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Coupled singular and non singular thermoelastic system and Double Laplace Decomposition method

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Abstract: In this paper, the double Laplace decomposition methods are applied to solve the non singular and singular one dimensional thermo-elasticity coupled system. The technique is described and illustrated with some examples.

Keywords: Double Laplace transform, inverse Laplace transform, nonlinear system partial differential equation, single Laplace transform,decomposition methods.

1 Introduction

The nonlinear one dimensional thermoelasticity coupled systems appear in many fields of science such as fluid mechanics, solid state physics and plasma physics. Thermoelasticity problems have gained a considerable attention for their importance and applications. Linear and nonlinear thermoelasticity provide a rich field of research for investigating the coupling between the thermal and the mechanical fields. The exact solutions for such system are difficult to find. Therefore, some numerical methods have been recently developed them analyticaly such as variational iteration method [1], Adomain's decomposition method [2,3], homotopy perturbation method [12,13,14] and iteration method [10,11] for finding analytical solutions of linear and nonlinear problems. The aim of this paper is to adopt the double Laplace transform and domain decomposition to obtain approximate solutions with high accuracy of singular and non singular one dimensional thermo-elasticity coupled system. For the illustration of our propesed method, two examples are given. The propesed technique is called modified double Laplace decomposition method and is performed by combining Laplace transform methods and decomposition methods see [6]. First of all, we recall the following definitions which are given in [9,7]. The double Laplace transform is defined as

$$L_xL_t\left[f(x,t)\right] = F(p,s) = \int_0^\infty e^{-px}\int_0^\infty e^{-st}f(x,t)dt\,dx,$$

where $x,t>0$ and p,s are complex values and further double Laplace transform of the first order partial derivatives is given by

$$L_xL_t\left[\frac{\partial f(x,t)}{\partial x}\right] = pF(p,s)-F(0,s).$$

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Similarly, the double Laplace transform for second partial derivative with respect to x and t is defined as follows

$$L_xL_t\left[\frac{\partial^2 f(x,t)}{\partial^2 x}\right] = p^2F(p,s) - pF(0,s) - \frac{\partial F(0,s)}{\partial x},$$
$$L_xL_t\left[\frac{\partial^2 f(x,t)}{\partial^2 t}\right] = s^2F(p,s) - sF(p,0) - \frac{\partial F(p,0)}{\partial t}.$$

2 Nonlinear one dimensional thermo-elasticity coupled system

In this section, we discuss the analytical solution of the regular nonlinear one dimensional thermo-elasticity coupled system [4,5]

$$\frac{\partial^2 u}{\partial t^2} - a\left(\frac{\partial u}{\partial x}, v\right)\frac{\partial^2 u}{\partial x^2} + b\left(\frac{\partial u}{\partial x}, v\right)\frac{\partial v}{\partial x} = f(x,t), \quad x \in \Omega \tag{1}$$

$$c\left(\frac{\partial u}{\partial x}, v\right)\frac{\partial v}{\partial t} + b\left(\frac{\partial u}{\partial x}, v\right)\frac{\partial^2 u}{\partial x \partial t} - d(v)\frac{\partial^2 v}{\partial x^2} = g(x,t), \tag{2}$$

with initial conditions

$$u(x,0) = f_1(x), \quad \frac{\partial u(x,0)}{\partial t} = f_2(x), \qquad v(x,0) = g_1(x), \tag{3}$$

where $u(x,t)$ and $v(x,t)$ are the body displacement from equilibrium and the displacement of the body temperature from reference $T_0 = 0$, subscripts denote partial derivatives, a, b, c and d are given smooth functions. For physical interpretation see [7]. Now let the us assume the following

$$a\left(\frac{\partial u}{\partial x}, v\right) = c\left(\frac{\partial u}{\partial x}, v\right) = d(v) = 1, \quad b\left(\frac{\partial u}{\partial x}, v\right) = \frac{\partial u}{\partial x}v,$$

using the above assumptions, the nonlinear system in Eq. (1) and Eq.(2) becomes

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + N_1(u,v) = f(x,t), \tag{4}$$

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + N_2(u,v) = g(x,t), \qquad t > 0 \tag{5}$$

where $N_1 = \left(\frac{\partial u}{\partial x}v\right)\frac{\partial v}{\partial x}$ and $N_2 = \left(\frac{\partial u}{\partial x}v\right)\frac{\partial^2 u}{\partial x \partial t}$ are nonlinear operators. In the following theorem we apply modified double Laplace decomposition methods.

Theorem 1. *We claim that the solution of the system given in Eq.(4), Eq.(5) and Eq.(3) is given by*

$$\sum_{n=0}^{\infty} u_n(x,t) = f_1(x) + t f_2(x) + L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s^2}\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2} - \sum_{n=0}^{\infty}A_n\right]\right], \tag{6}$$

$$\sum_{n=0}^{\infty} v_n(x,t) = g_1(x) + L_p^{-1}L_s^{-1}\left[\frac{G(p,s)}{s}\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial x^2} - \sum_{n=0}^{\infty}B_n\right]\right]. \tag{7}$$

or

$$u(x,t) = u_0 + u_1 + \ldots \text{ and } v(x,t) = v_0 + v_1 + \ldots,$$

where L_xL_t is the double Laplace transform with respect to x, t and $L_p^{-1}L_s^{-1}$ is the double inverse Laplace transform with respect to p, s. In addtion A_n and B_n are nonlinear terms. Here ,we provide double inverse Laplace transform with respect to p and s exist for each terms in the right hand side of Eq.(6), Eq.(7) .

Proof. By using the definition of partial derivatives of the double Laplace transform and single Laplace transform for Eq.(4), Eq.(5) and Eq.(3) respectively, we get

$$L_xL_t\left[u(x,t)\right]=\frac{F_1(p)}{s}+\frac{F_1(p)}{s^2}+\frac{F(p,s)}{s^2}+\frac{1}{s^2}L_xL_t\left[\frac{\partial^2 u}{\partial x^2}-N_1(u,v)\right],\tag{8}$$

$$L_xL_t\left[v(x,t)\right]=\frac{G_1(p)}{s}+\frac{G(p,s)}{s}+\frac{1}{s}L_xL_t\left[\frac{\partial^2 v}{\partial x^2}-N_2(u,v)\right].\tag{9}$$

The double Laplace a domain decomposition methods (DLADM) defines the solution of the system as $u(x,t)$ and $v(x,t)$ by an infinite series,

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t),\quad v(x,t)=\sum_{n=0}^{\infty}v_n(x,t).\tag{10}$$

The nonlinear operators can be defined as follows

$$N_1(u,v)=\sum_{n=0}^{\infty}A_n,\quad N_2(u,v)=\sum_{n=0}^{\infty}B_n,\tag{11}$$

where A_n and B_n are denoted by:

$$A_n=\frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_1\sum_{i=0}^{\infty}(\lambda^n u_n)\right]\right)_{\lambda=0},\quad B_n=\frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_2\sum_{i=0}^{\infty}(\lambda^n v_n)\right]\right)_{\lambda=0}.\tag{12}$$

Here, few terms of a domain's polynomials A_n and B_n are given by:

$$\begin{aligned}A_0&=u_{0x}v_0v_{0x}\\A_1&=(u_{0x}v_0)v_{1x}+(u_{0x}v_1)v_{0x}+(u_{1x}v_0)v_{0x}\\A_3&=(u_{1x}v_1)v_{1x}+(u_{0x}v_1)v_{2x}+(u_{0x}v_1)v_{1x}+(u_{0x}v_2)v_{0x}+(u_{1x}v1)v_{0x}+(u_{2x}v_0)v_{0x}\end{aligned}\tag{13}$$

and

$$\begin{aligned}B_0&=u_{0x}v_0v_{0tx}\\B_1&=(u_{0x}v_0)v_{1tx}+(u_{0x}v_1)v_{0tx}+(u_{1x}v_0)v_{0tx}\\B_3&=(u_{1x}v_1)v_{1xt}+(u_{0x}v_1)v_{2xt}+(u_{0x}v_1)v_{1xt}+(u_{0x}v_2)v_{0xt}+(u_{1x}v1)v_{0xt}+(u_{2x}v_0)v_{0xt}.\end{aligned}\tag{14}$$

By applying double inverse Laplace transform for Eq.(8) and Eq.(9) and use Eq.(10) and Eq.(11) we have

$$\sum_{n=0}^{\infty}u_n(x,t)=f_1(x)+t f_2(x)+L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s^2}\right]+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2}-\sum_{n=0}^{\infty}A_n\right]\right],\tag{15}$$

$$\sum_{n=0}^{\infty}v_n(x,t)=g_1(x)+L_p^{-1}L_s^{-1}\left[\frac{G(p,s)}{s}\right]+L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial x^2}-\sum_{n=0}^{\infty}B_n\right]\right].\tag{16}$$

in particular, we have

$$
\begin{aligned}
&u_0 = f_1(x) + t f_2(x), \; v_0 = g_1(x) \\
&v_1 = L_p^{-1} L_s^{-1}\left[\frac{G(p,s)}{s}\right] + L_p^{-1} L_s^{-1}\left[\frac{1}{s} L_x L_t\left[\sum_{n=0}^{\infty} \frac{\partial^2 v_n}{\partial x^2} - \sum_{n=0}^{\infty} B_n\right]\right] \\
&u_1 = L_p^{-1} L_s^{-1}\left[\frac{F(p,s)}{s^2}\right] + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2} L_x L_t\left[\frac{\partial^2 u_0}{\partial x^2} - A_0\right]\right],
\end{aligned}
\tag{17}
$$

and generally we have

$$
u_{n+1}(x,t) = L_p^{-1} L_s^{-1}\left[\frac{1}{s^2} L_x L_t\left[\sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2} - \sum_{n=1}^{\infty} A_n\right]\right] \tag{18}
$$

$$
v_{n+1}(x,t) = L_p^{-1} L_s^{-1}\left[\frac{1}{s} L_x L_t\left[\sum_{n=1}^{\infty} \frac{\partial^2 v_n}{\partial x^2} - \sum_{n=1}^{\infty} B_n\right]\right], \tag{19}
$$

by calculating the terms $u_0, u_1, \ldots$ and $v_0, v_1, \ldots$, we obtain the solution of the system as

$$
u(x,t) = u_0 + u_1 + \ldots \text{ and } v(x,t) = v_0 + v_1 + \ldots
$$

3 Applications

To validate our method for systems of nonlinear partial differential equations we consider some illustrated examples of nonlinear one dimensional thermo-elasticity coupled systems as follows:

Example 1. Consider the following nonlinear one dimensional thermo-elasticity coupled system:

$$
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x} v\right)\frac{\partial v}{\partial x} = -e^{-x+t}, \quad x \in \Omega \tag{20}
$$

$$
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial u}{\partial x} v\right)\frac{\partial^2 u}{\partial t \partial x} = -e^{x-t}, \quad t > 0 \tag{21}
$$

with initial conditions

$$
u(x,0) = e^x, \; \frac{\partial u(x,0)}{\partial t} = -e^x, \qquad v(x,0) = e^{-x}. \tag{22}
$$

By taking the double and single Laplace transform for Eq. (20),Eq. (21) and Eq.(22) respectively, we obtain

$$
U(p,s) = \frac{1}{s(p-1)} - \frac{1}{s^2(p-1)} - \frac{1}{s^2(p+1)(s-1)} + \frac{1}{s^2} L_x L_t\left[\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x} v\right)\frac{\partial v}{\partial x}\right] \tag{23}
$$

$$
V(p,s) = \frac{1}{s(p+1)} - \frac{1}{s(p-1)(s+1)} + \frac{1}{s} L_x L_t\left[\frac{\partial^2 v}{\partial x^2} - \left(\frac{\partial u}{\partial x} v\right)\frac{\partial^2 u}{\partial t \partial x}\right], \tag{24}
$$

On using double inverse Laplace transform, we have

$$
u(x,t) = e^x - te^x + te^{-x} - e^{-x+t} + e^{-x} + L_p^{-1} L_s^{-1}\left(\frac{1}{s^2} L_x L_t\left[\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x} v\right)\frac{\partial v}{\partial x}\right]\right) \tag{25}
$$

$$
v(x,t) = e^{-x} - e^x + e^{x-t} + L_p^{-1} L_s^{-1}\left(\frac{1}{s} L_x L_t\left[\frac{\partial^2 v}{\partial x^2} - \left(\frac{\partial u}{\partial x} v\right)\frac{\partial^2 u}{\partial t \partial x}\right]\right), \tag{26}
$$

by using Eq.(6) and Eq.(7) we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = e^x - te^x + te^{-x} - e^{-x+t} + e^{-x} + L_p^{-1} L_s^{-1} \left(\frac{1}{s^2} L_x L_t \left[\frac{\partial^2 u_n}{\partial x^2} - \sum_{n=0}^{\infty} A_n \right] \right) \tag{27}$$

$$\sum_{n=0}^{\infty} v_n(x,t) = e^{-x} - e^x + e^{x-t} + L_p^{-1} L_s^{-1} \left(\frac{1}{s} L_x L_t \left[\frac{\partial^2 v_n}{\partial x^2} - \sum_{n=0}^{\infty} B_n \right] \right), \tag{28}$$

where A_n and B_n are Adomain polynomials given by Eq.(13) and Eq.(14). By applying equations Eq.(17),Eq.(18) and Eq.(19), we have

$$\begin{aligned} u_0 &= e^x - te^x, \; v_0 = e^{-x} \\ u_1 &= te^{-x} - e^{-x+t} + e^{-x} + L_p^{-1} L_s^{-1} \left(\frac{1}{s^2} L_x L_t \left[\frac{\partial^2 u_0}{\partial x^2} - A_0 \right] \right) \\ &= te^{-x} - e^{-x+t} + e^{-x} + L_p^{-1} L_s^{-1} \left(\frac{1}{s^2} L_x L_t \left[e^x - te^x + e^{-x} - te^{-x} \right] \right) \\ u_1 &= te^{-x} - e^{-x+t} + e^{-x} + \frac{t^2}{2} e^x - \frac{t^3}{6} e^x + \frac{t^2}{2} e^{-x} - \frac{t^3}{6} e^{-x}, \end{aligned}$$

and

$$\begin{aligned} v_1 &= -e^x + e^{x-t} + L_p^{-1} L_s^{-1} \left(\frac{1}{s} L_x L_t \left[\frac{\partial^2 v_0}{\partial x^2} - B_0 \right] \right) \\ &= -e^x + e^{x-t} + L_p^{-1} L_s^{-1} \left(\frac{1}{s} L_x L_t \left[e^{-x} + e^x - te^x \right] \right) \\ &= e^{x-t} - e^x + te^{-x} + te^x - \frac{t^2}{2} e^x, \end{aligned}$$

the other components given by

$$u_{n+1}(x,t) = L_p^{-1} L_s^{-1} \left(\frac{1}{s^2} L_x L_t \left[\frac{\partial^2 u_n}{\partial x^2} - \sum_{n=1}^{\infty} A_n \right] \right) \tag{29}$$

$$v_{n+1}(x,t) = L_p^{-1} L_s^{-1} \left(\frac{1}{s} L_x L_t \left[\frac{\partial^2 v_n}{\partial x^2} - \sum_{n=1}^{\infty} B_n \right] \right). \tag{30}$$

Applying Eq. (29), Eq.(30), Eq.(13) and Eq.(14), we obtain

$$\begin{aligned} u_2 = &-\frac{t^3}{6} e^{-x} + te^{-x} - e^{-x+t} + e^{-x} + \frac{t^2}{2} e^{-x} + \frac{t^4}{4!} e^x - \frac{t^5}{5!} e^{-x} \\ &- \frac{2t^4}{4!} e^{-x} + \frac{t^3}{6} e^{-3x} + te^{-3x} - e^{-3x+t} + e^{-3x} + \frac{t^2}{2} e^{-3x} + \frac{t^4}{4!} e^{-3x} - \frac{t^5}{5!} e^{-3x}, \\ v_2 = &\, e^x - e^{x-t} + \frac{t^3}{6} e^x - te^x + \frac{t^2}{2} e^x + \frac{t^2}{2} e^{-x} - \frac{4t^3}{3!} e^{-x} - \frac{4t^4}{4!} e^x + \frac{4t^4}{4!} e^{-x} \\ &+ te^{-x+t} - e^{-x+t} + e^{-x} - \frac{t^3}{3!} e^{3x} + \frac{2t^2}{2} e^{3x} + \frac{t^3}{6} e^x + 3\frac{t^4}{4!} e^{3x} \\ &+ e^{3x} - e^{3x-t} - te^{3x} + e^{3x+t} - te^{3x+t} - e^{3x}, \end{aligned}$$

it is obvious that the self-cancelling terms appear between various components and connected by coming terms, as follows

$$u(x,t) = u_0 + u_1 + \ldots \text{ and } v(x,t) = v_0 + v_1 + \ldots,$$

therefore, the exact solution is given by

$$u(x,t) = e^{x-t}, \quad v(x,t) = e^{-x+t}.$$

Example 2. Consider the nonlinear coupled system one dimensional thermo-elasticity given by

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right) + \frac{\partial v}{\partial x} = 2x - 6x^2 - 2t^2 - 2, \;\; x \in \Omega \tag{31}$$

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial^2 u}{\partial t \partial x} = 2t^2 + 2t - 6x^2, \quad t > 0 \tag{32}$$

subject to

$$u(x,0) = x^2, \; \frac{\partial u(x,0)}{\partial t} = 0, \qquad v(x,0) = x^2. \tag{33}$$

By applying the modified double Laplace decomposition methods and the inverse double Laplace transform as in previous example one can obtain

$$u_0 = x^2, \, v_0 = x^2 \tag{34}$$

$$u_1 = xt^2 - 3x^2t^2 - \frac{1}{6}t^4 - t^2 + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[A_0 - \frac{\partial v_0}{\partial x}\right]\right] \tag{35}$$

$$v_1 = \frac{2}{3}t^3 + t^2 - 6x^2t + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[B_0 - \frac{\partial^2 u_0}{\partial t \partial x}\right]\right], \tag{36}$$

and

$$u_{n+1} = L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[\sum_{n=1}^{\infty} A_n - \frac{\partial v_n}{\partial x}\right]\right] \tag{37}$$

$$v_{n+1} = L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\sum_{n=1}^{\infty} B_n - \frac{\partial^2 u_n}{\partial t \partial x}\right]\right], \tag{38}$$

where A_n and B_n are Adomain's polynomials given by:

$$\begin{aligned} A_0 &= v_{0x}u_0 \\ A_1 &= v_{0x}u_1 + v_{1x}u_0 \\ A_3 &= v_{0x}u_2 + v_{1x}u_1 + v_{2x}u_0, \end{aligned} \tag{39}$$

and

$$\begin{aligned} B_0 &= u_{0x}v_0 \\ B_1 &= u_{0x}v_1 + u_{1x}v_0 \\ B_3 &= u_{0x}v_2 + u_{1x}v_1 + u_{2x}v_0. \end{aligned} \tag{40}$$

The other components of the solution can easily found by using above recursive relation and Eq. (39) and Eq.(40),

$$u_1 = -\frac{1}{6}t^4 - t^2$$
$$v_1 = \frac{2}{3}t^3 + t^2,$$

and

$$u_2 = \frac{1}{15}t^5 + \frac{1}{6}t^4$$
$$v_2 = -\frac{1}{15}t^5 - \frac{2}{3}t^3,$$

then

$$u(x,t) = u_0 + u_1 + \ldots \text{ and } v(x,t) = v_0 + v_1 + \ldots,$$

we get the following exact solution

$$u(x,t) = x^2 - t^2, \;\; v(x,t) = x^2 + t^2.$$

4 Linear singular one dimensional thermo-elasticity coupled system

In this part of the paper, we apply our technique to solve the linear singular one dimensional thermo-elasticity coupled system given below

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^2}\left(x^2\frac{\partial u}{\partial x}\right)_x + x\frac{\partial v}{\partial x} = f(x,t), \;\; x \in \Omega$$
$$\frac{\partial v}{\partial t} - \frac{1}{x^2}\left(x^2\frac{\partial v}{\partial x}\right)_x + x\frac{\partial^2 u}{\partial x \partial t} = g(x,t), \qquad t > 0 \tag{41}$$

subject to

$$u(x,0) = f_1(x), \;\; \frac{\partial u(x,0)}{\partial t} = f_2(x), \qquad v(x,0) = g_1(x), \tag{42}$$

where, $\frac{1}{x^2}\left(x^2\frac{\partial u}{\partial x}\right)_x$ and $\frac{1}{x^2}\left(x^2\frac{\partial v}{\partial x}\right)_x$ are called bessel operator, $f(x,t)$, $g(x,t)$, $f_1(x)$, $f_2(x)$ and $g_1(x)$ are known function. The following definition is used in this section,

Definition 1. *Double Laplace transform of the non constant coefficient second order partial derivative $x^2\frac{\partial^2 u}{\partial t^2}$ and the function $x^2 f(x,t)$ are given by*

$$\frac{d^2}{dp^2}\left[L_x L_t\left(\frac{\partial^2 u}{\partial t^2}\right)\right] = L_x L_t\left(x^2\frac{\partial^2 u}{\partial t^2}\right), \tag{43}$$

and

$$L_x L_t\left(x^2 f(x,t)\right) = \frac{d^2}{dp^2}\left[L_x L_t\left(f(x,t)\right)\right] = \frac{d^2 F(p,s)}{dp^2}. \tag{44}$$

To obtain the solution of Linear singular one dimensional thermo-elasticity coupled system of Eq.(41), multiplying Eq.(41) by x^2 and taking the double Laplace transform and single Laplace transform for Eq.(41) and Eq.(42) respectively and use definition 1, we get

$$\frac{d^2U(p,s)}{dp^2} = \frac{d^2F_1(p)}{sdp^2} + \frac{d^2F_2(p)}{s^2dp^2} + \frac{d^2F(p,s)}{s^2dp^2} + \frac{1}{s^2}L_xL_t\left[\left(x^2\frac{\partial u}{\partial x}\right)_x - x^3\frac{\partial v}{\partial x}\right] \tag{45}$$

and

$$\frac{d^2V(p,s)}{dp^2} = \frac{d^2G_1(p)}{sdp^2} + \frac{d^2G(p,s)}{sdp^2} + \frac{1}{s}L_xL_t\left[\left(x^2\frac{\partial v}{\partial x}\right)_x - x^3\frac{\partial^2 u}{\partial x\partial t}\right], \tag{46}$$

by integrating 2 time for both sides of Eq.(45) and Eq.(45) from 0 to p with respect to p, we have

$$U(p,s) = \int\int\left(\frac{d^2F_1(p)}{sdp^2} + \frac{d^2F_2(p)}{s^2dp^2} + \frac{d^2F(p,s)}{s^2dp^2}\right)dpdp\frac{1}{s^2}\int\int\left(L_xL_t\left[\left(x^2\frac{\partial u}{\partial x}\right)_x - x^3\frac{\partial v}{\partial x}\right]\right)dpdp, \tag{47}$$

and

$$V(p,s) = \int\int\left(\frac{d^2G_1(p)}{sdp^2} + \frac{d^2G(p,s)}{sdp^2}\right)dpdp + \frac{1}{s}\int\int L_xL_t\left[\left(x^2\frac{\partial v}{\partial x}\right)_x - x^3\frac{\partial^2 u}{\partial x\partial t}\right]dpdp. \tag{48}$$

The double Laplace A domain decomposition methods (DLADM) defines the solution of the system as $u(x,t)$ and $v(x,t)$ by the infinite series,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \tag{49}$$

By applying double inverse Laplace transform for Eq.(47) and Eq.(48) and use Eq.(49), we have

$$\begin{aligned} u(x,t) &= L_p^{-1}L_s^{-1}\left[\int\int\left(\frac{d^2F_1(p)}{sdp^2} + \frac{d^2F_2(p)}{s^2dp^2} + \frac{d^2F(p,s)}{s^2dp^2}\right)dpdp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int\int\left(L_xL_t\left[\left(x^2\frac{\partial u}{\partial x}\right)_x - x^3\frac{\partial v}{\partial x}\right]\right)dpdp\right] \end{aligned} \tag{50}$$

$$v(x,t) = L_p^{-1}L_s^{-1}\left[\int\int\left(\frac{d^2G_1(p)}{sdp^2} + \frac{d^2G(p,s)}{sdp^2}\right)dpdp\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int\int L_xL_t\left[\left(x^2\frac{\partial v}{\partial x}\right)_x - x^3\frac{\partial^2 u}{\partial x\partial t}\right]dpdp\right], \tag{51}$$

Using Eq.(49) into Eq.(50) and Eq.(51), one gets

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= L_p^{-1}L_s^{-1}\left[\int\int\left(\frac{d^2F_1(p)}{sdp^2} + \frac{d^2F_2(p)}{s^2dp^2} + \frac{d^2F(p,s)}{s^2dp^2}\right)dpdp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int\int L_xL_t\left(x^2\sum_{n=0}^{\infty} u_{nx}(x,t)\right)_x dpdp\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int\int L_xL_t\left[x^3\sum_{n=0}^{\infty} v_{nx}(x,t)\right]dpdp\right], \end{aligned} \tag{52}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} v_n(x,t) &= L_p^{-1}L_s^{-1}\left[\int\int\left(\frac{d^2G_1(p)}{sdp^2} + \frac{d^2G(p,s)}{sdp^2}\right)dpdp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int\int\left(L_xL_t\left[\left(x^2\sum_{n=0}^{\infty} v_{nx}(x,t)\right)_x\right]\right)dpdp\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int\left(L_xL_t\left[x^3\left(\sum_{n=0}^{\infty} u_{nxt}(x,t)\right)\right]\right)dpdp\right], \end{aligned} \tag{53}$$

in particular,

$$u_0(x,t) = L_p^{-1} L_s^{-1}\left[\int\int\left(\frac{d^2F_1(p)}{sdp^2}+\frac{d^2F_2(p)}{s^2dp^2}+\frac{d^2F(p,s)}{s^2dp^2}\right)dpdp\right]$$
$$v_0(x,t) = L_p^{-1} L_s^{-1}\left[\int\int\left(\frac{d^2G_1(p)}{sdp^2}+\frac{d^2G(p,s)}{sdp^2}\right)dpdp\right]. \tag{54}$$

Generally, we have

$$u_{n+1}(x,t) = L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int\int L_xL_t\left(x^2\sum_{n=0}^{\infty}u_{nx}(x,t)\right)_x dpdp\right] - L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int\int L_xL_t\left[x^3\sum_{n=0}^{\infty}v_{nx}(x,t)\right]dpdp\right], \tag{55}$$

$$v_{n+1}(x,t) = +L_p^{-1} L_s^{-1}\left[\frac{1}{s}\int\int\left(L_xL_t\left[\left(x^2\sum_{n=0}^{\infty}v_{nx}(x,t)\right)_x\right]\right)dpdp\right]$$
$$-L_p^{-1} L_s^{-1}\left[\frac{1}{s}\int\left(L_xL_t\left[x^3\left(\sum_{n=0}^{\infty}u_{nxt}(x,t)\right)\right]\right)dpdp\right], \tag{56}$$

we provide double inverse Laplace transform with respect to p and s exist for each terms in the right hand side of Eq. (54), Eq.(55) and Eq.(56), by calculate the terms $u_0, u_1, ..., u_n$ and $v_0, v_1, ..., v_n$, we obtain the solution of the system as follows:

$$u(x,t) = u_0 + u_1 + ... \text{ and } v(x,t) = v_0 + v_1 + ...$$

Example 3. Consider the following linear singular one dimensional thermo-elasticity coupled system

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^2}\left(x^2\frac{\partial u}{\partial x}\right)_x + x\frac{\partial v}{\partial x} = 2x^2t6 - 6t$$
$$\frac{\partial v}{\partial t} - \frac{1}{x^2}\left(x^2\frac{\partial v}{\partial x}\right)_x + x\frac{\partial^2 u}{\partial x\partial t} = 3x^2 - 6t, \quad t > 0 \tag{57}$$

subject to

$$u(x,0) = x^2, \ \frac{\partial u(x,0)}{\partial t} = x^2, \quad v(x,0) = 0. \tag{58}$$

By using modified double Laplace decomposition methods for Eq.(57), Eq.(58) and apply Eq.(50), Eq.(51) we have

$$u(x,t) = x^2 + x^2t + \frac{1}{3}x^2t^3 - 3t^2 - t^3 + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[\left(x^2\frac{\partial u}{\partial x}\right)_x - x^3\frac{\partial v}{\partial x}\right]dpdp\right], \tag{59}$$

and

$$v(x,t) = 3x^2t - 3t^2 + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\int_0^p\left(L_xL_t\left[\left(x^2\frac{\partial v}{\partial x}\right)_x - x^3\frac{\partial^2 u}{\partial x\partial t}\right]\right)dpdp\right]. \tag{60}$$

On using Eq.(54), Eq.(55) and Eq.(56), we get

$$u_0(x,t) = x^2 + x^2t + \frac{1}{3}x^2t^3 - 3t^2 - t^3$$
$$v_0(x,t) = 3x^2t - 3t^2, \tag{61}$$

$$u_1(x,t) = 3t^2 + t^3 + \frac{1}{10}t^5 - x^2t^3$$
$$v_1(x,t) = 9t^2 - 2x^2t - \frac{2}{3}x^2t^3,$$

$$u_2(x,t) = -\frac{3}{10}t^5 + \frac{2}{3}x^2t^3 + \frac{1}{15}x^2t^5$$
$$v_2(x,t) = -6t^2 - t^4 + 2x^2t^3,$$

and

$$u_3(x,t) = \frac{1}{5}t^5 + \frac{1}{105}t^7 - \frac{1}{5}x^2t^5$$
$$v_3(x,t) = 3t^4 - \frac{4}{3}x^2t^3 - \frac{2}{5}x^2t^5,$$

therefore, the approximate solution is

$$u(x,t) = u_0 + u_1 + \ldots + u_n \text{ and } v(x,t) = v_0 + v_1 + \ldots + v_n,$$

the solution of the system given by

$$u(x,t) = x^2 + x^2t \text{ and } v(x,t) = x^2t.$$

5 Conclusion

In this paper, we proposed new modified double Laplace decomposition methods to solve linear regular and singular one dimensional singular one dimensional thermo-elasticity coupled system. The results obtained by double laplace decomposition method are compared with those of the exact solution, which shows very good agreement, even using only few terms of the recursive relations. This method can be applied to many complicated linear and non-linear PDEs.does not require linearization.

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