_{1/b+}(g\circ h)(1/a) \\ +J^{\alpha}_{1/a-}(g\circ h)(1/b)\end{bmatrix} \tag{7}$$

with $\alpha>0$ and $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. Since f is a harmonically convex function on $[a,b]$, we write

$$f\left(\frac{2ab}{a+b}\right)=f\left(\frac{2\left(\frac{ab}{ta+(1-t)b}\right)\left(\frac{ab}{tb+(1-t)a}\right)}{\left(\frac{ab}{ta+(1-t)b}\right)+\left(\frac{ab}{tb+(1-t)a}\right)}\right)\le\frac{f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)}{2}. \tag{8}$$

for all $t\in[0,1]$. Multiplying both sides of (8) by $2t^{\alpha-1}g\left(\frac{ab}{tb+(1-t)a}\right)$ then integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)\int_0^1 t^{\alpha-1}g\left(\frac{ab}{tb+(1-t)a}\right)dt\\
&\le\int_0^1 t^{\alpha-1}\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)\right]g\left(\frac{ab}{tb+(1-t)a}\right)dt\\
&=\int_0^1 t^{\alpha-1}f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt+\int_0^1 t^{\alpha-1}f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt.
\end{aligned}$$

Since g is harmonically symmetric with respect to $2ab/a+b$, from Definition 3, we have $g\left(\frac{1}{x}\right)=g\left(\frac{1}{\left(\frac{1}{a}\right)+\left(\frac{1}{b}\right)-x}\right)$, for all $x\in\left[\frac{1}{b},\frac{1}{a}\right]$. Setting $x=\frac{tb+(1-t)a}{ab}$, and $dx=\left(\frac{b-a}{ab}\right)dt$ gives

$$\begin{aligned}
&2\left(\frac{ab}{b-a}\right)^{\alpha}f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}g\left(\frac{1}{x}\right)dx\\
&\le\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g\left(\frac{1}{x}\right)dx+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx\right\}\\
&=\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx\right\}\\
&=\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx\right\}.
\end{aligned}$$

Using Lemma 2, we have

$$\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)f\left(\frac{2ab}{a+b}\right)\begin{bmatrix} J^{\alpha}_{1/b+}(g\circ h)(1/a) \\ +J^{\alpha}_{1/a-}(g\circ h)(1/b)\end{bmatrix}\le\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\begin{bmatrix} J^{\alpha}_{1/b+}(fg\circ h)(1/a) \\ +J^{\alpha}_{1/a-}(fg\circ h)(1/b)\end{bmatrix}.$$

This inequality gives the left hand side of (7).

On the other hand, since f is a harmonically convex function, then, for all $t\in[0,1]$, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)\le f(a)+f(b). \tag{9}$$

Then multiplying both sides of (9) by $t^{\alpha-1}g\left(\frac{ab}{tb+(1-t)a}\right)$ and integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 t^{\alpha-1}f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt+\int_0^1 t^{\alpha-1}f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\&\le[f(a)+f(b)]\int_0^1 t^{\alpha-1}g\left(\frac{ab}{tb+(1-t)a}\right)dt.\end{aligned}$$

It means that

$$\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\begin{bmatrix}J^{\alpha}_{1/b+}(fg\circ h)(1/a)\\+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\end{bmatrix}\le\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\left(\frac{f(a)+f(b)}{2}\right)\begin{bmatrix}J^{\alpha}_{1/b+}(g\circ h)(1/a)\\+J^{\alpha}_{1/a-}(g\circ h)(1/b)\end{bmatrix}.$$

This inequality gives the right hand side of (7). The proof is completed.

Remark. In Theorem 5, one can see the following.

(i) If one takes $\alpha=1$, then inequality (7) becomes inequality (6) of Theorem 4.
(ii) If one takes $g(x)=1$, then inequality (7) becomes inequality (5) of Theorem 3.
(iii) If one takes $\alpha=1$ and $g(x)=1$, then inequality (7) becomes inequality (4) of Theorem 2.

Lemma 3. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $g:[a,b]\to\mathbb{R}$ is integrable and harmonically symmetric with respect to $2ab/a+b$, then the following equality for fractional integrals holds*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]-\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\\&=\frac{1}{\Gamma(\alpha)}\int_{\frac1b}^{\frac1a}\left[\int_{\frac1b}^{t}\left(\frac1a-s\right)^{\alpha-1}(g\circ h)(s)ds-\int_t^{\frac1a}\left(s-\frac1b\right)^{\alpha-1}(g\circ h)(s)ds\right](f\circ h)'(t)dt\end{aligned}\tag{10}$$

with $\alpha>0$ and $h(x)=1/x$, $x\in\left[\frac1b,\frac1a\right]$.

Proof. It suffices to note that

$$\begin{aligned}I&=\int_{\frac1b}^{\frac1a}\begin{bmatrix}\int_{\frac1b}^{t}\left(\frac1a-s\right)^{\alpha-1}(g\circ h)(s)ds\\-\int_t^{\frac1a}\left(s-\frac1b\right)^{\alpha-1}(g\circ h)(s)ds\end{bmatrix}(f\circ h)'(t)dt\\&=\int_{\frac1b}^{\frac1a}\left[\int_{\frac1b}^{t}\left(\frac1a-s\right)^{\alpha-1}(g\circ h)(s)ds\right](f\circ h)'(t)dt-\int_{\frac1b}^{\frac1a}\left[\int_t^{\frac1a}\left(s-\frac1b\right)^{\alpha-1}(g\circ h)(s)ds\right](f\circ h)'(t)dt\\&=I_1-I_2.\end{aligned}\tag{11}$$

By integration by parts and using Lemma 2, we have

$$\begin{aligned}I_1&=\left(\int_{\frac1b}^{t}\left(\frac1a-s\right)^{\alpha-1}(g\circ h)(s)ds\right)(f\circ h)(t)\Big|_{\frac1b}^{\frac1a}-\int_{\frac1b}^{\frac1a}\left(\frac1a-t\right)^{\alpha-1}(g\circ h)(t)(f\circ h)(t)dt\\&=\left(\int_{\frac1b}^{\frac1a}\left(\frac1a-s\right)^{\alpha-1}(g\circ h)(s)ds\right)f(a)-\int_{\frac1b}^{\frac1a}\left(\frac1a-t\right)^{\alpha-1}(g\circ h)(t)(f\circ h)(t)dt\\&=\Gamma(\alpha)\left[f(a)J^{\alpha}_{1/b+}(g\circ h)(1/a)-J^{\alpha}_{1/b+}(fg\circ h)(1/a)\right]\\&=\Gamma(\alpha)\left[\frac{f(a)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]-J^{\alpha}_{1/b+}(fg\circ h)(1/a)\right].\end{aligned}\tag{12}$$

Similarly we have

$$\begin{aligned}
I_2 &= \left(\int_t^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(s)ds \right)(f\circ h)(t)\Bigg|_{\frac{1}{b}}^{\frac{1}{a}} - \int_{\frac{1}{b}}^{\frac{1}{a}} \left(t-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(t)(f\circ h)(t)dt \\
&= -\left(\int_{\frac{1}{b}}^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(s)ds \right) f(b) + \int_{\frac{1}{b}}^{\frac{1}{a}} \left(t-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(t)(f\circ h)(t)dt \\
&= \Gamma(\alpha)\left[-f(b)J^{\alpha}_{1/a-}(g\circ h)(1/b) + J^{\alpha}_{1/a-}(fg\circ h)(1/b) \right] \\
&= \Gamma(\alpha)\left[-\frac{f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a) + J^{\alpha}_{1/a-}(g\circ h)(1/b) \right] + J^{\alpha}_{1/a-}(fg\circ h)(1/b) \right]. \qquad (13)
\end{aligned}$$

A combination of (11), (12) and (13) gives

$$I = I_1 - I_2 = \Gamma(\alpha)\left\{ \left(\frac{f(a)+f(b)}{2} \right) \begin{bmatrix} J^{\alpha}_{1/b+}(g\circ h)(1/a) \\ +J^{\alpha}_{1/a-}(g\circ h)(1/b) \end{bmatrix} - \begin{bmatrix} J^{\alpha}_{1/b+}(fg\circ h)(1/a) \\ +J^{\alpha}_{1/a-}(fg\circ h)(1/b) \end{bmatrix} \right\}. \qquad (14)$$

Multiplying both sides of (14) by $(\Gamma(\alpha))^{-1}$ we have (10). This completes the proof.

Remark. In Lemma 3, if one takes $g(x)=1$, then equality (10) becomes equality in [10, Lemma 3].

Theorem 6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ and $a<b$. If $|f'|$ is harmonically convex on $[a,b]$, $g : [a,b] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$, then the following inequality for fractional integrals holds.*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a) + J^{\alpha}_{1/a-}(g\circ h)(1/b) \right] - \left[J^{\alpha}_{1/b+}(fg\circ h)(1/a) + J^{\alpha}_{1/a-}(fg\circ h)(1/b) \right] \right| \\
&\leq \frac{\|g\|_{\infty} ab(b-a)}{\Gamma(\alpha+1)} \left(\frac{b-a}{ab} \right)^{\alpha} \left[C_1(\alpha)\left|f'(a)\right| + C_2(\alpha)\left|f'(b)\right| \right] \qquad (15)
\end{aligned}$$

where

$$C_1(\alpha) = \begin{bmatrix} \frac{b^{-2}}{\alpha+2}\, {}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\, {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right) \\ +\frac{2(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\, {}_2F_1\left(2,\alpha+1;\alpha+3;\frac{b-a}{b+a}\right) \end{bmatrix},$$

$$C_2(\alpha) = \begin{bmatrix} \frac{b^{-2}}{(\alpha+1)(\alpha+2)}\, {}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{\alpha+2}\, {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right) \\ +\left(\frac{a+b}{2}\right)^{-2} \frac{1}{\alpha+1}\, {}_2F_1\left(2,\alpha+1;\alpha+2;\frac{b-a}{b+a}\right) \\ -\frac{2(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\, {}_2F_1\left(2,\alpha+1;\alpha+3;\frac{b-a}{b+a}\right) \end{bmatrix}$$

with $0<\alpha\leq 1$ and $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. From Lemma 3 we have

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a) + J^{\alpha}_{1/a-}(g\circ h)(1/b) \right] - \left[J^{\alpha}_{1/b+}(fg\circ h)(1/a) + J^{\alpha}_{1/a-}(fg\circ h)(1/b) \right] \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{b}}^{\frac{1}{a}} \left| \int_{\frac{1}{b}}^{t} \left(\frac{1}{a}-s\right)^{\alpha-1} (g\circ h)(s)ds - \int_t^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(s)ds \right| \left|(f\circ h)'(t)\right| dt. \qquad (16)
\end{aligned}$$

Since g is harmonically symmetric with respect to $2ab/a+b$, using Definition 3 we have $g\left(\frac{1}{x}\right)=g(\frac{1}{(\frac{1}{a})+(\frac{1}{b})-x})$, for all $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

$$\begin{aligned}
&\left|\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}(g\circ h)(s)ds-\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right| \\
&=\left|\int_{\frac{1}{a}+\frac{1}{b}-t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds+\int_{\frac{1}{a}}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right| \\
&=\left|\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right| \\
&\leq\begin{cases}\int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\ ,\ t\in\left[\frac{1}{b},\frac{a+b}{2ab}\right]\\[2ex] \int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\ ,\ t\in\left[\frac{a+b}{2ab},\frac{1}{a}\right]\end{cases}.
\end{aligned}\tag{17}$$

If we use (17) in (16), we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}(g\circ h)(1/a)+J_{1/a-}^{\alpha}(g\circ h)(1/b)\right]\right. \\
&\left.-\left[J_{1/b+}^{\alpha}(fg\circ h)(1/a)+J_{1/a-}^{\alpha}(fg\circ h)(1/b)\right]\right| \\
&\leq\frac{1}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\right)\left|(f\circ h)'(t)\right|dt\\[2ex] +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\right)\left|(f\circ h)'(t)\right|dt\end{array}\right] \\
&\leq\frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\frac{1}{t^{2}}\left|f'(\frac{1}{t})\right|dt\right. \\
&\left.+\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\frac{1}{t^{2}}\left|f'(\frac{1}{t})\right|dt\right].
\end{aligned}$$

Setting $t=\frac{ub+(1-u)a}{ab}$, and $dt=\left(\frac{b-a}{ab}\right)du$ gives

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}(g\circ h)(1/a)+J_{1/a-}^{\alpha}(g\circ h)(1/b)\right]\right. \\
&\left.-\left[J_{1/b+}^{\alpha}(fg\circ h)(1/a)+J_{1/a-}^{\alpha}(fg\circ h)(1/b)\right]\right| \\
&\leq\frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{l}\int_{0}^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^{2}}\left|f'(\frac{ab}{ub+(1-u)a})\right|du\\[2ex] +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^{2}}\left|f'(\frac{ab}{ub+(1-u)a})\right|du\end{array}\right].
\end{aligned}\tag{18}$$

Since $|f'|$ is harmonically convex on $[a,b]$, we have

$$\left|f'(\frac{ab}{ub+(1-u)a})\right|\leq u\left|f'(a)\right|+(1-u)\left|f'(b)\right|.\tag{19}$$

If we use (19) in (18), we have

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]-\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\right|$$
$$\leq \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c}\left[\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}udu\right]|f'(a)| \\ +\left[\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}(1-u)du+\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}(1-u)du\right]|f'(b)|\end{array}\right]. \tag{20}$$

Using Lemma 1, we have

$$\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}udu$$
$$=\int_0^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}udu+2\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}udu$$
$$=\int_0^1\frac{u^{\alpha+1}}{(ub+(1-u)a)^2}du-\int_0^1\frac{u(1-u)^{\alpha}}{(ub+(1-u)a)^2}du+2\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}udu$$
$$\leq\int_0^1\frac{u^{\alpha+1}}{(ub+(1-u)a)^2}du-\int_0^1\frac{u(1-u)^{\alpha}}{(ub+(1-u)a)^2}du+2\int_0^{\frac{1}{2}}\frac{(1-2u)^{\alpha}}{(ub+(1-u)a)^2}udu \tag{21}$$

and

$$\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}(1-u)du+\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}(1-u)du$$
$$=\int_0^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}(1-u)du+2\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}(1-u)du$$
$$\leq\int_0^1\frac{u(1-u)^{\alpha}}{(ua+(1-u)b)^2}du-\int_0^1\frac{u^{\alpha+1}}{(ua+(1-u)b)^2}du$$
$$+\int_0^1\frac{(1-u)^{\alpha}}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^2}du-\frac{1}{2}\int_0^1\frac{u(1-u)^{\alpha}}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^2}du. \tag{22}$$

Calculating following integrals, we have

$$\int_0^1\frac{(1-u)^{\alpha+1}}{(ua+(1-u)b)^2}du-\int_0^1\frac{(1-u)u^{\alpha}}{(ua+(1-u)b)^2}du+\frac{1}{2}\int_0^1\frac{u(1-u)^{\alpha}}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^2}du$$
$$=\int_0^1(1-u)^{\alpha+1}b^{-2}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}du-\int_0^1(1-u)u^{\alpha}b^{-2}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}du$$
$$+\frac{1}{2}\int_0^1(1-v)v^{\alpha}\left(\frac{a+b}{2}\right)^{-2}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}dv=\left[\begin{array}{c}\frac{b^{-2}}{\alpha+2}\,{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right) \\ +\frac{2(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;\frac{b-a}{b+a}\right)\end{array}\right]=C_1(\alpha) \tag{23}$$

and

$$\int_0^1 \frac{u(1-u)^\alpha}{(ua+(1-u)b)^2}du-\int_0^1 \frac{u^{\alpha+1}}{(ua+(1-u)b)^2}du+\int_0^1 \frac{(1-u)^\alpha}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^2}du-\frac{1}{2}\int_0^1 \frac{u(1-u)^\alpha}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^2}du$$

$$=\int_0^1 \frac{u(1-u)^\alpha}{(ua+(1-u)b)^2}du-\int_0^1 \frac{u^{\alpha+1}}{(ua+(1-u)b)^2}du+\left(\frac{a+b}{2}\right)^{-2}\int_0^1 v^\alpha\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}dv$$

$$-\frac{1}{2}\left(\frac{a+b}{2}\right)^{-2}\int_0^1 (1-v)v^\alpha\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}dv=\left[\begin{array}{c}\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)\\ -\frac{b^{-2}}{\alpha+2}\,{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)\\ +\left(\frac{a+b}{2}\right)^{-2}\frac{1}{\alpha+1}\,{}_2F_1\left(2,\alpha+1;\alpha+2;\frac{b-a}{b+a}\right)\\ -\frac{2(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;\frac{b-a}{b+a}\right)\end{array}\right]=C_2(\alpha). \quad (24)$$

If we use (21), (22), (23) and (24) in (20), we have (15). This completes the proof.

Corollary 1. *In Theorem 6, one has the following.*

(1) *If one takes $\alpha=1$, one has the following Hermite-Hadamard-Fejér inequality for harmonically convex functions which is related the right-hand side of (6):*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right|\le\frac{\|g\|_\infty(b-a)^2}{2}\left[C_1(1)\left|f'(a)\right|+C_2(1)\left|f'(b)\right|\right],$$

(2) *If one takes $g(x)=1$, one has the following Hermite-Hadamard type inequality for harmonically convex function in fractional integral forms which is related the right-hand side of (5):*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^\alpha\left\{\begin{array}{c}J^\alpha_{1/a-}(f\circ h)(1/b)\\ +J^\alpha_{1/b+}(f\circ h)(1/a)\end{array}\right\}\right|\le\frac{ab(b-a)}{2}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(b)\right|\right],$$

(3) *If one takes $\alpha=1$ and $g(x)=1$, one has the following Hermite-Hadamard type inequality for harmonically convex function which is related the right-hand side of (4):*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\le\frac{ab(b-a)}{2}\left[C_1(1)\left|f'(a)\right|+C_2(1)\left|f'(b)\right|\right].$$

Theorem 7. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $|f'|^q, q\ge 1$, is harmonically convex on $[a,b]$, $g:[a,b]\to\mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2}\left[J^\alpha_{1/b+}(g\circ h)(1/a)+J^\alpha_{1/a-}(g\circ h)(1/b)\right]-\left[J^\alpha_{1/b+}(fg\circ h)(1/a)+J^\alpha_{1/a-}(fg\circ h)(1/b)\right]\right|$$

$$\le\frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha\left[C_3^{1-\frac{1}{q}}(\alpha)\left[\left(\begin{array}{c}C_4(\alpha)|f'(a)|^q\\ +C_5(\alpha)|f'(b)|^q\end{array}\right)\right]^{\frac{1}{q}}+C_6^{1-\frac{1}{q}}(\alpha)\left[\left(\begin{array}{c}C_7(\alpha)|f'(a)|^q\\ +C_8(\alpha)|f'(b)|^q\end{array}\right)\right]^{\frac{1}{q}}\right] \quad (25)$$

where

$$C_3(\alpha)=\frac{2(a+b)^{-2}}{(\alpha+1)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;\tfrac{b-a}{b+a}\right),$$

$$C_4(\alpha)=\frac{(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;\tfrac{b-a}{b+a}\right),$$

$$C_5(\alpha)=C_3(\alpha)-C_4(\alpha),$$

$$C_6(\alpha)=\left[\begin{array}{c}\frac{b^{-2}}{(\alpha+1)}\,{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)\\ -\frac{b^{-2}}{(\alpha+1)}\,{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)+C_3(\alpha)\end{array}\right],$$

$$C_7(\alpha)=\left[\begin{array}{c}\frac{b^{-2}}{(\alpha+2)}\,{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right)\\ -\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)+C_4(\alpha)\end{array}\right],$$

$$C_8(\alpha)=C_6(\alpha)-C_7(\alpha),$$

with $0<\alpha\le 1$ *and* $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. By using power mean inequality and the harmonically convexity of $|f'|^q$ in (18), we have

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]-\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\right|$$

$$\le\frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c}\int_0^{\frac12}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du\\ +\int_{\frac12}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du\end{array}\right]$$

$$\le\frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\left(\int_0^{\frac12}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac1q}\left(\int_0^{\frac12}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|^q du\right)^{\frac1q}\right.$$

$$\left.+\left(\int_{\frac12}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac1q}\left(\int_{\frac12}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|^q du\right)^{\frac1q}\right]$$

$$\le\frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\left(\int_0^{\frac12}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac1q}\left(\int_0^{\frac12}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left[u\left|f'(a)\right|^q+(1-u)\left|f'(b)\right|^q\right]du\right)^{\frac1q}\right.$$

$$\left.+\left(\int_{\frac12}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac1q}\left(\int_{\frac12}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left[u\left|f'(a)\right|^q+(1-u)\left|f'(b)\right|^q\right]du\right)^{\frac1q}\right]$$

$$\le\frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}$$

$$\times\left[\begin{array}{c}\left(\int_0^{\frac12}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac1q}\left(\int_0^{\frac12}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}u\,du\,|f'(a)|^q\int_0^{\frac12}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}(1-u)\,du\,|f'(b)|^q\right)^{\frac1q}\\ +\left(\int_{\frac12}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac1q}\left(\int_{\frac12}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}u\,du\,|f'(a)|^q\int_{\frac12}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}(1-u)\,du\,|f'(b)|^q\right)^{\frac1q}\end{array}\right]\tag{26}$$

Calculating following integrals by Lemma 1, we have

$$\int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}du \le \int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub+(1-u)a)^2}du = \frac{1}{2}\int_0^1 \frac{(1-u)^\alpha}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^2}du$$
$$= 2(a+b)^{-2}\int_0^1 v^\alpha \left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv$$
$$= \frac{2(a+b)^{-2}}{(\alpha+1)}\, {}_2F_1\left(2,\alpha+1;\alpha+2;\tfrac{b-a}{b+a}\right) = C_3(\alpha), \tag{27}$$

$$\int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}udu \le \int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub+(1-u)a)^2}udu = \frac{1}{4}\int_0^1 \frac{u(1-u)^\alpha}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^2}du$$
$$= (a+b)^{-2}\int_0^1 (1-v)v^\alpha \left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2} du$$
$$= \frac{(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\, {}_2F_1\left(2,\alpha+1;\alpha+3;\tfrac{b-a}{b+a}\right) = C_4(\alpha), \tag{28}$$

$$\int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}(1-u)du \le C_3(\alpha) - C_4(\alpha) = C_5(\alpha), \tag{29}$$

$$\int_{\frac{1}{2}}^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}du = \int_0^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}du + \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}du$$
$$\le \left[\begin{array}{c} \frac{b^{-2}}{(\alpha+1)}\, {}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)}\, {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) + C_3(\alpha) \end{array}\right] = C_6(\alpha), \tag{30}$$

$$\int_{\frac{1}{2}}^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}udu = \int_0^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}udu + \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}udu$$
$$\le \left[\begin{array}{c} \frac{b^{-2}}{(\alpha+2)}\, {}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\, {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right) + C_4(\alpha) \end{array}\right] = C_7(\alpha), \tag{31}$$

$$\int_{\frac{1}{2}}^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}(1-u)du \le C_6(\alpha) - C_7(\alpha) = C_8(\alpha). \tag{32}$$

If we use $(27-32)$ in (26), we have (25).This completes the proof.

Corollary 2. *In Theorem 7, one has the following.*

(1) *If one takes $\alpha = 1$, one has the following Hermite-Hadamard-Fejér inequality for harmonically convex functions which is related the right-hand side of (6):*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right| \le \frac{\|g\|_\infty (b-a)^2}{2}$$
$$\times \left[C_3^{1-\frac{1}{q}}(1)\left[\left(\begin{array}{c} C_4(1)|f'(a)|^q \\ +C_5(1)|f'(b)|^q \end{array}\right)\right]^{\frac{1}{q}} + C_6^{1-\frac{1}{q}}(1)\left[\left(\begin{array}{c} C_7(1)|f'(a)|^q \\ +C_8(1)|f'(b)|^q \end{array}\right)\right]^{\frac{1}{q}}\right],$$

(2) *If one takes $g(x)=1$, one has the following Hermite-Hadamard type inequality for harmonically convex function in fractional integral forms which is related the right-hand side of (5):*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{1/a-}(f\circ h)(1/b)+J^{\alpha}_{1/b+}(f\circ h)(1/a)\right\}\right|\leq\frac{ab(b-a)}{2}$$
$$\times\left[C_3^{1-\frac{1}{q}}(\alpha)\left[\begin{pmatrix}C_4(\alpha)|f'(a)|^q\\+C_5(\alpha)|f'(b)|^q\end{pmatrix}\right]^{\frac{1}{q}}+C_6^{1-\frac{1}{q}}(\alpha)\left[\begin{pmatrix}C_7(\alpha)|f'(a)|^q\\+C_8(\alpha)|f'(b)|^q\end{pmatrix}\right]^{\frac{1}{q}}\right],$$

(3) *If one takes $\alpha=1$ and $g(x)=1$, one has the following Hermite-Hadamard type inequality for harmonically convex function which is related the right-hand side of (4):*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab(b-a)}{2}$$
$$\times\left[C_3^{1-\frac{1}{q}}(1)\left[\begin{pmatrix}C_4(1)|f'(a)|^q\\+C_5(1)|f'(b)|^q\end{pmatrix}\right]^{\frac{1}{q}}+C_6^{1-\frac{1}{q}}(1)\left[\begin{pmatrix}C_7(1)|f'(a)|^q\\+C_8(1)|f'(b)|^q\end{pmatrix}\right]^{\frac{1}{q}}\right].$$

We can state another inequality for $q>1$ as follows.

Theorem 8. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $|f'|^q, q>1$, is harmonically convex on $[a,b]$, $g:[a,b]\to\mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]-\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\right|$$
$$\leq\frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[C_9^{\frac{1}{p}}(\alpha)\left[\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right]^{\frac{1}{q}}+C_{10}^{\frac{1}{p}}(\alpha)\left[\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right]^{\frac{1}{q}}\right]\tag{33}$$

where

$$C_9(\alpha)=\left(\frac{a+b}{2}\right)^{-2p}\frac{1}{2(\alpha p+1)}\,{}_2F_1\left(2p,\alpha p+1;\alpha p+2;\tfrac{b-a}{b+a}\right),$$
$$C_{10}(\alpha)=b^{-2p}\frac{1}{2(\alpha p+1)}\,{}_2F_1\left(2p,1;\alpha p+2;\tfrac{1}{2}\left(1-\tfrac{a}{b}\right)\right),$$

with $0<\alpha\leq1$, $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$ and $1/p+1/q=1$.

Proof. Using (18), Hölder's inequality and the harmonically convexity of $|f'|^q$, we have

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]-\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\right|$$
$$\leq\frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du\right.$$
$$\left.+\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du\right]$$

$$\leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha\left[\left(\int_0^{\frac{1}{2}}\frac{\left[(1-u)^\alpha-u^\alpha\right]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|^q du\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^1\frac{\left[u^\alpha-(1-u)^\alpha\right]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|^q du\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha\left[\left(\int_0^{\frac{1}{2}}\frac{\left[(1-u)^\alpha-u^\alpha\right]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}u\left|f'(a)\right|^q+(1-u)\left|f'(b)\right|^q du\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^1\frac{\left[u^\alpha-(1-u)^\alpha\right]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1u\left|f'(a)\right|^q+(1-u)\left|f'(b)\right|^q du\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\|g\|_\infty ab(b-a)}{2\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha\times\left[\left(\int_0^{\frac{1}{2}}\frac{\left[(1-u)^\alpha-u^\alpha\right]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right]^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^1\frac{\left[u^\alpha-(1-u)^\alpha\right]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left[\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right]^{\frac{1}{q}}\right] \tag{34}$$

Calculating following integrals by Lemma 1, we have

$$\int_0^{\frac{1}{2}}\frac{\left[(1-u)^\alpha-u^\alpha\right]^p}{(ub+(1-u)a)^{2p}}du\leq\int_0^{\frac{1}{2}}\frac{(1-2u)^{\alpha p}}{(ub+(1-u)a)^{2p}}du=\frac{1}{2}\int_0^1\frac{(1-u)^{\alpha p}}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^{2p}}du$$

$$=\frac{1}{2}\int_0^1 v^{\alpha p}\left(\frac{a+b}{2}\right)^{-2p}\left[1-v\left(\frac{b-a}{b+a}\right)\right]^{-2p}dv$$

$$=\left(\frac{a+b}{2}\right)^{-2p}\frac{1}{2(\alpha p+1)}\,{}_2F_1\left(2p,\alpha p+1;\alpha p+2;\tfrac{b-a}{b+a}\right)=C_9(\alpha), \tag{35}$$

and similarly

$$\int_{\frac{1}{2}}^1\frac{\left[u^\alpha-(1-u)^\alpha\right]^p}{(ub+(1-u)a)^{2p}}du\leq\int_{\frac{1}{2}}^1\frac{(2u-1)^{\alpha p}}{(ub+(1-u)a)^{2p}}du=\int_0^{\frac{1}{2}}\frac{(1-2u)^{\alpha p}}{(ua+(1-u)b)^{2p}}du$$

$$=\frac{1}{2}\int_0^1\frac{(1-v)^{\alpha p}}{\left(\frac{v}{2}a+(1-\frac{v}{2})b\right)^{2p}}dv=\frac{1}{2}\int_0^1(1-v)^{\alpha p}b^{-2p}\left(1-\frac{v}{2}\left(1-\frac{a}{b}\right)\right)^{-2p}dv$$

$$=b^{-2p}\frac{1}{2(\alpha p+1)}\,{}_2F_1\left(2p,1;\alpha p+2;\tfrac{1}{2}\left(1-\tfrac{a}{b}\right)\right)=C_{10}(\alpha). \tag{36}$$

If we use (35) and (36) in (34), we have (33).This completes the proof.

Corollary 3. *In Theorem 8, one has the following.*

(1) *If one takes $\alpha=1$, one has the following Hermite-Hadamard-Fejér inequality for harmonically convex functions which is related the right-hand side of (6):*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right|\leq\frac{\|g\|_\infty(b-a)^2}{2}\left[\begin{array}{l}C_9^{\frac{1}{p}}(1)\left[\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right]^{\frac{1}{q}}\\+C_{10}^{\frac{1}{p}}(1)\left[\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right]^{\frac{1}{q}}\end{array}\right],$$

(2) *If one takes* $g(x)=1$*, one has the following Hermite-Hadamard type inequality for harmonically convex function in fractional integral forms which is related the right-hand side of* (5)*:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{1/a-}(f\circ h)(1/b)+J^{\alpha}_{1/b+}(f\circ h)(1/a)\right\}\right|\leq\frac{ab(b-a)}{2}$$
$$\times\left[C_9^{\frac{1}{p}}(\alpha)\left[\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right]^{\frac{1}{q}}+C_{10}^{\frac{1}{p}}(\alpha)\left[\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right]^{\frac{1}{q}}\right],$$

(3) *If one takes* $\alpha=1$ *and* $g(x)=1$*,one has the following Hermite-Hadamard type inequality for harmonically convex function which is related the right-hand side of* (4)*:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab(b-a)}{2}\left[\begin{array}{c}C_9^{\frac{1}{p}}(1)\left[\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right]^{\frac{1}{q}}\\+C_{10}^{\frac{1}{p}}(1)\left[\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right]^{\frac{1}{q}}\end{array}\right].$$

3 Conclusions

In this paper, Hermite-Hadamard-Fejer type inequalities for harmonically convex functions in fractional integral forms are given. Also, an integral identity and some trapezoidal Hermite-Hadamard-Fejer type integral inequalities for harmonically convex functions in fractional integral forms are obtained.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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