BISKA
New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2016318837

Biharmonic maps on kenmotsu manifolds

Abdelkader Zagane[1] *and Seddik Ouakkas*[2]

[1] Department of mathematics - University of Mascara, Algeria
[2] Department of mathematics - University of Saida, Algeria

Received: 27 January 2016, Accepted: 4 March 2016
Published online: 22 June 2016.

Abstract: In this paper we study biharmonic maps on Kenmotsu manifolds. An example for biharmonic map of a three-Kenmotsu manifold is constructed for illustration.

Keywords: Biharmonic map, Kenmotsu manifold, three-Kenmotsu manifold.

1 Introduction

1.1 Kenmotsu manifolds.

The notion of Kenmotsu manifolds is defined by K. Kenmotsu (see [15]). Let $\left(M^{2n+1},\phi,\xi,\eta,g\right)$ be an almost contact Riemannian manifold, where ϕ is an $(1,1)$ tensor field, η is a 1-form and g is the Riemannian metric. We have for any $X,Y\in\Gamma(TM)$ (see [7] and [21])

$$\phi(\xi)=0,\ \ \eta(\phi(X))=0,\ \ \eta(\xi)=1, \tag{1}$$

$$\phi^2(X)=-X+\eta(X)\xi, \tag{2}$$

$$g(X,\xi)=\eta(X) \tag{3}$$

and

$$g(\phi(X),\phi(Y))=g(X,Y)-\eta(X)\eta(Y). \tag{4}$$

If, moreover

$$(\nabla_X\phi)Y=-\eta(Y)\phi(X)-g(X,\phi(Y))\xi \tag{5}$$

and

$$\nabla_X\xi=X-\eta(X)\xi, \tag{6}$$

where ∇ denotes the Riemannian connexion of g, then $\left(M^{2n+1},\phi,\xi,\eta,g\right)$ is called an almost Kenmotsu manifold. In kenmotsu manifolds, we have the following relations (see [15]) :

$$(\nabla_X\eta)Y=g(\phi(X),\phi(Y)), \tag{7}$$

* Corresponding author e-mail: a.zaagane@yahoo.com

© 2016 BISKA Bilisim Technology

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{8}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{9}$$

$$R(\xi,X)\xi = X - \eta(X)\xi \tag{10}$$

and

$$S(X,\xi) = -(n-1)\eta(X), \tag{11}$$

where R is the Riemannian curvature tensor and S is the Ricci tensor. Kenmotsu manifolds have been studied by many authors, for example see [8], [14] and [20].

1.2 Harmonic maps.

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between riemannian manifolds. Then ϕ is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g \tag{12}$$

with respect to compactly supported variations. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations :

$$\tau(\phi) = Tr_g \nabla d\phi = 0, \tag{13}$$

$\tau(\phi)$ is called the tension field of ϕ. One can refer to [1], [5], [9], [10], [11] and [12] for background on harmonic maps. Indeed, the Euler-Lagrange equation associated to the energy is the vanishing of the tension field $\tau(\phi) = Tr_g \nabla d\phi$. In the context of harmonic maps, the stress-energy tensor was studied in details by Baird and Eells in [2]. Indeed, the Euler-Lagrange equation associated to the energy is the vanishing of the tension field $\tau(\phi) = Tr_g \nabla d\phi$, and the stress-energy tensor for a map $\phi : (M^m, g) \longrightarrow (N^n, h)$ defined by

$$S(\phi) = e(\phi)g - \phi^* h.$$

The relation between $S(\phi)$ and $\tau(\phi)$ is given by

$$divS(\phi) = -h(\tau(\phi), d\phi).$$

1.3 Biharmonic maps.

The map ϕ is said to be biharmonic if it is a critical point of the bi-energy functional :

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g \tag{14}$$

Equivalently, ϕ is biharmonic if it satisfies the associated Euler-Lagrange equations :

$$\tau_2(\phi) = -Tr_g \left(\nabla^\phi\right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0, \tag{15}$$

© 2016 BISKA Bilisim Technology

where ∇^{ϕ} is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if e_i is a local orthonormal frame field on M, then

$$Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi)=\left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}-\nabla^{\phi}_{\nabla_{e_i}e_i}\right)\tau(\phi),$$

where we sum over repeated indices. We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ. In analogy with harmonic maps, Jiang In [13] has constructed for a map ϕ the stress bi-energy tensor defined by

$$S_2(\phi)=\left(\frac{-1}{2}|\tau(\phi)|^2+divh(\tau(\phi),d\phi)\right)g-2symh(\nabla\tau(\phi),d\phi),$$

where

$$symh(\nabla\tau(\phi),d\phi)(X,Y)=\frac{1}{2}\{h(\nabla_X\tau(\phi),d\phi(Y))+h(\nabla_Y\tau(\phi),d\phi(X))\},$$

for any $X,Y\in\Gamma(TM)$. The stress bi-energy tensor of ϕ satisfies the following relationship

$$divS_2(\phi)=h(\tau_2(\phi),d\phi).$$

Clearly any harmonic map is biharmonic, therefore it is interesting to construct non-harmonic biharmonic maps. In [4] the authors found new examples of biharmonic maps by conformally deforming the domain metric of harmonic ones. While in [6] the author analyzed the behavior of the biharmonic equation under the conformal change the domain metric, he obtained metrics $\widetilde{g}=e^{2\gamma}$ such that the idendity map $Id:(M,g)\longrightarrow(M,\widetilde{g})$ is biharmonic non-harmonic. Moreover, in [19] the author gave some extensions of the result in [6] together with some further constructions of biharmonic maps. The author in [18] deform conformally the codomain metric in order to render a semi-conformal harmonic map biharmonic. In [3] the authors studied the case where $\phi:(M^n,g)\longrightarrow(N^n,h)$ is a conformal mapping between equidimensional manifolds where they show that a conformal mapping ϕ is biharmonic if and only if the gradient of its dilation satisfies a second order elliptic partial differential equation. We can refer the reader to [17], for a survey of biharmonic maps. In [16], the author obtain some results concerning harmonic maps and harmonic morphism on Kenmotsu manifolds, he prove that any structure preserving map from a Kenmotsu manifold to a Kahler manifold is harmonic and that there are no nonconstant harmonic holomorphic maps from a Kahler manifold to a Kenmotsu manifold. In this paper, we calculate the bitension field of $\psi:\left(N^{2n},J,h\right)\longrightarrow(M,\phi,\xi,\eta,g)$, where $\left(N^{2n},J,h\right)$ is a Kahlerian manifold and (M,ϕ,ξ,η,g) is a Kenmotsu manifold (Theorem 1), with this setting we obtain new example of biharmonic non-harmonic maps.

2 Statement of results.

As first result, we give another formula for the stress bi-energy tensor for a smooth map $\phi:(M^m,g)\to(N^n,h)$.

Proposition 1. *Let $\phi:(M^m,g)\to(N^n,h)$ be a smooth map, then we have*

$$S_2(\phi)=\left(\frac{1}{2}|\tau(\phi)|^2+Tr_gh(\nabla\tau(\phi),d\phi)\right)g-2symh(\nabla\tau(\phi),d\phi), \tag{16}$$

and the trace of $S_2(\phi)$ is given by

$$Tr_gS_2(\phi)=\frac{m}{2}|\tau(\phi)|^2+(m-2)Tr_gh(\nabla\tau(\phi),d\phi). \tag{17}$$

Let $\left(N^{2n},J,h\right)$ be a Kahlerian manifold, (M,ϕ,ξ,η,g) be a Kenmotsu manifold and $\psi:N\longrightarrow M$ be a (J,ϕ)-holomorphic map. To give a necessary and sufficient condition for the biharmonicity of the map ψ, we will calculate the bi-tension field $\tau_2(\psi)$. We get the following theorem :

© 2016 BISKA Bilisim Technology

Theorem 1. *Let $\left(N^{2n},J,h\right)$ be a Kahlerian manifold, (M,ϕ,ξ,η,g) be a Kenmotsu manifold and $\psi:N\longrightarrow M$ be a (J,ϕ)-holomorphic map. Then, the bi-tension field of ψ is given by*

$$\tau_2(\psi)=-2\left(\Delta\left(e(\psi)\right)\xi+2d\psi\left(grad\left(e(\psi)\right)\right)\right). \tag{18}$$

The biharmonicity of ψ is given by the following result :

Corollary 1. *Let $\left(N^{2n},J,h\right)$ be a Kahlerian manifold, (M,ϕ,ξ,η,g) be a Kenmotsu manifold and $\psi:N\longrightarrow M$ be a (J,ϕ)-holomorphic map. Then ψ is biharmonic if and only if*

$$\Delta\left(e(\psi)\right)\xi+2d\psi\left(grad\left(e(\psi)\right)\right)=0.$$

By calculating of the term $\eta\circ\tau_2(\psi)$, we obtain immediately the following Corollary :

Corollary 2. *Let $\left(N^{2n},J,h\right)$ be a Kahlerian manifold, (M,ϕ,ξ,η,g) be a Kenmotsu manifold and $\psi:N\longrightarrow M$ be a (J,ϕ)-holomorphic map. If the map ψ is biharmonic, then the function $e(\psi)$ is harmonic.*

In particular, if the function $e(\psi)$ is constant we obtain the following result :

Corollary 3. *Let $\left(N^{2n},J,h\right)$ be a Kahlerian manifold, (M,ϕ,ξ,η,g) be a Kenmotsu manifold and $\psi:N\longrightarrow M$ be a (J,ϕ)-holomorphic map. If the function $e(\psi)$ is constant, then ψ is biharmonic.*

As an application of Theorem 1, we construct an example of biharmonic non-harmonic map.

Example 1. Let $\left(N=\mathbb{R}^2,J,h=dx^2+dy^2\right)$ be a complex manifold with complex structure $J\left(\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial y}$, $J\left(\frac{\partial}{\partial y}\right)=-\frac{\partial}{\partial x}$ and $\left(M=\mathbb{R}^3\backslash\{(0,0,0)\},\phi,\xi,\eta,g\right)$ be a Kenmotsu manifold with orthonormal basis

$$e_1=x_3\frac{\partial}{\partial x_1},$$

$$e_2=x_3\frac{\partial}{\partial x_2},$$

$$e_3=-x_3\frac{\partial}{\partial x_3},$$

$$\xi=e_3,$$

$$\eta(X)=g(X,\xi),$$

where

$$g=\frac{1}{x_3^2}\left(dx_1^2+dx_2^2+dx_3^2\right)$$

and

$$\phi=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

Let $\psi:N\longrightarrow M$ be a (J,ϕ)-holomorphic map defined by

$$\psi(x,y)=\left(\psi_1(x,y),\psi_2(x,y),\psi_3(x,y)\right).$$

We have

$$\frac{\partial\psi_1}{\partial x}=-\frac{\partial\psi_2}{\partial y},$$

© 2016 BISKA Bilisim Technology

$$\frac{\partial \psi_1}{\partial y} = \frac{\partial \psi_2}{\partial x}$$

and

$$\frac{\partial \psi_3}{\partial x} = \frac{\partial \psi_3}{\partial y} = 0.$$

The last equation give $\psi_3(x,y) = C$, where C is the real constant. A simple calculate give

$$e(\psi) = \frac{1}{C^2}\left(\left(\frac{\partial \psi_1}{\partial x}\right)^2 + \left(\frac{\partial \psi_1}{\partial y}\right)^2\right),$$

$$grad(e(\psi)) = \frac{2}{C^2}\left(\frac{\partial \psi_1}{\partial x}\left(\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial x \partial y}\right)\frac{\partial}{\partial x} + \frac{\partial \psi_1}{\partial y}\left(\frac{\partial^2 \psi_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial x \partial y}\right)\frac{\partial}{\partial y}\right),$$

$$\begin{aligned} d\psi(grad(e(\psi))) = &\frac{2}{C^2}\left(\left(\frac{\partial \psi_1}{\partial x}\right)^2\left(\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial x \partial y}\right) + \left(\frac{\partial \psi_1}{\partial y}\right)^2\left(\frac{\partial^2 \psi_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial x \partial y}\right)\right)\frac{\partial}{\partial x_1} \\ &+ \frac{2}{C^2}\left(\frac{\partial \psi_1}{\partial x}\frac{\partial \psi_2}{\partial x}\left(\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial x \partial y}\right) + \frac{\partial \psi_1}{\partial y}\frac{\partial \psi_2}{\partial y}\left(\frac{\partial^2 \psi_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial x \partial y}\right)\right)\frac{\partial}{\partial x_2} \end{aligned}$$

and

$$\begin{aligned} \Delta(e(\psi)) = &\frac{2}{C^2}\left(\left(\frac{\partial^2 \psi_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2 \psi_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2 \psi_1}{\partial y^2}\right)^2 + \left(\frac{\partial^2 \psi_2}{\partial y^2}\right)^2\right) \\ &+ \frac{2}{C^2}\left(\frac{\partial \psi_1}{\partial x}\left(\frac{\partial^3 \psi_1}{\partial x^3} - \frac{\partial^3 \psi_2}{\partial y^3}\right) + \frac{\partial \psi_2}{\partial x}\left(\frac{\partial^3 \psi_1}{\partial y^3} + \frac{\partial^3 \psi_2}{\partial x^3}\right)\right) \end{aligned}$$

In particular, if we consider $\psi_1(x,y) = \psi_1(x)$ and $\psi_2(x,y) = \psi_2(y)$, we obtain

$$\psi(x,y) = (Ax + B, -Ay + D, C),$$

where A,B,C and D are a real constants $(A,C \neq 0)$. In this case $e(\psi) = \frac{A^2}{C^2}$ and the map ψ is biharmonic non-harmonic.

Proposition 2. *Let (N^{2n},J,h) be a Kahlerian manifold, (M,ϕ,ξ,η,g) be a Kenmotsu manifold and $\psi : N \longrightarrow M$ be a (J,ϕ)-holomorphic map. Then, the stress bi-energy tensor of ψ is given by*

$$S_2(\psi) = 6(e(\psi))^2 h - 4e(\psi)g(d\psi(\cdot), d\psi(\cdot)), \tag{19}$$

where $g(d\psi(\cdot),d\psi(\cdot))(X,Y) = g(d\psi(X),d\psi(Y))$, and the trace of $S_2(\psi)$ is given by

$$Tr_h S_2(\psi) = (12n-8)(e(\psi))^2.$$

3 Proof of results.

Proof of Proposition 1. Let $\phi : (M^m,g) \to (N^n,h)$ be a smooth map. By definition, we have

$$S_2(\phi) = \left(\frac{-1}{2}|\tau(\phi)|^2 + divh(\tau(\phi),d\phi)\right)g - 2symh(\nabla\tau(\phi),d\phi). \tag{20}$$

© 2016 BISKA Bilisim Technology

Let $(e_i)_{i=1}^m$ is a local orthonormal frame field on M. For the term $divh(\tau(\phi),d\phi)$, we have

$$\begin{aligned} divh(\tau(\phi),d\phi) &= (\nabla_{e_i}h(\tau(\phi),d\phi))(e_i) \\ &= e_i(h(\tau(\phi),d\phi(e_i))) - h(\tau(\phi),d\phi(\nabla_{e_i}e_i)) \\ &= h\left(\nabla^{\phi}_{e_i}\tau(\phi),d\phi(e_i)\right) + h\left(\tau(\phi),\nabla^{\phi}_{e_i}d\phi(e_i)\right) \\ &\quad - h(\tau(\phi),d\phi(\nabla_{e_i}e_i)) \\ &= h\left(\nabla^{\phi}_{e_i}\tau(\phi),d\phi(e_i)\right) + h\left(\tau(\phi),\nabla^{\phi}_{e_i}d\phi(e_i) - d\phi(\nabla_{e_i}e_i)\right). \end{aligned}$$

But we know that

$$h\left(\nabla^{\phi}_{e_i}\tau(\phi),d\phi(e_i)\right) = Tr_g h(\nabla\tau(\phi),d\phi)$$

and

$$h\left(\tau(\phi),\nabla^{\phi}_{e_i}d\phi(e_i) - d\phi(\nabla_{e_i}e_i)\right) = h(\tau(\phi),\tau(\phi)) = |\tau(\phi)|^2,$$

it follows that

$$divh(\tau(\phi),d\varphi) = Tr_g h(\nabla\tau(\phi),d\phi) + |\tau(\phi)|^2 \tag{21}$$

If we replace (21) in (20), we obtain

$$S_2(\phi) = \left(\frac{1}{2}|\tau(\phi)|^2 + Tr_g h(\nabla\tau(\phi),d\phi)\right)g - 2symh(\nabla\tau(\phi),d\phi).$$

By definition the trace of $S_2(\phi)$, we have

$$\begin{aligned} Tr_g S_2(\phi) &= S_2(\phi)(e_i,e_i) \\ &= \left(\frac{1}{2}|\tau(\phi)|^2 + Tr_g h(\nabla\tau(\phi),d\phi)\right)g(e_i,e_i) \\ &\quad - 2symh(\nabla\tau(\phi),d\phi)(e_i,e_i) \\ &= m\left(\frac{1}{2}|\tau(\phi)|^2 + Tr_g h(\nabla\tau(\phi),d\phi)\right) - 2h(\nabla_{e_i}\tau(\phi),d\phi(e_i)) \\ &= \frac{m}{2}|\tau(\phi)|^2 + mTr_g h(\nabla\tau(\phi),d\phi) - 2Tr_g h(\nabla\tau(\phi),d\phi) \end{aligned}$$

which gives

$$Tr_g S_2(\phi) = \frac{m}{2}|\tau(\phi)|^2 + (m-2)Tr_g h(\nabla\tau(\phi),d\phi).$$

The proof of Proposition 1 is complete. To prove Theorem 1, we need the following Lemma :

Lemma 1. *Let (N^{2n},J,h) be a Kahlerian manifold, (M,ϕ,ξ,η,g) be a Kenmotsu manifold and $\psi : N \longrightarrow M$ be a (J,ϕ)-holomorphic map. Then, we have for any $X \in \Gamma(TM)$*

$$(\eta \circ d\psi)(X) = 0.$$

Proof of Lemma 1. Since $\psi : N \longrightarrow M$ is (J,ϕ)-holomorphic map, then for any $X \in \Gamma(TN)$, we have

$$(d\psi \circ J)(X) = (\phi \circ d\psi)(X).$$

© 2016 BISKA Bilisim Technology

Using the fact that $\left(N^{2n},J,h\right)$ is a Kahlerian manifold, then

$$J^2(X)=-X.$$

As (M,ϕ,ξ,η,g) is a Kenmotsu manifold and by using the equation 1, we have

$$\eta\circ\phi=0.$$

Then, we obtain

$$\begin{aligned}(\eta\circ d\psi)(X)&=(\eta\circ d\psi)\left(-J^2(X)\right)\\&=-\eta\circ d\psi\circ J(J(X))\\&=-\eta\circ\phi\circ d\psi(J(X))\\&=0.\end{aligned}$$

We are now able to prove Theorem 1.

Proof of Theorem 1. By definition, we have

$$\tau_2(\psi)=-Tr_h\left(\nabla^{\psi}\right)^2\tau(\psi)-Tr_hR^M(\tau(\psi),d\psi)d\psi.$$

The tension field of ψ is given by (see [16])

$$\tau(\psi)=2e(\psi)\xi,$$

then

$$\tau_2(\psi)=-2\left(Tr_h\left(\nabla^{\psi}\right)^2e(\psi)\xi+Tr_hR^M(e(\psi)\xi,d\psi)d\psi\right). \tag{22}$$

In the first, we will simplify the term $Tr_h\left(\nabla^{\psi}\right)^2e(\psi)\xi$. Let $(e_i)_{i=1}^{2n}$ is a local orthonormal frame field on N, we have

$$Tr_g\left(\nabla^{\psi}\right)^2e(\psi)\xi=\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i}e(\psi)\xi-\nabla^{\psi}_{\nabla^N_{e_i}e_i}e(\psi)\xi. \tag{23}$$

A direct calculation gives

$$\begin{aligned}\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i}e(\psi)\xi&=\nabla^{\psi}_{e_i}\left(e(\psi)\nabla^{\psi}_{e_i}\xi\right)+\nabla^{\psi}_{e_i}(e_i(e(\psi))\xi)\\&=e(\psi)\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i}\xi+e_i(e(\psi))\nabla^{\psi}_{e_i}\xi+e_i(e(\psi))\nabla^{\psi}_{e_i}\xi\\&+e_i(e_i(e(\psi)))\xi\\&=e(\psi)\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i}\xi+2\nabla^{\psi}_{\mathrm{grad}(e(\psi))}\xi+e_i(e_i(e(\psi)))\xi\end{aligned}$$

and

$$\nabla^{\psi}_{\nabla_{e_i}e_i}e(\psi)\xi=e(\psi)\nabla^{\psi}_{\nabla_{e_i}e_i}\xi+\nabla_{e_i}e_i(e(\psi))\xi.$$

Using the fact that

$$\Delta(e(\psi))=e_i(e_i(e(\psi)))-\left(\nabla^N_{e_i}e_i\right)(e(\psi))$$

and

$$Tr_g\left(\nabla^{\psi}\right)^2\xi=\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i}\xi-\nabla^{\psi}_{\nabla^N_{e_i}e_i}\xi,$$

© 2016 BISKA Bilisim Technology

we deduce that

$$Tr_h\,(\nabla^{\psi})^2\, e(\psi)\,\xi = e(\psi)\,Tr_h\,(\nabla^{\psi})^2\,\xi + \Delta\,(e(\psi))\,\xi + 2\nabla^{\psi}_{\mathrm{grad}(e(\psi))}\xi. \tag{24}$$

Calculate the term $Tr_h\,(\nabla^{\psi})^2\,\xi$, we have

$$Tr_h\,(\nabla^{\psi})^2\,\xi = \nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i}\xi - \nabla^{\psi}_{\nabla_{e_i}e_i}\xi.$$

By using equation (6), we obtain

$$\nabla^{\psi}_{e_i}\xi = \nabla_{d\psi(e_i)}\xi = d\psi\,(e_i) - (\eta \circ d\psi)\,(e_i)\,\xi,$$

and by Lemma 1, we have

$$(\eta \circ d\psi)\,(e_i) = 0,$$

then

$$\nabla^{\psi}_{e_i}\xi = d\psi\,(e_i)$$

wish gives us

$$\nabla^{\psi}_{e_i}\nabla^{\psi}_{e_i}\xi = \nabla^{\psi}_{e_i}d\psi\,(e_i)$$

and

$$\begin{aligned}\nabla^{\psi}_{\nabla_{e_i}e_i}\xi &= \nabla_{d\psi(\nabla_{e_i}e_i)}\xi \\ &= d\psi\,(\nabla_{e_i}e_i) - (\eta \circ d\psi)\,(\nabla_{e_i}e_i)\,\xi \\ &= d\psi\,(\nabla_{e_i}e_i)\,,\end{aligned}$$

we conclude that

$$Tr_h\,(\nabla^{\psi})^2\,\xi = \nabla^{\psi}_{e_i}d\psi\,(e_i) - d\psi\,(\nabla_{e_i}e_i) = \tau\,(\psi)\,.$$

Then

$$Tr_h\,(\nabla^{\psi})^2\,\xi = 2e\,(\psi)\,\xi.$$

Finally, we obtain

$$Tr_h\,(\nabla^{\psi})^2\, e(\psi)\,\xi = 2\,(e\,(\psi))^2\,\xi + \Delta\,(e\,(\psi))\,\xi + 2\nabla^{\psi}_{\mathrm{grad}(e(\psi))}\xi$$

Now simplify the term $\nabla^{\psi}_{\mathrm{grad}(e(\psi))}\xi$, we have

$$\begin{aligned}\nabla^{\psi}_{\mathrm{grad}(e(\psi))}\xi &= \nabla^{M}_{d\psi(\mathrm{grad}(e(\psi)))}\xi \\ &= d\psi\,(\mathrm{grad}\,(e\,(\psi))) - \eta\,(d\psi\,(\mathrm{grad}\,(e\,(\psi))))\,\xi \\ &= d\psi\,(\mathrm{grad}\,(e\,(\psi))) - (\eta \circ d\psi)\,(\mathrm{grad}\,(e\,(\psi)))\,\xi \\ &= d\psi\,(\mathrm{grad}\,(e\,(\psi)))\,.\end{aligned}$$

which finally gives us

$$Tr_h\,(\nabla^{\psi})^2\, e(\psi)\,\xi = 2\,(e\,(\psi))^2\,\xi + \Delta\,(e\,(\psi))\,\xi + 2d\psi\,(\mathrm{grad}\,(e\,(\psi)))\,. \tag{25}$$

To complete the proof, look at the term $Tr_g R^M\,(e\,(\psi)\,\xi, d\psi)\,d\psi$, we have

© 2016 BISKA Bilisim Technology

$$\begin{aligned} Tr_h R^M(e(\psi)\xi, d\psi)d\psi &= e(\psi) Tr_h R^M(\xi, d\psi)d\psi \\ &= e(\psi) R^N(\xi, d\psi(e_i))d\psi(e_i). \end{aligned}$$

Using equation (9) , we obtain

$$\begin{aligned} R^N(\xi, d\psi(e_i))d\psi(e_i) &= \eta(d\psi(e_i)) - g(d\psi(e_i), d\psi(e_i))\xi \\ &= (\eta \circ d\psi)(e_i) - 2e(\psi)\xi \\ &= -2e(\psi)\xi. \end{aligned}$$

Then

$$Tr_h R^N(e(\psi)\xi, d\psi)d\psi = -2(e(\psi))^2\xi. \tag{26}$$

If we replace (25) and (26) in (22), wa arrive at

$$\tau_2(\psi) = -2(\Delta(e(\psi))\xi + 2d\psi(\text{grad}(e(\psi)))).$$

This completes the proof of Theorem 1. Then $\psi : N \longrightarrow M$ is biharmonic if and only if

$$\Delta(e(\psi))\xi + 2d\psi(\text{grad}(e(\psi))) = 0.$$

Proof of Corollary 2. If we assume that the map $\psi : N \longrightarrow M$ is biharmonic, by Theorem 1, we have

$$\Delta(e(\psi))\xi + 2d\psi(\text{grad}(e(\psi))) = 0.$$

Then

$$\eta \circ (\Delta(e(\psi))\xi + 2d\psi(\text{grad}(e(\psi)))) = 0,$$

it follows that

$$\Delta(e(\psi))\eta(\xi) + 2(\eta \circ d\psi)(\text{grad}(e(\psi))) = 0.$$

By using the equation (1), we have

$$\eta(\xi) = 1$$

and by Lemma 1, we have

$$(\eta \circ d\psi)(\text{grad}(e(\psi))) = 0.$$

Which finally gives

$$\Delta(e(\psi)) = 0.$$

We deduce that the function $e(\psi)$ is harmonic.

Proof of Proposition 2. By definition, for any $X, Y \in \Gamma(TN)$ we have

$$\begin{aligned} S_2(\psi)(X,Y) = &\left(\frac{1}{2}|\tau(\psi)|^2 + Tr_h g(\nabla\tau(\psi), d\psi)\right)h(X,Y) \\ &- 2sym g(\nabla\tau(\psi), d\psi)(X,Y). \end{aligned}$$

© 2016 BISKA Bilisim Technology

Since

$$\tau(\psi) = 2e(\psi)\xi,$$

it follows that

$$|\tau(\psi)|^2 = 4(e(\psi))^2.$$

For the term $Tr_h g(\nabla\tau(\psi), d\psi)$, we have

$$\begin{aligned} Tr_h g(\nabla\tau(\psi), d\psi) &= 2g(\nabla_{e_i} e(\psi)\xi, d\psi(e_i)) \\ &= 2e(\psi) g\left(\nabla^{\psi}_{e_i}\xi, d\psi(e_i)\right) \\ &\quad + 2e_i(e(\psi)) g(\xi, d\psi(e_i)), \end{aligned}$$

where $(e_i)_{i=1}^{2n}$ is a local orthonormal frame field on N. By equation (6), we obtain

$$\nabla^{\psi}_{e_i}\xi = \nabla^{M}_{d\psi(e_i)}\xi = d\psi(e_i) - \eta(d\psi(e_i))\xi = d\psi(e_i)$$

and by (3), we have

$$g(\xi, d\psi(e_i)) = \eta(d\psi(e_i)) = 0,$$

then

$$Tr_h g(\nabla\tau(\psi), d\psi) = 2e(\psi) g(d\psi(e_i), d\psi(e_i)) = 4(e(\psi))^2.$$

it follow that

$$\frac{1}{2}|\tau(\psi)|^2 + Tr_h g(\nabla\tau(\psi), d\psi) = 6(e(\psi))^2.$$

Finally, for the term $symg(\nabla\tau(\psi), d\psi)(X,Y)$, we have

$$symg(\nabla\tau(\psi), d\psi)(X,Y) = \frac{1}{2}g\left(\nabla^{\psi}_X\tau(\psi), d\psi(Y)\right) + \frac{1}{2}g\left(\nabla^{\psi}_Y\tau(\psi), d\psi(X)\right).$$

A simple calculate gives

$$\begin{aligned} \frac{1}{2}g\left(\nabla^{\psi}_X\tau(\psi), d\psi(Y)\right) &= g\left(\nabla^{\psi}_X e(\psi)\xi, d\psi(Y)\right) \\ &= e(\psi) g\left(\nabla^{\psi}_X\xi, d\psi(Y)\right) + X(e(\psi)) g(\xi, d\psi(Y)), \end{aligned}$$

but by (6), we have

$$\nabla^{\psi}_X\xi = \nabla^{M}_{d\psi(X)}\xi = d\psi(X) - \eta(d\psi(X))\xi = d\psi(X)$$

and by (3), we obtain

$$g(\xi, d\psi(Y)) = \eta(d\psi(Y)) = 0,$$

then

$$\frac{1}{2}g\left(\nabla^{\psi}_X\tau(\psi), d\psi(Y)\right) = g(d\psi(X), d\psi(Y)).$$

A similar calculate gives

$$\frac{1}{2}g\left(\nabla^{\psi}_Y\tau(\psi), d\psi(X)\right) = g(d\psi(X), d\psi(Y)).$$

We deduce that

$$symg(\nabla\tau(\psi), d\psi)(X,Y) = 2g(d\psi(X), d\psi(Y)),$$

© 2016 BISKA Bilisim Technology

then

$$S_2(\psi)(X,Y) = 6(e(\psi))^2 g(X,Y) - 4g(d\psi(X), d\psi(Y)).$$

finally, the stress bi-energy tensor is given by

$$S_2(\psi) = 6(e(\psi))^2 g - 4g(d\psi(\cdot), d\psi(\cdot)),$$

where $g(d\psi(\cdot), d\psi(\cdot))(X,Y) = g(d\psi(X), d\psi(Y))$. Now calculate $Tr_h S_2(\psi)$, we have

$$\begin{aligned} Tr_h S_2(\psi) &= S_2(\psi)(e_i, e_i) \\ &= 6(e(\psi))^2 g(e_i, e_i) - 4g(d\psi(e_i), d\psi(e_i)) \\ &= 12n(e(\psi))^2 - 4|d\psi|^2 \\ &= 12n(e(\psi))^2 - 8(e(\psi))^2 \\ &= (12n-8)(e(\psi))^2. \end{aligned}$$

We conclude that $Tr_g S_2(\psi) = 0$ if and only if the map ψ is constant.

References

[1] P. Baird, *Harmonic maps with symmetry, harmonic morphisms and deformation of metrics*, Pitman Books Limited, 27-39, (1983).
[2] P. Baird and J. Eells, *A conservation law for harmonic maps*, Lecture Notes in Math. 894, Springer, 1-25, (1981).
[3] P.Baird, A. Fardoun and S. Ouakkas, *Conformal and semi-conformal biharmonic maps*, Ann. Glob Anal Geom 34, 403-414 (2008).
[4] P. Baird and D. Kamissoko, *On constructing biharmonic maps and metrics*, Annals of Global Analysis and Geometry 23, 65-75, (2003).
[5] P. Baird and J.C. Wood, *Harmonic morphisms between Riemannain manifolds*, Oxford Sciences Publications (2003).
[6] A. Balmus, *Biharmonic properties and conformal changes*, An. Stiint. Univ. Al.I. Cuza Iasi Mat. (N.S.) 50, 367-372, (2004).
[7] D.E. Blair, *Riemannian geometry of contact and Symplectic Manifolds*, Birkhauser. Boston, Second Edition (2010).
[8] U.C.De and G. Pathok, *On 3-dimensional Kenmotsu manifolds*, Indian J. Pure Appl. Math. 35, 159-165, (2004).
[9] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 16, 1-68, (1978).
[10] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. 20, 385-524, (1988).
[11] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, CNMS Regional Conference Series of the National Sciences Foundation, November 1981.
[12] J. Eells and A. Ratto, *Harmonic Maps and Minimal Immersions with Symmetries*, Princeton University Press 1993.
[13] G. Y. Jiang, *2-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A 7, 389-402, (1986).
[14] J.B Jun, U.C. De and G. Pathak, *On Kenmotsu Manifolds*, J. Korean Math. Soc. 42, No. 3, 435-445, (2005).
[15] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. II Ser. 24, 93-103, (1972).
[16] A. Najma, *Harmonic Maps on Kenmotsu Manifolds*, An. St. Univ. Ovidius Constanta. Vol. 21(3), 197-208, 2013.
[17] C. Oniciuc, *New examples of biharmonic maps in spheres*, Colloq. Math., 97, 131-139, (2003).
[18] Ouakkas, S, *Biharmonic maps, conformal deformations and the Hopf maps*, Diff. Geom. Appl, 26, 495-502, (2008).
[19] Y.-L. Ou, *p-harmonic morphisms, biharmonic morphisms, and non-harmonic biharmonic maps*, J. Geom. Phys. Volume 56, 3, 358-374, (2006).
[20] G. Pitis¸, *Geometry of Kenmotsu manifolds*, Publishing House of Transilvania University of Bra¸sov, Bra¸sov, (2007).
[21] K. Yano and M. Kon, *Structures on manifolds*, vol. 3, Series in pure Math., World Scientifc, Singapore, 1984.

© 2016 BISKA Bilisim Technology