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Ostrowski type inequalities for p-convex functions

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Abstract: In this paper, we give a different version of the concept of p-convex functions and obtain some new properties of p-convex functions. Moreover we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are p-convex.

Keywords: p-convex function, Ostrowski type inequality, hypergeometric function.

1 Introduction

Let $f: I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in I° (the interior of I) and let $a,b \in I^\circ$ with $a<b$. If $|f'(x)| \le M$, for all $x \in [a,b]$, then the following inequality holds

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right] \tag{1}$$

for all $x \in [a,b]$. In the literature, the inequality (1) is known as Ostrowski inequality (see [18]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a,b]$. In [3,5,6,9,10,11], the reader can find generalizations, improvements and extensions for the inequality (1).
For $p \in \mathbb{R}$ the power mean $M_p(a,b)$ of order p of two positive numbers a and b is defined by

$$M_p = M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0 \\ \sqrt{ab}, & p = 0 \end{cases}.$$

It is well-known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b>0$ with $a \neq b$.

Let $L = L(a,b) = (b-a)/(\ln b - \ln a)$, $I = I(a,b) = \frac{1}{e}\left(a^a/b^b\right)^{1/a-b}$, $A = A(a,b) = (a+b)/2$, $G = G(a,b) = \sqrt{ab}$ and $H = H(a,b) = 2ab/(a+b)$ be the logarithmic, identric, arithmetic, geometric, and harmonic means of two positive real numbers a and b with $a \neq b$, respectively. Then

$$\min\{a,b\} < \quad H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) < I(a,b) < A(a,b) = M_1(a,b) < \max\{a,b\}.$$

Let $\mathfrak{M}$ be the family of all mean values of two numbers in $\mathbb{R}_+ = (0,\infty)$. Given $M,N \in \mathfrak{M}$, we say that a function $f: \mathbb{R}_+ \to \mathbb{R}_+$ is (M,N)-convex if $f(M(x,y)) \le N(f(x), f(y))$ for all $x,y \in \mathbb{R}_+$. The concept of (M,N)-convexity has been studied extensively in the literature from various points of view (see e.g. [1,4,12,15]).

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Let $A(a,b;t)=ta+(1-t)b$, $G(a,b;t)=a^{t}b^{1-t}$, $H(a,b;t)=ab/(ta+(1-t)b)$ and $M_{p}(a,b;t)=(ta^{p}+(1-t)b^{p})^{1/p}$ be the weighted arithmetic, weighted geometric, weighted harmonic , weighted power of order p means of two positive real numbers a and b with $a\neq b$ for $t\in[0,1]$, respectively. $M_{p}(a,b;t)$ is continuous and strictly increasing with respect to $t\in\mathbb{R}$ for fixed $p\in\mathbb{R}\backslash\{0\}$ and $a,b>0$ with $a>b$. See [8,14] for some kinds of convexity obtained by using weighted means.

In [8], the author gave definition Harmonically convex and concave functions as follow.

Definition 1. *Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\rightarrow\mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq tf(y)+(1-t)f(x) \tag{2}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality (2) is reversed, then f is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds for harmonically convex functions.

Theorem 1([8]). *Let $f:I\subset\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_{a}^{b}\frac{f(x)}{x^{2}}dx\leq\frac{f(a)+f(b)}{2}.$$

The above inequalities are sharp.

2 The Definition of p-convex Function

In [19], Zhang and Wan give the definition of p-convex function as follows:

Definition 2. *Let I be a p-convex set. A function $f:I\rightarrow\mathbb{R}$ is said to be a p-convex function or belongs to the class PC(I), if*

$$f\left(\left[tx^{p}+(1-t)y^{p}\right]^{1/p}\right)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in I$ and $t\in[0,1]$.

Remark.[19]. An interval I is said to be a p-convex set if $\left[tx^{p}+(1-t)y^{p}\right]^{1/p}\in I$ for all $x,y\in I$ and $t\in[0,1]$, where $p=2k+1$ or $p=n/m$, $n=2r+1$, $m=2t+1$ and $k,r,t\in\mathbb{N}$.

Remark. If $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\backslash\{0\}$, then

$$\left[tx^{p}+(1-t)y^{p}\right]^{1/p}\in I \text{ for all } x,y\in I \text{ and } t\in[0,1].$$

According to Remark 2, we can give a different version of the definition of p-convex function as follows:

Definition 3. *Let $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\backslash\{0\}$. A function $f:I\rightarrow\mathbb{R}$ is said to be a p-convex function, if*

$$f\left(\left[tx^{p}+(1-t)y^{p}\right]^{1/p}\right)\leq tf(x)+(1-t)f(y) \tag{3}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality (3) is reversed, then f is said to be p-concave.

According to Definition 3, It can be easily seen that for $p=1$ and $p=-1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I\subset(0,\infty)$, respectively.

Example 1. Let $f:(0,\infty)\to\mathbb{R},\ f(x)=x^p, p\neq 0$, and $g:(0,\infty)\to\mathbb{R},\ g(x)=c,\quad c\in\mathbb{R}$, then f and g are both p-convex and p-concave functions.

In [7, Theorem 5], if we take $I\subset(0,\infty)$, $h(t)=t$ and $p\in\mathbb{R}\setminus\{0\}$, then we have the following theorem.

Theorem 2. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a p-convex function, $p\in\mathbb{R}\setminus\{0\}$, and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then we have*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\leq\frac{f(a)+f(b)}{2}. \tag{4}$$

Remark. The inequalities (4) are sharp. Indeed we consider the function $f:(0,\infty)\to\mathbb{R},\ f(x)=1$. Thus

$$1=f\left([ta^p+(1-t)b^p]^{1/p}\right)=tf(y)+(1-t)f(x)=1$$

for all $x,y\in(0,\infty)$ and $t\in[0,1]$. Therefore f is p-convex on $(0,\infty)$. We also have

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)=1,\ \frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx=1,$$

and

$$\frac{f(a)+f(b)}{2}=1$$

which shows us that the inequalities (4) are sharp.

For some results related to p-convex functions and its generalizations, we refer the reader to see [7,**?**,**?**,17,19].

3 Main Results

Proposition 1. *Let $I\subset(0,\infty)$ be a real interval, $p\in\mathbb{R}\setminus\{0\}$ and $f:I\to\mathbb{R}$ is a function, then ;*

(1) *If $p\leq 1$ and f is convex and nondecreasing function then f is p-convex.*
(2) *If $p\geq 1$ and f is p-convex and nondecreasing function then f is convex.*
(3) *If $p\leq 1$ and f is p-concave and nondecreasing function then f is concave.*
(4) *If $p\geq 1$ and f is concave and nondecreasing function then f is p-concave.*
(5) *If $p\geq 1$ and f is convex and nonincreasing function then f is p-convex.*
(6) *If $p\leq 1$ and f is p-convex and nonincreasing function then f is convex.*
(7) *If $p\geq 1$ and f is p-concave and nonincreasing function then f is concave.*
(8) *If $p\leq 1$ and f is concave and nonincreasing function then f is p-concave.*

Proof. Since $g(x)=x^p,\ p\in(-\infty,0)\cup[1,\infty)$, is a convex function on $(0,\infty)$ and $g(x)=x^p,\ p\in(0,1]$, is a concave function on $(0,\infty)$,the proof is obvious from the following power mean inequalities

$$[tx^p+(1-t)y^p]^{1/p}\geq tx+(1-t)y,\ p\geq 1,$$

and

$$[tx^p+(1-t)y^p]^{1/p}\leq tx+(1-t)y,\ p\leq 1,$$

for all $x,y\in(0,\infty)$ and $t\in[0,1]$.

According to above Proposition, we can give the following examples for p-convex and p-concave functions.

Example 2. Let $f:(0,\infty)\to\mathbb{R},\ f(x)=x$, then f is p-convex function for $p\leq 1$ and f is p-concave function for $p\geq 1$.

Example 3. Let $f:(0,\infty)\to\mathbb{R},\ f(x)=x^{-p}$, $p\geq 1$, then f is p-convex function.

Example 4. Let $f:(0,\infty)\to\mathbb{R},\ f(x)=-\ln x$ and $p\geq 1$, then f is p-convex function.

Example 5. Let $f:(0,\infty)\to\mathbb{R},\ f(x)=\ln x$ and $p\geq 1$, then f is p-concave function.

The following proposition is obvious.

Proposition 2. *If $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ and if we consider the function $g:[a^p,b^p]\to\mathbb{R}$, defined by $g(t)=f\left(t^{1/p}\right)$, $p\in\mathbb{R}\backslash\{0\}$, then f is p-convex on $[a,b]$ if and only if g is convex on $[a^p,b^p]$, $p>0$ (or $[b^p,a^p]$, $p<0$).*

Remark. According to Proposition 2, as examples of p-convex functions we can take $f(t)=g(t^p)$, $p\in\mathbb{R}\backslash\{0\}$, where g is any convex function on $[a^p,b^p]$. Thus, we can obtain the inequality (4) in a different manner as follows:

If f is a is p-convex on $[a,b]$ then we write the Hermite-Hadamard inequality for the convex function $g(t)=f\left(t^{1/p}\right)$ on the closed interval $[a^p,b^p]$ as follows

$$g\left(\frac{a^p+b^p}{2}\right)\leq\frac{1}{b^p-a^p}\int_{a^p}^{b^p}g(t)dt\leq\frac{g(a^p)+g(b^p)}{2}$$

that is equivalent to

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq\frac{1}{b^p-a^p}\int_{a^p}^{b^p}f(t^{1/p})dt\leq\frac{f(a)+f(b)}{2}. \tag{5}$$

Using the change of variable $x=t^{1/p}$, then

$$\int_{a^p}^{b^p}f(t^{1/p})dt=p\int_a^b\frac{f(x)}{x^{1-p}}dx$$

and we get the inequality (4) by using the inequality (5).

Lemma 1. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and $a,b\in I$ with $a<b$ and $p\in\mathbb{R}\backslash\{0\}$. If $f'\in L[a,b]$ then*

$$f(x)-\frac{p}{b^p-a^p}\int_a^b\frac{f(u)}{u^{1-p}}du=\frac{1}{p\left(b^p-a^p\right)}\left\{(x^p-a^p)^2\int_0^1\frac{t}{(tx^p+(1-t)a^p)^{1-1/p}}f'\left([tx^p+(1-t)a^p]^{1/p}\right)dt\right.$$
$$\left.-\,(b^p-x^p)^2\int_0^1\frac{t}{(tx^p+(1-t)b^p)^{1-1/p}}f'\left([tx^p+(1-t)b^p]^{1/p}\right)dt\right\}.$$

Proof. Integrating by part and changing variables of integration yields

$$
\begin{aligned}
&\frac{1}{p\left(b^{p}-a^{p}\right)}\left\{\left(x^{p}-a^{p}\right)^{2}\int_{0}^{1}\frac{t}{\left(tx^{p}+(1-t)a^{p}\right)^{1-1/p}}f'\left(\left[tx^{p}+(1-t)a^{p}\right]^{1/p}\right)dt\right.\\
&\left.-\left(b^{p}-x^{p}\right)^{2}\int_{0}^{1}\frac{t}{\left(tx^{p}+(1-t)b^{p}\right)^{1-1/p}}f'\left(\left[tx^{p}+(1-t)b^{p}\right]^{1/p}\right)dt\right\}\\
&=\frac{1}{\left(b^{p}-a^{p}\right)}\left[\left(x^{p}-a^{p}\right)\int_{0}^{1}t\,d f\left(\left[tx^{p}+(1-t)a^{p}\right]^{1/p}\right)+\left(b^{p}-x^{p}\right)\int_{0}^{1}t\,d f\left(\left[tx^{p}+(1-t)b^{p}\right]^{1/p}\right)\right]\\
&=\frac{1}{\left(b^{p}-a^{p}\right)}\left[\left(x^{p}-a^{p}\right)\left\{t f\left(\left[tx^{p}+(1-t)a^{p}\right]^{1/p}\right)\Big|_{0}^{1}-\int_{0}^{1}f\left(\left[tx^{p}+(1-t)a^{p}\right]^{1/p}\right)dt\right\}\right]\\
&+\frac{1}{\left(b^{p}-a^{p}\right)}\left[\left(b^{p}-x^{p}\right)\left\{t f\left(\left[tx^{p}+(1-t)b^{p}\right]^{1/p}\right)\Big|_{0}^{1}-\int_{0}^{1}f\left(\left[tx^{p}+(1-t)b^{p}\right]^{1/p}\right)dt\right\}\right]\\
&=f(x)-\frac{p}{\left(b^{p}-a^{p}\right)}\int_{a}^{b}\frac{f(u)}{u^{1-p}}du.
\end{aligned}
$$

Lemma 2. *Let $0<a\leq x\leq b$, $p\in\mathbb{R}\setminus\{0\}$, $\lambda\geq 0,\mu\geq 0$ and $\eta\geq 1$. Then*

$$
\int_{0}^{1}\frac{t^{\lambda}(1-t)^{\mu}}{\left(tx^{p}+(1-t)a^{p}\right)^{\eta-\eta/p}}dt=C_{a,p}(x,\lambda,\mu,\eta)=\begin{cases}B_{a,p}(x,\lambda,\mu,\eta),p<0\\ A_{a,p}(x,\lambda,\mu,\eta),p>0\end{cases},
$$

$$
\int_{0}^{1}\frac{t^{\lambda}(1-t)^{\mu}}{\left(tx^{p}+(1-t)b^{p}\right)^{\eta-\eta/p}}dt=S_{b,p}(x,\lambda,\mu,\eta),=\begin{cases}T_{b,p}(x,\lambda,\mu,\eta),p<0\\ U_{b,p}(x,\lambda,\mu,\eta),p>0\end{cases},
$$

where

$$
\begin{aligned}
B_{a,p}(x,\lambda,\mu,\eta)&=\frac{\beta(\lambda+1,\mu+1)}{a^{\eta p-\eta}}\cdot{}_2F_1\left(\eta-\eta/p,\lambda+1;\lambda+\mu+2;1-\left(\frac{x}{a}\right)^{p}\right),\\
A_{a,p}(x,\lambda,\mu,\eta)&=\frac{\beta(\mu+1,\lambda+1)}{x^{\eta p-\eta}}\cdot{}_2F_1\left(\eta-\eta/p,\mu+1;\lambda+\mu+2;1-\left(\frac{a}{x}\right)^{p}\right),\\
T_{b,p}(x,\lambda,\mu,\eta)&=\frac{\beta(\mu+1,\lambda+1)}{x^{\eta p-\eta}}\cdot{}_2F_1\left(\eta-\eta/p,\mu+1;\lambda+\mu+2;1-\left(\frac{b}{x}\right)^{p}\right),\\
U_{b,p}(x,\lambda,\mu,\eta)&=\frac{\beta(\lambda+1,\mu+1)}{b^{\eta p-\eta}}\cdot{}_2F_1\left(\eta-\eta/p,\lambda+1;\lambda+\mu+2;1-\left(\frac{x}{b}\right)^{p}\right),
\end{aligned}
$$

β is Euler Beta function defined by

$$
\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_{0}^{1}t^{x-1}(1-t)^{y-1}dt,\;\;x,y>0,
$$

and ${}_2F_1$ is hypergeometric function defined by

$$ {}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\ c>b>0,\ |z|<1 \text{ (see [2]).} $$

Proof. (i) Let $p>0$. Then

$$ \begin{aligned} \int_0^1 \frac{t^{\lambda}(1-t)^{\mu}}{(tx^p+(1-t)a^p)^{\eta-\eta/p}}dt &= \frac{1}{x^{\eta p-\eta}}\int_0^1 \frac{t^{\mu}(1-t)^{\lambda}}{\left[1-t\left(1-\left(\frac{a}{x}\right)^p\right)\right]^{\eta-\eta/p}}dt \\ &= \frac{\beta(\mu+1,\lambda+1)}{x^{\eta p-\eta}}.{}_2F_1\left(\eta-\eta/p,\mu+1;\lambda+\mu+2;1-\left(\frac{a}{x}\right)^p\right) \end{aligned} $$

and

$$ \begin{aligned} \int_0^1 \frac{t^{\lambda}(1-t)^{\mu}}{(tx^p+(1-t)b^p)^{\eta-\eta/p}}dt &= \frac{1}{b^{\eta p-\eta}}\int_0^1 \frac{t^{\lambda}(1-t)^{\mu}}{\left[1-t\left(1-\left(\frac{x}{b}\right)^p\right)\right]^{\eta-\eta/p}}dt \\ &= \frac{\beta(\lambda+1,\mu+1)}{b^{\eta p-\eta}}.{}_2F_1\left(\eta-\eta/p,\lambda+1;\lambda+\mu+2;1-\left(\frac{x}{b}\right)^p\right). \end{aligned} $$

(ii) Let $p<0$. Then

$$ \begin{aligned} \int_0^1 \frac{t^{\lambda}(1-t)^{\mu}}{(tx^p+(1-t)a^p)^{\eta-\eta/p}}dt &= \frac{1}{a^{\eta p-\eta}}\int_0^1 \frac{t^{\lambda}(1-t)^{\mu}}{\left[1-t\left(1-\left(\frac{x}{a}\right)^p\right)\right]^{\eta-\eta/p}}dt \\ &= \frac{\beta(\lambda+1,\mu+1)}{a^{\eta p-\eta}}.{}_2F_1\left(\eta-\eta/p,\lambda+1;\lambda+\mu+2;1-\left(\frac{x}{a}\right)^p\right) \end{aligned} $$

and

$$ \begin{aligned} \int_0^1 \frac{t^{\lambda}(1-t)^{\mu}}{(tx^p+(1-t)b^p)^{\eta-\eta/p}}dt &= \frac{1}{x^{\eta p-\eta}}\int_0^1 \frac{t^{\mu}(1-t)^{\lambda}}{\left[1-t\left(1-\left(\frac{b}{x}\right)^p\right)\right]^{\eta-\eta/p}}dt \\ &= \frac{\beta(\mu+1,\lambda+1)}{x^{\eta p-\eta}}.{}_2F_1\left(\eta-\eta/p,\mu+1;\lambda+\mu+2;1-\left(\frac{b}{x}\right)^p\right). \end{aligned} $$

By using Lemma 1 and Lemma 2, we obtained the following some new Ostrowski type inequalities for p-convex functions.

Theorem 3. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^{\circ}$ with $a<b$, $p\in\mathbb{R}\setminus\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q\geq1$, then for all $x\in[a,b]$, we have*

$$ \begin{aligned} \left|f(x)-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| &\leq \frac{1}{p(b^p-a^p)} \\ &\times\left\{(x^p-a^p)^2 C_{a,p}^{1-1/q}(x,1,0,1)\left[C_{a,p}(x,2,0,1)\left|f'(x)\right|^q+C_{a,p}(x,1,1,1)\left|f'(a)\right|^q\right]^{1/q}\right. \\ &\left.+(b^p-x^p)^2 S_{b,p}^{1-1/q}(x,1,0,1)\left[S_{b,p}(x,2,0,1)\left|f'(x)\right|^q+S_{b,p}(x,1,1,1)\left|f'(b)\right|^q\right]^{1/q}\right\}. \end{aligned} \tag{6} $$

Proof. From Lemma 1, Power mean integral inequality and the p-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|f(x)-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| \\
&\le \frac{1}{p(b^p-a^p)}\left\{(x^p-a^p)^2\int_0^1 \frac{t}{(tx^p+(1-t)a^p)^{1-1/p}}\left|f'\left([tx^p+(1-t)a^p]^{1/p}\right)\right|dt\right. \\
&\left.+(b^p-x^p)^2\int_0^1 \frac{t}{(tx^p+(1-t)b^p)^{1-1/p}}\left|f'\left([tx^p+(1-t)b^p]^{1/p}\right)\right|dt\right\} \\
&\le \frac{(x^p-a^p)^2}{p(b^p-a^p)}\left(\int_0^1 \frac{t}{(tx^p+(1-t)a^p)^{1-1/p}}dt\right)^{1-1/q}\times\left(\int_0^1 \frac{t}{(tx^p+(1-t)a^p)^{1-1/p}}\left|f'\left((tx^p+(1-t)a^p)^{1-1/p}\right)\right|^q dt\right)^{1/q} \\
&+\frac{(b^p-x^p)^2}{p(b^p-a^p)}\left(\int_0^1 \frac{t}{(tx^p+(1-t)b^p)^{1-1/p}}dt\right)^{1-1/q}\times\left(\int_0^1 \frac{t}{(tx^p+(1-t)b^p)^{1-1/p}}\left|f'\left((tx^p+(1-t)b^p)^{1-1/p}\right)\right|^q dt\right)^{1/q} \\
&\le \frac{(x^p-a^p)^2}{p(b^p-a^p)}\left(\int_0^1 \frac{t}{(tx^p+(1-t)a^p)^{1-1/p}}dt\right)^{1-1/q}\times\left(\int_0^1 \frac{t^2|f'(x)|^q+t(1-t)|f'(a)|^q}{(tx^p+(1-t)a^p)^{1-1/p}}dt\right)^{1/q} \\
&+\frac{(b^p-x^p)^2}{p(b^p-a^p)}\left(\int_0^1 \frac{t}{(tx^p+(1-t)b^p)^{1-1/p}}dt\right)^{1-1/q}\times\left(\int_0^1 \frac{t^2|f'(x)|^q+t(1-t)|f'(b)|^q}{(tx^p+(1-t)b^p)^{1-1/p}}dt\right)^{1/q}. \qquad (7)
\end{aligned}$$

Hence, If we use (7) and the equalities in Lemma 2 , we obtain the desired result. This completes the proof.

Theorem 4. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$, $p \in \mathbb{R}\setminus\{0\}$ and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q \ge 1$, then for all $x \in [a,b]$, we have*

$$\begin{aligned}
\left|f(x)-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| &\le \frac{1}{p(b^p-a^p)}\left\{(x^p-a^p)^2 C_{a,p}^{1-1/q}(x,0,0,1)\right. \qquad (8) \\
&\times\left[C_{a,p}(x,q+1,0,1)|f'(x)|^q+C_{a,p}(x,q,1,1)|f'(a)|^q\right]^{1/q} \\
&\left.+(b^p-x^p)^2 S_{b,p}^{1-1/q}(x,0,0,1)\left[S_{b,p}(x,q+1,0,1)|f'(x)|^q+S_{b,p}(x,q,1,1)|f'(b)|^q\right]^{1/q}\right\}.
\end{aligned}$$

Proof. From Lemma 1 and Lemma 2, Power mean integral inequality and the p-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}
\left|f(x)-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| &\le \frac{1}{p(b^p-a^p)}\left\{(x^p-a^p)^2\int_0^1 \frac{t}{(tx^p+(1-t)a^p)^{1-1/p}}\left|f'\left([tx^p+(1-t)a^p]^{1/p}\right)\right|dt\right. \\
&\left.+(b^p-x^p)^2\int_0^1 \frac{t}{(tx^p+(1-t)b^p)^{1-1/p}}\left|f'\left([tx^p+(1-t)b^p]^{1/p}\right)\right|dt\right\}
\end{aligned}$$

$$\leq \frac{(x^p-a^p)^2}{p(b^p-a^p)}\left(\int_0^1 \frac{1}{(tx^p+(1-t)a^p)^{1-1/p}}dt\right)^{1-1/q} \times \left(\int_0^1 \frac{t^q}{(tx^p+(1-t)a^p)^{1-1/p}}\left|f'\left([tx^p+(1-t)a^p]^{1/p}\right)\right|^q dt\right)^{1/q}$$
$$+\frac{(b^p-x^p)^2}{p(b^p-a^p)}\left(\int_0^1 \frac{1}{(tx^p+(1-t)b^p)^{1-1/p}}dt\right)^{1-1/q} \times \left(\int_0^1 \frac{t^q}{(tx^p+(1-t)b^p)^{1-1/p}}\left|f'\left([tx^p+(1-t)a^p]^{1/p}\right)\right|^q dt\right)^{1/q}$$
$$\leq \frac{(x^p-a^p)^2}{p(b^p-a^p)}\left(\int_0^1 \frac{1}{(tx^p+(1-t)a^p)^{1-1/p}}dt\right)^{1-1/q} \times \left(\int_0^1 \frac{t^{q+1}|f'(x)|^q+t^q(1-t)|f'(a)|^q}{(tx^p+(1-t)a^p)^{1-1/p}}dt\right)^{1/q}$$
$$+\frac{(b^p-x^p)^2}{p(b^p-a^p)}\left(\int_0^1 \frac{1}{(tx^p+(1-t)b^p)^{1-1/p}}dt\right)^{1-1/q} \times \left(\int_0^1 \frac{t^{q+1}|f'(x)|^q+t^q(1-t)|f'(b)|^q}{(tx^p+(1-t)b^p)^{1-1/p}}dt\right)^{1/q}$$
$$\leq \frac{1}{p(b^p-a^p)}\left\{(x^p-a^p)^2 C_{a,p}^{1-1/q}(x,0,0,1)\times\left[C_{a,p}(x,q+1,0,1)\left|f'(x)\right|^q+C_{a,p}(x,q,1,1)\left|f'(a)\right|^q\right]^{1/q}\right.$$
$$\left.+(b^p-x^p)^2 S_{b,p}^{1-1/q}(x,0,0,1)\left[S_{b,p}(x,q+1,0,1)\left|f'(x)\right|^q+S_{b,p}(x,q,1,1)\left|f'(b)\right|^q\right]^{1/q}\right\}.$$

This completes the proof.

For $q\geq 1$, we can give the following result:

Corollary 1. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}\backslash\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q\geq 1$. If $|f'(x)|\leq M$, $x\in[a,b]$ then*

$$\left|f(x)-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| \leq \frac{M}{p(b^p-a^p)}\min\{I_1,I_2\}$$

where

$$I_1=\left\{(x^p-a^p)^2 C_{a,p}^{1-1/q}(x,1,0,1)\left[C_{a,p}(x,2,0,1)+C_{a,p}(x,1,1,1)\right]^{1/q}\right.$$
$$\left.+(b^p-x^p)^2 S_{b,p}^{1-1/q}(x,1,0,1)\left[S_{b,p}(x,2,0,1)+S_{b,p}(x,1,1,1)\right]^{1/q}\right\},$$
$$I_2=(x^p-a^p)^2 C_{a,p}^{1-1/q}(x,0,0,1)\left[C_{a,p}(x,q+1,0,1)+C_{a,p}(x,q,1,1)\right]^{1/q}$$
$$+(b^p-x^p)^2 S_{b,p}^{1-1/q}(x,0,0,1)\left[S_{b,p}(x,q+1,0,1)+S_{b,p}(x,q,1,1)\right]^{1/q}.$$

Theorem 5. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $r\in\mathbb{R}\backslash\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|f(x)-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| \leq \frac{1}{p(b^p-a^p)}\left(\frac{1}{q+2}\right)^{1/q}\left\{(x^p-a^p)^2 C_{a,p}^{1/r}(x,0,0,r)\right. \tag{9}$$
$$\times\left[\left|f'(x)\right|^q+\frac{1}{q+1}\left|f'(a)\right|^q\right]^{1/q}$$
$$\left.+(b^p-x^p)^2 S_{b,p}^{1/r}(x,0,0,r)\left[\left|f'(x)\right|^q+\frac{1}{q+1}\left|f'(b)\right|^q\right]^{1/q}\right\}.$$

Proof. From Lemma 1 and Lemma 2, Hölder's inequality and the p-convexity of $|f'|^q$ on $[a,b]$, we have

$$\left|f(x)-\frac{p}{b^p-a^p}\int_a^b\frac{f(u)}{u^{1-p}}du\right|\le\frac{1}{p(b^p-a^p)}$$
$$\times\left\{(x^p-a^p)^2\left(\int_0^1\frac{1}{(tx^p+(1-t)a^p)^{r-r/p}}dt\right)^{1/r}\left(\int_0^1 t^q\left|f'\left([tx^p+(1-t)a^p]^{1/p}\right)\right|^q dt\right)^{1/q}\right.$$
$$\left.+(b^p-x^p)^2\left(\int_0^1\frac{1}{(tx^p+(1-t)b^p)^{r-r/p}}dt\right)^{1/r}\left(\int_0^1 t^q\left|f'\left([tx^p+(1-t)b^p]^{1/p}\right)\right|^q dt\right)^{1/q}\right\}$$
$$\le\frac{1}{p(b^p-a^p)}\left(\frac{1}{q+2}\right)^{1/q}\left\{(x^p-a^p)^2C_{a,p}^{1/r}(x,0,0,r)\times\left[|f'(x)|^q+\frac{1}{q+1}|f'(a)|^q\right]^{1/q}\right.$$
$$\left.+(b^p-x^p)^2S_{b,p}^{1/r}(x,0,0,r)\left[|f'(x)|^q+\frac{1}{q+1}|f'(b)|^q\right]^{1/q}\right\}.$$

This completes the proof.

Theorem 6. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $r\in\mathbb{R}\setminus\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|f(x)-\frac{p}{b^p-a^p}\int_a^b\frac{f(u)}{u^{1-p}}du\right|\le\frac{1}{p(b^p-a^p)}\left\{(x^p-a^p)^2C_{a,p}^{1/r}(x,r,0,r)\times\left[\frac{|f'(x)|^q+|f'(a)|^q}{2}\right]^{1/q}\right.$$
$$\left.+(b^p-x^p)^2S_{b,p}^{1/r}(x,r,0,r)\left[\frac{|f'(x)|^q+|f'(b)|^q}{2}\right]^{1/q}\right\}.$$

Proof. From Lemma 1 and Lemma 2, Hölder's inequality and the p-convexity of $|f'|^q$ on $[a,b]$, we have

$$\left|f(x)-\frac{p}{b^p-a^p}\int_a^b\frac{f(u)}{u^{1-p}}du\right|\le\frac{1}{p(b^p-a^p)}$$
$$\times\left\{(x^p-a^p)^2\left(\int_0^1\frac{t^r}{(tx^p+(1-t)a^p)^{r-r/p}}dt\right)^{1/r}\left(\int_0^1\left|f'\left([tx^p+(1-t)a^p]^{1/p}\right)\right|^q dt\right)^{1/q}\right.$$
$$\left.+(b^p-x^p)^2\left(\int_0^1\frac{t^r}{(tx^p+(1-t)b^p)^{r-r/p}}dt\right)^{1/r}\left(\int_0^1\left|f'\left([tx^p+(1-t)b^p]^{1/p}\right)\right|^q dt\right)^{1/q}\right\}$$
$$\le\frac{1}{p(b^p-a^p)}\left\{(x^r-a^r)^2C_{a,p}^{1/r}(x,r,0,r)\times\left[\frac{|f'(x)|^q+|f'(a)|^q}{2}\right]^{1/q}\right.$$
$$\left.+(b^p-x^p)^2S_{b,p}^{1/r}(x,r,0,r)\left[\frac{|f'(x)|^q+|f'(b)|^q}{2}\right]^{1/q}\right\}.$$

This completes the proof.

Theorem 7. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$, $r \in \mathbb{R}\setminus\{0\}$ and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|f(x)-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| \le \frac{1}{p(b^p-a^p)}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left\{(x^p-a^p)^2\left[C_{a,p}(x,1,0,q)\left|f'(x)\right|^q+C_{a,p}(x,0,1,q)\left|f'(a)\right|^q\right]^{1/q}\right.$$
$$\left.+(b^p-x^p)^2\left[S_{b,p}(x,1,0,q)\left|f'(x)\right|^q+S_{b,p}(x,0,1,q)\left|f'(b)\right|^q\right]^{1/q}\right\}.$$

Proof. From Lemma 1 and Lemma 2, Hölder's inequality and the p-convexity of $|f'|^q$ on $[a,b]$, we have

$$\left|f(x)-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| \le \frac{1}{p(b^p-a^p)}$$
$$\times\left\{(x^p-a^p)^2\left(\int_0^1 t^r dt\right)^{1/r}\left(\int_0^1 \frac{\left|f'\left(\left[tx^p+(1-t)a^p\right]^{1/p}\right)\right|^q}{(tx^p+(1-t)a^p)^{q-q/p}}dt\right)^{1/q}\right.$$
$$\left.+(b^p-x^p)^2\left(\int_0^1 t^r dt\right)^{1/r}\left(\int_0^1 \frac{\left|f'\left(\left[tx^p+(1-t)b^p\right]^{1/p}\right)\right|^q}{(tx^p+(1-t)b^p)^{q-q/p}}dt\right)^{1/q}\right\}$$
$$\le \frac{1}{p(b^p-a^p)}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left\{(x^p-a^p)^2\left[C_{a,p}(x,1,0,q)\left|f'(x)\right|^q+C_{a,p}(x,0,1,q)\left|f'(a)\right|^q\right]^{1/q}\right.$$
$$\left.+(b^p-x^p)^2\left[S_{b,p}(x,1,0,q)\left|f'(x)\right|^q+S_{b,p}(x,0,1,q)\left|f'(b)\right|^q\right]^{1/q}\right\}.$$

This completes the proof.

For $q>1$, we can give the following result:

Corollary 2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$, $r \in \mathbb{R}\setminus\{0\}$ and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, if $|f'(x)|\le M$, $x\in[a,b]$ then*

$$\left|f(x)-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| \le \frac{M}{p(b^p-a^p)}\min\{J_1,J_2,J_3\} \tag{10}$$

where

$$J_1=\left(\frac{1}{q+1}\right)^{1/q}\left\{(x^p-a^p)^2C_{a,p}^{1/r}(x,0,0,r)+(b^p-x^p)^2S_{b,p}^{1/r}(x,0,0,r)\right\},$$
$$J_2=(x^p-a^p)^2C_{a,p}^{1/r}(x,r,0,r)+(b^p-x^p)^2S_{b,p}^{1/r}(x,r,0,r),$$
$$J_3=\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left\{(x^p-a^p)^2\left[C_{a,p}(x,1,0,q)+C_{a,p}(x,0,1,q)\right]^{1/q}\right.$$
$$\left.+(b^p-x^p)^2\left[S_{b,p}(x,1,0,q)+S_{b,p}(x,0,1,q)\right]^{1/q}\right\}.$$

4 Conclusion

The paper deals with Ostrowski type inequalities for p-convex functions. Firstly, we give a different version of the concept of p-convex functions and get some new properties of p-convex functions. Later, by using a new identity, we obtain several new Ostrowski type inequalities for this class of functions via hypergeometric functions.

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