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Numerical studies for solving system of linear fractional integro-differential equations by using least squares method and shifted Chebyshev polynomials

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Abstract: In this paper, a new numerical method for solving a linear system of fractional integro-differential equations is presented. The fractional derivative is considered in the Caputo sense. The proposed method is least squares method aid of shifted Chebyshev polynomials of the third kind. The suggested method reduces this type of systems to the solution of system of linear algebraic equations. To demonstrate the accuracy and applicability of the presented method some test examples are provided. Numerical results show that this approach is easy to implement and accurate when applied to integro-differential equations. We show that the solutions approach to classical solutions as the order of the fractional derivatives approach.

Keywords: Linear system of fractional Fredholm integro-differential equations; Caputo fractional derivative; Least squares method; Chebyshev Polynomials.

1 Introduction

Many problems can be models by fractional integro-differential equations from various sciences and engineering applications [12]. Furthermore most problems cannot be solved analytically, and hence finding good approximate solutions, using numerical methods, will be very helpful. Recently, several numerical methods to solve system of fractional differential equations (SFDEs) and fractional integro-differential equations (FIDEs) have been given. The authors in ([3], [21]) applied collocation method for solving the following: nonlinear fractional Langevin equation involving two fractional orders in different intervals and fractional Fredholm integro-differential equations. Chebyshev polynomials method is introduced in ([4], [5], [7]) for solving multi-term fractional orders differential equations and nonlinear Volterra and Fredholm integro-differential equations of fractional order, respectively. The authors in [8] applied variational iteration method for solving fractional integro-differential equations with the nonlocal boundary conditions. Adomian decomposition method is introduced in ([9], [11], [15], [16]) to solve fractional integro-differential equations. References ([13], [14]) used homotopy perturbation method for solving nonlinear Fredholm integro-differential equations of fractional order and system of linear Fredholm fractional integro-differential equations. Taylor series method is introduced in [1] for solving linear fractional integro-differential equations of Volterra type. In [23] numerical solution of fractional integro-differential equations by least squares method and shifted Laguerre polynomials pseudo-spectral method. Considerable attention has been given to the solutions of fractional differential equations (FDEs) and integral equations of physical interest ([2], [8], [10], [16], [20]). Most non-linear FDEs do not have exact analytic solutions, so approximate and numerical techniques ([17]-[19]) must be used. Many mathematical

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problems in science and engineering are set in unbounded domains. There is a need to consider practical design and implementation issues in scientific computing for reliable and efficient solutions of these problems. Several numerical methods to solve the FDEs have been given such as variational iteration method [8], homotopy perturbation method ([13], [18]) and homotopy analysis method [6].

In this paper, least squares method with aid of shifted Chebyshev polynomials of the third kind method is applied to solving system of fractional integro-differential equations. Least squares method has been studied in [22].

In this paper, we present numerical solution of the system of integro-differential equation with fractional derivative of the type [16]:

$$D^{\alpha}y_i(x) = f_i(x) + \int_0^1 k_i(x,t)\left(\sum_{k=1}^{n}\alpha_{ik}y_k(t)\right)dt, \quad i=1,2,\cdots,n, \quad 0\le x,t\le 1, \tag{1}$$

with initial conditions

$$y_i^{(j)}(x_0) = y_{ij} \quad i=1,2,\cdots,n,$$

where $D^{\alpha}y_i(x)$ indicates the αth Caputo fractional derivative of $y_i(x)$. $f_i(x)$ and $k_i(x,t)$ are given functions, x,t are real varying in the interval $[0,1]$ and $y_i(x)$ is the unknown functions to be determined.

2 Preliminaries and notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

Definition 1. *The Caputo fractional derivative operator D^{v} of order v is defined in the following form:*

$$D^{v}f(x) = \frac{1}{\Gamma(m-v)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{v-m+1}}\,dt, \quad x>0,$$

where, $m-1<v\le m, \quad m\in\mathbb{N}.$
Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$D^{v}(\lambda f(x)+\mu g(x)) = \lambda D^{v}f(x)+\mu D^{v}g(x),$$

where, λ and μ are constants. For the Caputo's derivative we have [12]:

$$D^{v}C = 0, \qquad C \text{ is a constant}, \tag{2}$$

$$D^{v}x^{n} = \begin{cases} 0, & \text{for } n\in\mathbb{N}_0 \text{ and } n<\lceil v\rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-v)}x^{n-v} & \text{for } n\in\mathbb{N}_0 \text{ and } n\ge\lceil v\rceil. \end{cases} \tag{3}$$

We use the ceiling function $\lceil v\rceil$ to denote the smallest integer greater than or equal to v, and $\mathbb{N}_0=\{0,1,2,\cdots\}$. Recall that for $v\in\mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

3 Some properties of Chebyshev polynomials of the third kind

3.1 Chebyshev polynomials of the third kind

The Chebyshev polynomials $V_n(x)$ of the third kind are orthogonal polynomials of degree n in x defined on the interval $[-1,1]$ [19],

$$V_n = \frac{\cos(n+\frac{1}{2})\Theta}{\cos(\frac{\Theta}{2})},$$

where, $x = \cos\Theta$ and $\Theta \in [0,\pi]$. They can be obtained explicitly using the Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$, for the special case $\beta = -\alpha = \frac{1}{2}$. These are given by:

$$V_k(x) = \frac{2^{2k} P_k^{(-1/2,1/2)}(x)(\Gamma(k+1))^2}{\Gamma(2k+1)}. \tag{4}$$

Also, these polynomials $V_n(x)$ are orthogonal on $[-1,1]$ with respect to the inner product:

$$\langle V_n(x), V_m(x)\rangle = \int_0^1 \sqrt{\frac{1+x}{1-x}}\, V_n(x)V_m(x)dx = \begin{cases} \pi, & \text{for } n=m; \\ 0, & \text{for } n \neq m, \end{cases} \tag{5}$$

where $\sqrt{\frac{1+x}{1-x}}$ is a weight function corresponding to $V_n(x)$. The polynomials $V_n(x)$ may be generated by using the recurrence relations

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x), \qquad V_0(x) = 1, \qquad V_1(x) = 2x-1, \qquad n = 1,2,\cdots.$$

The analytical form of the Chebyshev polynomials of the third kind $V_n(x)$ of degree n, using Eq.(4) and properties of Jacobi polynomials are given as:

$$V_n(x) = \sum_{k=0}^{[\frac{2n+1}{2}]} (-1)^k (2)^{n-k} \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)} (x+1)^{n-k}, \quad n \in \mathbb{Z}^+, \tag{6}$$

where $[\frac{2n+1}{2}]$ denotes the integer part of $\frac{(2n+1)}{2}$.

3.2 The shifted Chebyshev polynomials of the third kind

In order to use these polynomials on the interval $[0,1]$, we define the so called shifted Chebyshev polynomials of the third kind by introducing the change of variable $s = 2x-1$ [19]. The shifted Chebyshev polynomials of the third kind are defined as $V_n^*(x) = V_n(2x-1)$.

These polynomials are orthogonal on the support interval $[0,1]$ as the following inner product:

$$\langle V_n^*(x), V_m^*(x)\rangle = \int_0^1 \sqrt{\frac{x}{1-x}}\, V_n^*(x)V_m^*(x)dx = \begin{cases} \frac{\pi}{2}, & \text{for } n=m; \\ 0, & \text{for } n \neq m, \end{cases} \tag{7}$$

where $\sqrt{\frac{x}{1-x}}$ is weight function corresponding to $V_n^*(x)$, and normalized by the requirement that $V_n^*(1)=1$. The polynomials $V_n^*(x)$ may be generated by using the recurrence relations

$$V_{n+1}^*(x)=2(2x-1)V_n^*(x)-V_{n-1}^*(x),\quad V_0^*(x)=1,\quad V_1^*(x)=4x-3,\quad n=1,2,\cdots.$$

The analytically form of the shifted Chebyshev polynomials of the third kind $V_n^*(x)$ of degree n in x is given by:

$$V_n^*(x)=\sum_{k=0}^{n}(-1)^k(2)^{2n-2k}\frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)}(x)^{n-k},\qquad n\in\mathbb{Z}^+. \tag{8}$$

In a spectral method, in contrast, the function $g(x)$, square integrable in $[0,1]$ is represented by an infinite expansion of the shifted Chebyshev polynomials of the third kind as follows:

$$g(x)=\sum_{i=0}^{\infty}b_iV_i^*(x), \tag{9}$$

where b_i is a chosen sequence of prescribed basis functions. One then proceeds some how to estimate as many as possible of the coefficients b_i, thus approximating $g(x)$ by a finite sum of $(m+1)$-terms such as:

$$g_m(x)=\sum_{i=1}^{m}b_iV_i^*(x), \tag{10}$$

where the coefficients $b_i; i=0,1,\cdots$ are given by

$$b_i=\frac{2}{\pi}\int_0^1 g(x)V_i^*(x)\sqrt{\frac{x}{1-x}}\,dx. \tag{11}$$

Theorem 1. *(Chebyshev truncation theorem) [17] The error in approximating $g(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$g_m(x)=\sum_{i=0}^{m}b_iV_i(x), \tag{12}$$

then

$$E_T(m)\equiv\Big|g(x)-g_m(x)\Big|\le\sum_{k=m+1}^{\infty}|b_i|, \tag{13}$$

for all $g(x)$, all m, and all $x\in[-1,1]$.

4 Solution of system of linear fractional integro-differential equation

In this section, the least squares method with aid of shifted Chebyshev polynomial is applied to study the numerical solution of these systems of fractional integro-differential (1).

The method is based on approximating the unknown functions $y_i(x)$ as

$$y_i(x)=\sum_{j=0}^{m}a_j^iV_j^*(x),\qquad 0\le x\le 1, \tag{14}$$

where $V_j^*(x)$ is shifted Chebyshev polynomial of the third kind and a_j^i, $i=1,2,\cdots,n$, are constants.
Substituting (14) into (1), we obtain

$$D^{\alpha}\sum_{j=0}^{m} a_j^i V_j^*(x) = f_i(x) + \int_0^1 k_i(x,t)\left(\sum_{k=1}^{n}\alpha_{ik}\left[\sum_{j=0}^{m} a_j^i V_j^*(t)\right]\right)dt. \tag{15}$$

Hence the residual equation is defined as

$$R_i(x,a_0^i,a_1^i,\cdots,a_m^i) = \sum_{j=0}^{m} a_j^i D^{\alpha} V_j^*(x) - \int_0^1 k_i(x,t)\left(\sum_{k=1}^{n}\alpha_{ik}\left[\sum_{j=0}^{m} a_j^i V_j^*(t)\right]\right)dt - f_i(x). \tag{16}$$

Let

$$S_i(a_0^i,a_1^i,\cdots,a_m^i) = \int_0^1 [R_1(x,a_0^i,a_1^i,\cdots,a_m^i)]^2\, w(x)\, dx, \tag{17}$$

where $w(x)$ is the positive weight function defined on the interval $[0,1]$, in this work we take $w(x)=1$, then

$$\begin{aligned} &S_i(a_0^i,a_1^i,\cdots,a_m^i) = \\ &\int_0^1 \left\{\sum_{j=0}^{m} a_j^i D^{\alpha} V_j^*(x) - \int_0^1 k_i(x,t)\left(\sum_{k=1}^{n}\alpha_{ik}\left[\sum_{j=0}^{m} a_j^i V_j^*(t)\right]\right) dt - f_i(x)\right\}^2 dx. \end{aligned} \tag{18}$$

So finding the values of a_j^i, $j=0,1,\cdots,m$ which minimize S_i is equivalent to finding the best approximation for the solution of the SLFIDE (1).

The minimum value of S_i is obtained by setting

$$\frac{\partial S_i}{\partial a_j^i} = 0, \qquad j=0,1,\cdots,m, \tag{19}$$

$$\begin{aligned} &\int_0^1 \left\{\sum_{j=0}^{m} a_j^i D^{\alpha} V_j^*(x) - \int_0^1 k_i(x,t)\left[\sum_{k=1}^{n}\alpha_{ik}\sum_{j=0}^{m} a_j^i V_j^*(t)\right] dt - f_i(x)\right\}\times \\ &\left\{D^{\alpha} V_j^*(x) - \int_0^1 k_i(x,t)\left[\sum_{k=1}^{n}\alpha_{ik}\sum_{j=0}^{m} V_j^*(t)\right] dt\right\} dx = 0. \end{aligned} \tag{20}$$

By evaluating the above equation for $j=0,1,\cdots,n$ we can obtain a system of $(n+1)$ linear equations with $(n+1)$ unknown coefficients a_j^i. This system can be formed by using matrices form as follows:

$$A = \begin{pmatrix} \int_0^1 R_i(x,a_0^i)h_0^i dx & \int_0^1 R_i(x,a_1^i)h_0^i dx & \cdots & \int_0^1 R_i(x,a_n^i)h_0^i dx \\ \int_0^1 R_i(x,a_0^i)h_1^i dx & \int_0^1 R_i(x,a_1^i)h_1^i dx & \cdots & \int_0^1 R_i(x,a_n^i)h_1^i dx \\ \vdots & \vdots & \vdots & \vdots \\ \int_0^1 R_i(x,a_0^i)h_n^i dx & \int_0^1 R_i(x,a_1^i)h_n^i dx & \cdots & \int_0^1 R_i(x,a_n^i)h_n^i dx \end{pmatrix}, \tag{21}$$

$$B=\begin{pmatrix} \int_0^1 f_i(x)h_0^i dx \\ \\ \int_0^1 f_i(x)h_1^i dx \\ \\ \vdots \\ \\ \int_0^1 f_i(x)h_n^i dx \end{pmatrix}, \tag{22}$$

where

$$R_i(x,a_j^i)=\sum_{j=0}^{m} a_j^i D^{\alpha} V_j^*(x)-\int_0^1 k_i(x,t)\left[\sum_{k=1}^{n}\alpha_{ik}\sum_{j=0}^{m} a_j^i V_j^*(t)\right]dt, \tag{23}$$

$$h_j^i=D^{\alpha}V_j^*(x)-\int_0^1 k_i(x,t)\left[\sum_{k=1}^{n}\alpha_{ik}\sum_{j=0}^{m} V_j^*(t)\right]\ dt\ \ j=0,1,\cdots,m, i=1,2,\cdots,n. \tag{24}$$

By solving the above system we obtain the values of the unknown coefficients and the approximate solutions of (1).

5 Numerical examples

In this section, we have applied shifted Chebyshev polynomials of the third kind for solving system of linear fractional integro-differential equations with known exact solution. All results are obtained by using Mathematics Programming 10.

Example 1. Consider the following system of fractional integro-differential equations [16]

$$\begin{aligned} D^{\frac{2}{3}}y_1(x)&=\frac{-x}{6}+\frac{3x^{\frac{1}{3}}}{\Gamma(\frac{1}{3})}+\int_0^1 2xt[y_1(t)+y_2(t)]dt,\\ D^{\frac{2}{3}}y_2(x)&=\frac{5x^3}{6}+\frac{9x^{\frac{4}{3}}}{2\Gamma(\frac{1}{3})}+\int_0^1 x^3[y_1(t)-y_2(t)]dt. \end{aligned} \tag{25}$$

Subject to initial conditions $y_1(0)=-1$, $y_2(0)=0$ with the exact solution $y_1(x)=x-1$, $y_2(x)=x^2$.
Applying the least squares method with aid of shifted Chebyshev polynomials collocation of third kind $V_j^*(x)$, $j=0,1,\cdots,n$, at $n=4$ to system of the linear fractional integro-differential equation (25). The numerical results are showing in figure 1 and we obtain a system of linear equations with unknown coefficients a_j^i, $j=0,1,\cdots,m$, $i=1,2,\cdots,n$. The solution obtained using the suggested method is in excellent agreement with the already exact solution and show that this approach can be solved the problem effectively. It is evident that the overall errors can be made smaller by adding new terms from the series (14). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the proposed technique. Also, from our numerical results, we can see that these solutions are in more accuracy of those obtained in [16].

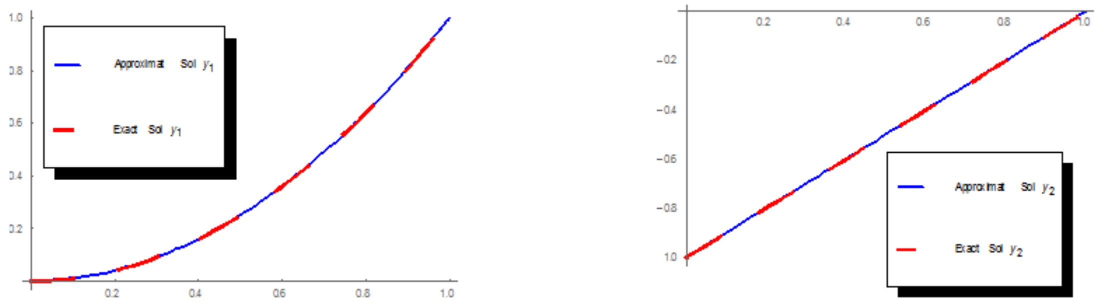

Fig. 1: Comparison between the approximate solution and the exact solution.

Example 2. Consider the following system of fractional integro-differential equations [16]

$$\begin{aligned} D^{\frac{3}{4}}y_1(x) &= -\frac{1}{20}-\frac{x}{12}+\frac{4x^{\frac{1}{4}}(15-23x^2)}{15\Gamma(\frac{1}{4})}+\int_0^1 (x+t)[y_1(t)+y_2(t)]dt, \\ D^{\frac{3}{4}}y_2(x) &= \frac{5x^3}{6}+\frac{9x^{\frac{4}{3}}}{2\Gamma(\frac{1}{3})}+\int_0^1 \sqrt{x}t^2[y_1(t)-y_2(t)]dt. \end{aligned} \tag{26}$$

Subject to to initial conditions $y_1(0)=0, y_2(0)=0$ with exact solution $y_1(x)=x-x^3, y_2(x)=x^2-x$.
Similarly, as in Example 5.1 applying the least squares method with aid of shifted Chebyshev polynomials collocation of third kind $V_j^*(x),\ j=0,1,\cdots,n$ at $n=4$ to the fractional integro-differential equation (26). The numerical results are showing figure 2 and we obtain the approximate solution which is the same the exact solution. The solution obtained using

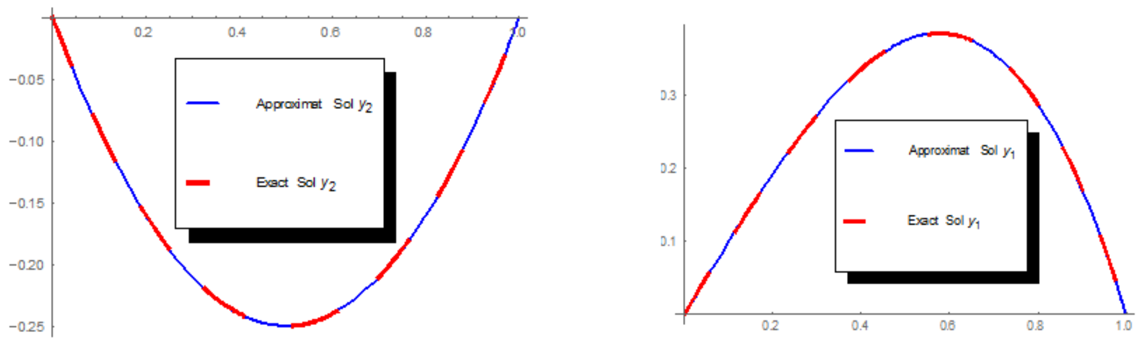

Fig. 2: Comparison between the approximate solution and the exact solution.

the suggested method is in excellent agreement with the already exact solution and show that this approach can be solved the problem effectively. It is evident that the overall errors can be made smaller by adding new terms from the series (14). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the proposed technique. Also, from our numerical results we can see that these solutions are in more accuracy of those obtained in [16].

Example 3. Consider the following system of fractional integro-differential equations [16]

$$
\begin{aligned}
D^{\frac{4}{5}}y_1(x) &= \frac{83x}{80} + \frac{25x^{\frac{6}{5}}(11+15x)}{33\Gamma(\frac{1}{5})} + \int_0^1 2xt[y_1(t)+y_2(t)]dt, \\
D^{\frac{4}{5}}y_2(x) &= \frac{5x^3}{6} + \frac{9x^{\frac{4}{3}}}{2\Gamma(\frac{1}{3})} + \int_0^1 (x+t)[y_1(t)-y_2(t)]dt.
\end{aligned} \tag{27}
$$

Subject to initial conditions $y_1(0)=0$, $y_2(0)=0$ with exact solution $y_1(x)=x^3-x^2$, $y_2(x)=\frac{15}{8}x^2$.
Similarly, as in examples 5.1 and 5.2 applying the least squares method with aid of shifted Chebyshev polynomials collocation of third kind $V_j^*(x)$, $j=0,1,\cdots,n$ at $n=4$ to the fractional integro-differential equation (27). The numerical results are showing in figure 3 and we obtain the approximation solution which is the same the exact solution.

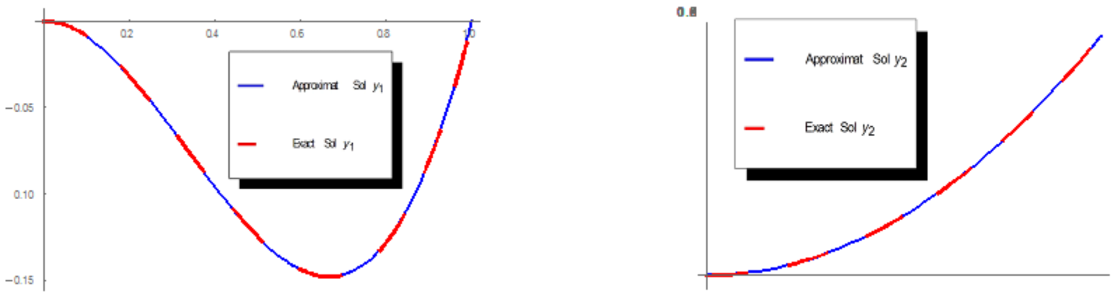

Fig. 3: Comparison between the approximate solution and the exact solution.

6 Conclusion and Remarks

In this article, we introduced an accurate numerical technique for solving system of linear fractional integro-differential equations. We have introduced an approximate formula for the Caputo fractional derivative of the shifted Chebyshev polynomials of the third kind in terms of classical shifted Chebyshev polynomials of the third kind. The results show that the proposed algorithm converges as the number of terms is increased. Some numerical examples are presented to illustrate the theoretical results and compared with the results obtained by other numerical methods. We have computed the numerical results using Mathematica programming 10.

References

[1] S. Ahmed and S. A. H. Salh, Generalized Taylor matrix method for solving linear integro-fractional differential equations of Volterra type, Applied Mathematical Sciences,vol 5 no 33-36, pp.(1765-1780), 2011.

[2] A. Arikoglu and I. Ozkol, Solution of fractional integro-dierential equations by using fractional diferential transform method, Chaos, Solitons and Fractals, 40(2), pp.(521-529), 2009.

[3] A. H. Bhrawy and M. A. Alghamdi, A shifted Jacobi-Gauss-Lobatto collocation method for solving nonlinear fractional Langevin equation involving two fractional orders in different intervals, Boundary Value Problems, 2012, article 62, 13 pages, 2012.

[4] A. H. Bhrawy and A. S. Alofi, The operational matrix of fractional integration for shifted Chebyshev polynomials, Applied Mathematics Letters, 26(1), pp.(25-31), 2013.

[5] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations, Applied Mathematical Modelling, 35(12), pp.(5662-5672), 2011.

[6] I. Hashim, O. Abdulaziz and S. Momani, Homotopy analysis method for fractional IVPs, Communications in Nonlinear Science and Numerical Simulations, 14, pp.(674-684), 2009.

[7] S. Irandoust-Pakchin, H. Kheiri, and S. Abdi-mazraeh, Chebyshev cardinal functions: an effective tool for solving nonlinear Volterra and Fredholm integro-differential equations of fractional order, Iranian Journal of Science and Technology Transaction A: Science, 37(1), pp.(53-62), 2013.

[8] S. Irandoust-Pakchin and S. Abdi-Mazraeh, Exact solutions for some of the fractional integro-differential equations with the nonlocal boundary conditions by using the modifcation of He's variational iteration method, International Journal of Advanced Mathematical Sciences, 1(3), pp.(139-144), 2013.

[9] H. Jafari and V. Daftardar-Gejji, Solving linear and non-linear fractional diffusion and wave equations by Adomian decomposition method, Appl. Math. and Comput., 180, pp.(488-497), 2006.

[10] K. Maleknejad, M. Shahrezaee, and H. Khatami, Numerical solution of integral equations system of the second kind by block pulse functions, Applied Mathematics and Computation, 166, pp.(15-24), 2005.

[11] R. C. Mittal and R. Nigam, Solution of fractional integrodifferential equations by Adomain decomposition method, International Journal of Applied Mathematics and Mechanics, 4(2), pp.(87-94), 2008.

[12] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[13] H. Saeedi and F. Samimi, He's homotopy perturbation method for nonlinear Ferdholm integro-differential equations of fractional order, International Journal of Engineering Research and Applications, 2(5), pp.(52-56), 2012.

[14] R. K. Saeed and H. M. Sdeq, Solving a system of linear F
redholm fractional integro-differential equations using homotopy perturbation method, Australian Journal of Basic and Applied Sciences, 4(4), pp.(633-638), 2010.

[15] S. Saha Ray, Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method, Communications in Nonlinear Science and Numerical Simulation, 14(4), pp.(1295-1306), 2009.

[16] M. H. Saleh, S. H. Mohamed, M. H. Ahmed and M. K. Marjan, System of linear fractional integro-differential equations by using Adomian decomposition method, International Journal of Computer Applications, 121(24), pp.(0975-8887), 2015.

[17] N. H. Sweilam and M. M. Khader, A Chebyshev pseudo-spectral method for solving fractional integro-diferential equations, ANZIAM, 51, pp.(464-475), 2010.

[18] N. H. Sweilam, M. M. Khader and R. F. Al-Bar, Homotopy perturbation method for linear and nonlinear system of fractional integro-diferential equations, International Journal of Computational Mathematics and Numerical Simulation, 1(1), pp.(73-87), 2008.

[19] N. H. Sweilam, A. M. Nagy and A. A. El-Sayed, On the numerical solution of space fractional order diffusion equation via shifted Chebyshev polynomials of the third kind, Journal of King Saud University Science, 28, pp.(41-47), 2016.

[20] F. Talay Akyildiz, Laguerre spectral approximation of Stokes; first problem for third-grade fluid, J. Comput. Math., 10, pp.(1029-1041), 2009.

[21] Y. Yang, Y. Chen, and Y. Huang, Spectral-collocation method for fractional Fredholm integro-differential equations, Journal of the Korean Mathematical Society, 51(1), pp.(203-224), 2014.

[22] D. Sh. Mohammed, Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomial, Mathematical Problems in Engineering, Volume 2014, Article ID 431965, 5 pages, 2014.

[23] A. M. S. Mahdy and R. T. Shwayye, Numerical solution of fractional integro-differential equations by least squares method and shifted Laguerre polynomials pseudo-spectral method, International Journal of Scientific & Engineering Research, 7(4), pp.(1589-1596), 2016.