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Tachibana and Vishnevskii operators applied to X^V and X^H in almost paracontact structure on tangent bundle $T(M)$

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Abstract: The differential geometry of tangent bundles was studied by several authors, for example: Yano and Ishihara [8], V. Oproiu [3], A.A. Salimov [5], D. E. Blair [1] and among others. It is well known that different structures defined on a manifold M can be lifted to the same type of structures on its tangent bundle. In addition, several authors was studied on operators too, for example: A.A. Salimov [5]. Our goal is to study Tachibana and Vishnevskii Operators Applied to X^V and X^H in almost paracontact structure on tangent bundle $T(M)$. In addition, this results which obtained shall be studied for some special values in almost paracontact structure.

Keywords: Tachibana Operators,Vishnevskii Operators, Almost Paracontact Structure, Horizontal Lift, Vertical Lift

1 Introduction

Let M be an $n-$dimensional differentiable manifold of class C^∞ and let $T_p(M)$ be the tangent space of M at a point p of M. Then the set [8]

$$T(M) = \underset{p\in M}{\cup} T_p(M) \tag{1}$$

is called the tangent bundle over the manifold M. For any point $\tilde{p}$ of $T(M)$, the correspondence $\tilde{p} \to p$ determines the bundle projection $\pi : T(M) \to M$, Thus $\pi(\tilde{p}) = p$, where $\pi : T(M) \to M$ defines the bundle projection of $T(M)$ over M. The set $\pi^{-1}(p)$ is called the fibre over $p \in M$ and M the base space.

Suppose that the base space M is covered by a system of coordinate neighbour-hoods $\{U; x^h\}$, where (x^h) is a system of local coordinates defined in the neighbour-hood U of M. The open set $\pi^{-1}(U) \subset T(M)$ is naturally differentially homeomorphic to the direct product $U \times R^n$, R^n being the $n-$dimensional vector space over the real field R, in such a way that a point $\tilde{p} \in T_p(M)(p \in U)$ is represented by an ordered pair (P,X) of the point $p \in U$, and a vector $X \in R^n$,whose components are given by the cartesian coordinates (y^h) of $\tilde{p}$ in the tangent space $T_p(M)$ with respect to the natural base $\{\partial_h\}$, where $\partial_h = \frac{\partial}{\partial x^h}$. Denoting by (x^h) the coordinates of $p = \pi(\tilde{p})$ in U and establishing the correspondence $(x^h, y^h) \to \tilde{p} \in \pi^{-1}(U)$, we can introduce a system of local coordinates (x^h, y^h) in the open set $\pi^{-1}(U) \subset T(M)$. Here we call (x^h, y^h) the coordinates in $\pi^{-1}(U)$ induced from (x^h) or simply, the induced coordinates in $\pi^{-1}(U)$.

We denote by $\Im^r_s(M)$ the set of all tensor fields of class $C^\infty$and of type (r,s) in M. We now put $\Im(M) = \sum_{r,s=0}^{\infty} \Im^r_s(M)$, which is the set of all tensor fields in M. Similarly, we denote by $\Im^r_s(T(M))$ and $\Im(T(M))$ respectively the corresponding sets of tensor fields in the tangent bundle $T(M)$.

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1.1 Vertical lifts

If f is a function in M, we write f^v for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \to M$ and $f : M \to R$, so that

$$f^v = fo\pi. \tag{2}$$

Thus, if a point $\tilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

$$f^v(\tilde{p}) = f^v(x,y) = fo\pi(\tilde{p}) = f(p) = f(x). \tag{3}$$

Thus the value of $f^v(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^v the vertical lift of the function f [8].

Let $\tilde{X} \in \Im_0^1(T(M))$ be such that $\tilde{X} f^v = 0$ for all $f \in \Im_0^0(M)$. Then we say that $\tilde{X}$ is a vertical vector field. Let $\binom{\tilde{X}^h}{\tilde{X}^{\bar{h}}}$ be components of $\tilde{X}$ with respect to the induced coordinates. Then $\tilde{X}$ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

$$\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ X^{\bar{h}} \end{pmatrix}. \tag{4}$$

Suppose that $X \in \Im_0^1(M)$, so that is a vector field in M. We define a vector field X^v in $T(M)$ by

$$X^v(\iota\ \omega) = (\omega X)^v \tag{5}$$

ω being an arbitrary $1-$form in M. We call X^v the vertical lift of X [8].

Let $\tilde{\omega} \in \Im_1^0(T(M))$ be such that $\tilde{\omega}(X)^v = 0$ for all $X \in \Im_0^1(M)$. Then we say that $\tilde{\omega}$ is a vertical $1-$form in $T(M)$. We define the vertical lift ω^v of the $1-$form ω by

$$\omega^v = (\omega_i)^v (dx^i)^v \tag{6}$$

in each open set $\pi^{-1}(U)$, where $(U; x^h)$ is coordinate neighbourhood in M and ω is given by $\omega = \omega_i dx^i$ in U. The vertical lift ω^v of ω with local expression $\omega = \omega_i dx^i$ has components of the form

$$\omega^v : (\omega^i, 0) \tag{7}$$

with respect to the induced coordinates in $T(M)$.

Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions

$$(P \otimes Q)^V = P^V \otimes Q^V,\ (P+R)^V = P^V + R^V \tag{8}$$

P, Q and R being arbitrary elements of $\Im(M)$. The vertical lifts F^V of an element $F \in \Im_1^1(M)$ with local components F_i^h has components of the form [8]

$$F^V : \begin{pmatrix} 0 & 0 \\ F_i^h & 0 \end{pmatrix}.$$

Vertical lift has the following formulas [4,8]:

$$(fX)^v = f^v X^v, I^v X^v = 0, \eta^v(X^v) = 0$$
$$(f\eta)^v = f^v \eta^v, [X^v, Y^v] = 0, \varphi^v X^v = 0 \qquad (9)$$
$$X^v f^v = 0, X^v f^v = 0$$

hold good, where $f \in \Im_0^0(M_n)$, $X,Y \in \Im_0^1(M_n)$, $\eta \in \Im_1^0(M_n)$, $\varphi \in \Im_1^1(M_n)$, $I = id_{M_n}$.

1.2 Complete lifts

If f is a function in M, we write f^c for the function in $T(M)$ defined by

$$f^c = \iota(df) \qquad (10)$$

and call f^c the complete lift of the function f. The complete lift f^c of a function f has the local expression

$$f^c = y^i \partial_i f = \partial f \qquad (11)$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \Im_0^1(M)$. Then we define a vector field X^c in $T(M)$ by

$$X^c f^c = (Xf)^c, \qquad (12)$$

f being an arbitrary function in M and call X^c the complete lift of X in $T(M)$ [2,8]. The complete lift X^c of X with components x^h in M has components

$$X^c = \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix} \qquad (13)$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\omega \in \Im_1^0(M)$, then a $1-$form ω^c in $T(M)$ defined by

$$\omega^c(X^c) = (\omega X)^c \qquad (14)$$

X being an arbitrary vector field in M. We call ω^c the complete lift of ω. The complete lift ω^c of ω with components ω_i in M has components of the form

$$\omega^c : (\partial \omega_{i,} \omega_i) \qquad (15)$$

with respect to the induced coordinates in $T(M)$ [2].

The complete lifts to a unique algebra isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients, is given by the conditions

$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C, \; (P+R)^C = P^C + R^C, \qquad (16)$$

where P, Q and R being arbitrary elements of $\Im(M)$. The complete lifts F^C of an element $F \in \Im_1^1(M)$ with local components F_i^h has components of the form

$$F^C : \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}.$$

In addition, we know that the complete lifts are defined by [4,8]:

$$\begin{aligned}
&(fX)^c = f^c X^v + f^v X^c = (X f)^c, \\
&X^c f^v = (X f)^v, \eta^v(x^c) = (\eta(x))^v, \\
&X^v f^c = (X f)^v, \varphi^v X^c = (\varphi X)^v, \\
&\varphi^c X^v = (\varphi X)^v, (\varphi X)^c = \varphi^c X^c, \\
&\eta^v(X^c) = (\eta(X))^c, \eta^c(X^v) = (\eta(X))^v, \\
&[X^v, Y^c] = [X,Y]^v, I^c = I, I^v X^c = X^v, [X^c, Y^c] = [X,Y]^c.
\end{aligned} \tag{17}$$

1.3 Horizontal lifts

The horizontal lift f^H of $f \in \Im_0^0(M)$ to the tangent bundle $T(M)$ is given by

$$f^H = f^C - \nabla_\gamma f, \tag{18}$$

where

$$\nabla_\gamma f = \gamma \nabla f. \tag{19}$$

Let $X \in \Im_0^1(M)$. Then the horizontal lift X^H of X defined by

$$X^H = X^C - \nabla_\gamma X \tag{20}$$

in $T(M)$, where

$$\nabla_\gamma X = \gamma \nabla X. \tag{21}$$

The horizontal lift X^H of X has the components

$$X^H : \begin{pmatrix} X^h \\ -\Gamma_i^h X^i \end{pmatrix} \tag{22}$$

with respect to the induced coordinates in $T(M)$, where

$$\Gamma_i^h = y^i \Gamma_{ji}^h. \tag{23}$$

Let $\omega \in \Im_1^0(M)$ with affine connection ∇. Then the horizontal lift ω^H of ω is defined by

$$\omega^H = \omega^C - \nabla_\gamma \omega \tag{24}$$

in $T(M)$, where $\nabla_\gamma \omega = \gamma \nabla \omega$. The horizontal lift ω^H of ω has component of the form

$$\omega^H : (\Gamma_i^h \omega_h, \omega_i) \tag{25}$$

with respect to the induced coordinates in $T(M)$.

Suppose there is given a tensor field

$$S = S^{i...h}_{k...j}\frac{\partial}{\partial x^i}\otimes ...\otimes\frac{\partial}{\partial x^h}\otimes dx^k\otimes ...\otimes dx^j \tag{26}$$

in M with affine connection ∇, and in $T(M)$ a tensor field $\nabla_\gamma S$ defined by

$$\nabla_\gamma S = y^l\nabla_l S^{i...h}_{k...j}\frac{\partial}{\partial y^i}\otimes ...\otimes\frac{\partial}{\partial y^h}\otimes dx^k\otimes ...\otimes dx^j \tag{27}$$

with respect to the induced coordinates (x^h, y^h) in $\pi^{-1}(U)$.

The horizontal lift S^H of a tensor field S of arbitrary type in M to $T(M)$ is defined by

$$S^H = S^C - \nabla_\gamma S. \tag{28}$$

For any $P, Q \in T(M)$, we have

$$\begin{aligned}\nabla_\gamma(P\otimes Q) &= (\nabla_\gamma P)\otimes Q^V + P^V\otimes(\nabla_\gamma Q),\\ (P\otimes Q)^H &= P^H\otimes Q^V + P^V\otimes Q^H.\end{aligned} \tag{29}$$

Let M be an $n-$dimensional differentiable manifold. Differantial transformation $D = L_X$ is called Lie derivation with respect to vector field $X \in \Im^1_0(M)$ if

$$\begin{aligned}L_X f &= Xf, \forall f\in\Im^0_0(M),\\ L_X Y &= [X,Y], \forall X,Y\in\Im^1_0(M).\end{aligned} \tag{30}$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [8]

$$(L_X F)Y = [X, FY] - F[X,Y]. \tag{31}$$

Let M be an $n-$dimensional differentiable manifold. Differantial transformation of algebra $T(M)$, defined by

$$D = \triangledown_X : T(M)\to T(M), X\in\Im^1_0(M),$$

is called as covariant derivation with respect to vector field X if

$$\begin{aligned}\nabla_{fX+gY}t &= f\nabla_X t + g\nabla_Y t,\\ \triangledown_X f &= Xf,\end{aligned} \tag{32}$$

where $\forall f,g\in\Im^0_0(M), \forall X,Y\in\Im^1_0(M), \forall t\in\Im(M)$.

On the other hand, a transformation defined by

$$\triangledown : \Im^1_0(M)\times\Im^1_0(M)\to\Im^1_0(M),$$

is called as an affine connection [5,8].

If we compare horizontal and complete lift, we obtain

$$X^H = (\hat{\nabla}_X)^C \tag{33}$$

for any $X \in \Im_0^1(M_n)$, where $\hat{\nabla}$ is an affine connection in M_n defined by

$$\hat{\nabla}_X Y = \nabla_Y X + [X,Y] \tag{34}$$

or

$$(\nabla_Y X)^v = (\hat{\nabla}_X Y)^v + [Y,X]^v. \tag{35}$$

$(\hat{\nabla}_X)^C$ is the complete lift of the derivation $\hat{\nabla}_X$. We also know that the horizontal lifts are defined by [4,8]

$$\begin{aligned} &I^H = I,\ I^H X^v = X^V,\ I^v X^H = X^v,\ I^H X^H = X^H, \\ &X^H f^v = (Xf)^v,\ (fX)^H = f^v X^H,\ \omega^H(X^H) = 0, \\ &\omega^v(X^H) = (\omega(X))^v,\ \omega^H(X^v) = (\omega(X))^v, \\ &F^H X^v = (FX)^v,\ F^H X^H = (FX)^H. \end{aligned} \tag{36}$$

Proposition 1. *For any $X,Y \in \Im_0^1(M)$ [8]*

(i) $[X^V,Y^H] = [X,Y]^V - (\nabla_X Y)^V = -(\hat{\nabla}_Y X)^V$,
(ii) $[X^C,Y^H] = [X,Y]^H - \gamma(L_X Y)$,
(iii) $[X^H,Y^V] = [X,Y]^V + (\nabla_Y X)^V$,
(iv) $[X^H,Y^H] = [X,Y]^H - \gamma\hat{R}(X,Y)$, where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$.

Proposition 2. *The horizontal lift ∇^H of an affine connection ∇ in M_n to $T(M)$ defined by the conditions of*

$$\begin{aligned} &\nabla^H_{X^V} Y^V = 0,\ \nabla^H_{X^V} Y^H = 0, \\ &\nabla^H_{X^H} Y^V = (\nabla_X Y)^V,\ \nabla^H_{X^H} Y^H = (\nabla_X Y)^H \end{aligned} \tag{37}$$

for any $X,Y \in \Im_0^1(M_n)$ [8].

2 Main results

2.1 Tachibana Operators Applied to X^V and X^H in Almost Paracontact Structure

Definition 1. *Let an $n-$dimensional differentiable manifold M_n be endowed with a tensor field φ of type $(1,1)$, a vector field ξ and a $1-$form η , I the identity and let them satisfy*

$$\varphi^2 = I - \eta \otimes \xi, \qquad \varphi(\xi) = 0, \qquad \eta o \varphi = 0, \qquad \eta(\xi) = 1. \tag{38}$$

Then (φ,ξ,η) define almost paracontact structure on M_n [7]. From (38), we get on taking complete and vertical lifts [4]

$$\begin{aligned}&(\varphi^H)^2 = I-\eta^v\otimes\xi^H-\eta^H\otimes\xi^v\\&\varphi^H\xi^v=0,\varphi^H\xi^H=0,\eta^v o\xi^H=0\\&\eta^H o\varphi^H=0,\eta^v(\xi^v)=0,\eta^v(\xi^H)=1\\&\eta^H(\xi^v)=1,\eta^H(\xi^H)=0.\end{aligned}\tag{39}$$

We now define a $(1,1)$ tensor field $\widetilde{J}$ on $T(M_n)$ by

$$\widetilde{J}=\varphi^H-\xi^v\otimes\eta^v-\xi^H\otimes\eta^H.\tag{40}$$

Then it is easy to show that $\widetilde{J}^2X^v=X^v$ and $\widetilde{J}^2X^c=X^c$, which give that $\widetilde{J}$ is an almost product structure on $T(M_n)$. We get from (40)

$$\begin{aligned}&\widetilde{J}X^v=(\varphi X)^v-(\eta(X)\xi)^H,\\&\widetilde{J}X^H=(\varphi X)^H-(\eta(X)\xi)^v\end{aligned}\tag{41}$$

for any $X\in\Im_0^1(M_n)$.

Definition 2. *Let $\varphi\in\Im_1^1(M_n)$, and $\Im(M_n)=\sum_{r,s=0}^{\infty}\Im_s^r(M_n)$ be a tensor algebra over R. A map $\phi_\varphi\mid_{r+s\rangle 0}:\overset{*}{\Im}(M_n)\to\Im(M_n)$ is called a Tachibana operator or ϕ_φ operator on M_n if*

(a) ϕ_φ is linear with respect to constant coefficient,
(b) $\phi_\varphi:\overset{*}{\Im}(M_n)\to\Im_{s+1}^r(M_n)$ for all r and s,
(c) $\phi_\varphi(K\overset{C}{\otimes}L)=(\phi_\varphi K)\otimes L+K\otimes\phi_\varphi L$ for all $K,L\in\overset{*}{\Im}(M_n)$,
(d) $\phi_{\varphi X}Y=-(L_Y\varphi)X$ for all $X,Y\in\Im_0^1(M_n)$ where L_Y is the Lie derivation with respect to Y,
(e) $(\phi_{\varphi X}\eta)Y=(d(\iota_Y\eta(\phi X)-(d(\iota_Y(\eta o\phi)X+\eta((L_Y\varphi)X)=(\phi X(\iota_Y\eta))(\phi X)-X(\iota_{\varphi Y}\eta)+\eta((L_Y\varphi)X)$
for all $\eta\in\Im_1^0(M_n)$ and $X,Y\in\Im_0^1(M_n)$, where $\iota_Y\eta=\eta(Y)=\eta\overset{C}{\otimes}Y,\overset{*}{\Im}{}_s^r(M_n)$ the module of all pure tensor fields of type (r,s) on M_n with respect to the affinor field φ [5].

Theorem 1. *For ϕ_φ Tachibana operator on M_n, L_X the operator Lie derivation with respect to X, $\widetilde{J}\in\Im_1^1(T(M_n))$ defined by (40) and $\eta(Y)=0$, we have*

$$\begin{aligned}&\text{(i)}\quad \phi_{\widetilde{J}Y^v}X^H=-((\hat{\nabla}_X\varphi)Y)^v+((\hat{\nabla}_X\eta)Y)^v\xi^H,\\&\text{(ii)}\quad \phi_{\widetilde{J}Y^H}X^H=-((L_X\varphi)Y)^H+\gamma\hat{R}(X,\varphi Y)+((L_X\eta)Y)^v\xi^v-\varphi^H\gamma\hat{R}(X,Y)+(\eta^v\gamma\hat{R}(X,Y))\xi^v+(\eta^H\gamma\hat{R}(X,Y))\xi^H,\\&\text{(iii)}\quad \phi_{\widetilde{J}Y^v}X^v=0,\\&\text{(iv)}\quad \phi_{\widetilde{J}Y^H}X^v=-((L_X\varphi)Y)^v+((\nabla_X\varphi)Y)^v+((L_X\eta)Y)^v\xi^H-((\nabla_X\eta)Y)^v\xi^H,\end{aligned}\tag{42}$$

where $X,Y\in\Im_0^1(M_n)$, a tensor field $\varphi\in\Im_1^1(M_n)$, a vector field ξ and a $1-$form $\eta\in\Im_1^0(M_n)$.

Proof. For $\widetilde{J} = \varphi^H - \xi^v \otimes \eta^v - \xi^H \otimes \eta^H$ and $\eta(Y) = 0$, we get

$$\begin{aligned}
\text{(i)}\quad \phi_{\widetilde{J}Y^v} X^H &= -(L_{X^H}\widetilde{J})Y^v = -(L_{X^H}\widetilde{J}Y^v - \widetilde{J}L_{X^H}Y^v) \\
&= -[X^H, (\varphi Y)^v - (\eta(Y)\xi)^H] + (\varphi^H - \xi^v \otimes \eta^v - \xi^H \otimes \eta^H)[X^H, Y^v] \\
&= -[X^H, (\varphi Y)^v] + [X^H, (\eta(Y)\xi)^H] + \varphi^H[X^H, Y^v] - \eta^v([X^H, Y^v])\xi^v - \eta^H([X^H, Y^v])\xi^H \\
&= -[X, \varphi Y]^v - (\nabla_{\varphi Y} X)^v + \varphi^H([X,Y]^v + (\nabla_Y X)^v) - \eta^v([X,Y]^v + (\nabla_Y X)^v)\xi^v \\
&\quad -\eta^H([X,Y]^v + (\nabla_Y X)^v)\xi^H \\
&= -((L_X\varphi)Y)^v - (\varphi(L_X Y))^v - (\hat{\nabla}_X \varphi Y)^v - [\varphi Y, X]^v + (\varphi L_X Y)^v + (\varphi \nabla_Y X)^v \\
&\quad -\eta^v([X,Y]^v)\xi^v - (\eta^v(\nabla_Y X)^v)\xi^v - (\eta[X,Y])^v\xi^H - \eta^H(\nabla_Y X)^v\xi^H \\
&= -((L_X\varphi)Y)^v - (\varphi(L_X Y))^v - ((\hat{\nabla}_X\varphi)Y)^v - (\varphi\hat{\nabla}_X Y)^v + ((L_X\varphi)Y)^v \\
&\quad +(\varphi(L_X Y))^v + (\varphi(L_X Y))^v + (\varphi\nabla_Y X)^v - (\eta[X,Y])^v\xi^H - (\eta^H(\nabla_Y X)^v)\xi^H \\
&= -((\hat{\nabla}_X\varphi)Y)^v - (\varphi\hat{\nabla}_X Y)^v + (\varphi(L_X Y))^v + \varphi^H(\nabla_Y X)^v + ((L_X\eta)Y)^v\xi^H \\
&\quad -(\eta^H((\hat{\nabla}_X Y)^v + [Y,X]^v))\xi^H \\
&= -((\hat{\nabla}_X\varphi)Y)^v - (\varphi\hat{\nabla}_X Y)^v + (\varphi(L_X Y))^v + \varphi^H((\hat{\nabla}_X Y)^v + [Y,X]^v) \\
&\quad +((L_X\eta)Y)^v\xi^H - (\eta(\hat{\nabla}_X Y)^v)\xi^H - (\eta(L_Y X))^v\xi^H \\
&= -((\hat{\nabla}_X\varphi)Y)^v - (\varphi\hat{\nabla}_X Y)^v + (\varphi(L_X Y))^v + (\varphi(\hat{\nabla}_X Y))^v - (\varphi(L_X Y))^v \\
&\quad +((L_X\eta)Y)^v\xi^H + ((\hat{\nabla}_X\eta)Y)^v)\xi^H + (\eta(L_X Y))^v\xi^H \\
&= -((\hat{\nabla}_X\varphi)Y)^v - (\varphi\hat{\nabla}_X Y)^v + (\varphi(\hat{\nabla}_X Y))^v + ((L_X\eta)Y)^v\xi^H + ((\hat{\nabla}_X\eta)Y)^v)\xi^H - ((L_X\eta)Y))^v\xi^H \\
&= -((\hat{\nabla}_X\varphi)Y)^v + ((\hat{\nabla}_X\eta)Y)^v)\xi^H,
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad \phi_{\widetilde{J}Y^H} X^H &= -(L_{X^H}\widetilde{J})Y^H = -L_{X^H}\widetilde{J}Y^H + \widetilde{J}L_{X^H}Y^H \\
&= -[X^H, (\varphi Y)^H - (\eta(Y)\xi)^v] + (\varphi^H - \xi^v \otimes \eta^v - \xi^H \otimes \eta^H)[X^H, Y^H] \\
&= -[X^H, (\varphi Y)^H] + [X^H, (\eta(Y)\xi)^v] + \varphi^H[X^H, Y^H] - \eta^v([X^H, Y^H])\xi^v - \eta^H([X^H, Y^H])\xi^H \\
&= -[X, \varphi Y]^H + \gamma\hat{R}(X, \varphi Y) + \varphi^H([X,Y]^H - \gamma\hat{R}(X,Y)) \\
&\quad -\eta^v([X,Y]^H - \gamma\hat{R}(X,Y))\xi^v - \eta^H([X,Y]^H - \gamma\hat{R}(X,Y))\xi^H \\
&= -((L_X\varphi)Y)^H - (\varphi(L_X Y))^H + \gamma\hat{R}(X,\varphi Y) + (\varphi(L_X Y))^H - \varphi^H\gamma\hat{R}(X,Y) \\
&\quad -(\eta L_X Y)^v\xi^v + (\eta^v\gamma\hat{R}(X,Y))\xi^v + \eta^H([X,Y]^H)\xi^H + (\eta^H\gamma\hat{R}(X,Y))\xi^H \\
&= -((L_X\varphi)Y)^H + \gamma\hat{R}(X,\varphi Y) + ((L_X\eta)Y)^v\xi^v - \varphi^H\gamma\hat{R}(X,Y) + (\eta^v\gamma\hat{R}(X,Y))\xi^v + (\eta^H\gamma\hat{R}(X,Y))\xi^H \\
&= -((L_X\varphi)Y)^H + \gamma\hat{R}(X,\varphi Y) + ((L_X\eta)Y)^v\xi^v - \widetilde{J}(\gamma\hat{R}(X,Y)),
\end{aligned}$$

$$\begin{aligned}
\text{(iii)}\quad \phi_{\widetilde{J}Y^V} X^v &= -(L_{X^v}\widetilde{J})Y^v = -L_{X^v}\widetilde{J}Y^v + \widetilde{J}L_{X^v}Y^v \\
&= -[X^v, (\varphi Y)^v - (\eta(Y)\xi)^H] + (\varphi^H - \xi^v \otimes \eta^v - \xi^H \otimes \eta^H)[X^v, Y^v] \\
&= -[X^v, (\varphi Y)^v] + [X^v, (\eta(Y)\xi)^H] \\
&= 0,
\end{aligned}$$

(iv) $\phi_{\widetilde{J}Y^H}X^v = -(L_{X^v}\widetilde{J})Y^H = -L_{X^v}\widetilde{J}Y^H + \widetilde{J}L_{X^v}Y^H$
$= -[X^v,(\varphi Y)^H - (\eta(Y)\xi)^v] + (\varphi^H - \xi^v\otimes\eta^v - \xi^H\otimes\eta^H)[X^v,Y^H]$
$= -[X^v,(\varphi Y)^H] + [X^v,(\eta(Y)\xi)^v] + \varphi^H[X^v,Y^H] - \eta^v([X^v,Y^H])\xi^v - \eta^H([X^v,Y^H])\xi^H$
$= -[X,\varphi Y]^v + (\nabla_X\varphi Y)^v + \varphi^H([X,Y]^v - (\nabla_X Y)^v) - \eta^v([X,Y]^v - (\nabla_X Y)^v)\xi^v - \eta^H([X,Y]^v - (\nabla_X Y)^v)\xi^H$
$= -((L_X\varphi)Y)^v - (\varphi(L_X Y))^v + ((\nabla_X\varphi)Y)^v + (\varphi\nabla_X Y)^v + (\varphi(L_X Y))^v$
$-(\varphi\nabla_X Y)^v - \eta^v([X,Y]^v)\xi^v + \eta^v(\nabla_X Y)^v\xi^v - (\eta L_X Y)^v\xi^H + (\eta\nabla_X Y)^v\xi^H$
$= -((L_X\varphi)Y)^v + ((\nabla_X\varphi)Y)^v + ((L_X\eta)Y)^v\xi^H - ((\nabla_X\eta)Y)^v\xi^H,$

where $\eta L_X Y = L_X\eta(Y) - (L_X\eta)Y)$ and $\eta\nabla_X Y = \nabla_X\eta(Y) - (\nabla_X\eta)Y), \varphi Y \in \Im_0^1(M_n)$.

Corollary 1. *If we put $Y = \xi$, i.e. $\eta(\xi) = 1$ and ξ has the conditions of (38), then we have*

(i) $\phi_{\widetilde{J}\xi^v}X^H = (L_X\xi)^H - \gamma\hat{R}(X,\xi) - ((\hat{\nabla}_X\varphi)\xi)^v + ((\hat{\nabla}_X\eta)\xi)^v\xi^H,$

(ii) $\phi_{\widetilde{J}\xi^H}X^H = (\hat{\nabla}_X\xi)^v - ((L_X\varphi)\xi)^H + ((L_X\eta)\xi)^v\xi^v - \varphi^H\gamma\hat{R}(X,\xi) + (\eta^v\gamma\hat{R}(X,\xi))\xi^v + (\eta^H\gamma\hat{R}(X,\xi))\xi^H,$

(iii) $\phi_{\widetilde{J}\xi^v}X^v = -(\hat{\nabla}_\xi X)^v,$

(iv) $\phi_{\widetilde{J}\xi^H}X^v = -((L_X\varphi)\xi)^v + ((\nabla_X\varphi)\xi)^v + ((L_X\eta)\xi)^v\xi^H - ((\nabla_X\eta)\xi)^v\xi^H.$

2.2 Vishnevskii Operators Applied to X^V and X^H in Almost Paracontact Structure

Definition 3. *Suppose now that ∇ is a linear connection on M, and let $\varphi \in \Im_1^1(M_n)$. We can replace the condition $d)$ of defination 2 by*

$$d')\quad \psi_{\varphi X}Y = \nabla_{\varphi X}Y - \varphi\nabla_X Y$$

for any $X,Y \in \Im_0^1(M_n)$. Then we can consider a new operator by a Vishnevskii operator or $\psi_\varphi-$operator on M, we shall mean a map $\psi_\varphi : \overset{*}{\Im}(M_n) \to \Im(M_n)$, which satisfies conditions $a), b), c), e)$ of definition 2 and the condition (d') [5].

Theorem 2. *For ψ_φ Vishnevskii operator on M_n , ∇^H the horizontal lift of an affine connection ∇ in M_n to $T(M_n)$, $\widetilde{J} \in \Im_1^1(T(M_n))$ defined by (40), we have*

$$\begin{aligned}
&\text{(i)} && \psi_{\widetilde{J}X^v}Y^H = -(\eta(X)\nabla_\xi Y)^H, && (43)\\
&\text{(ii)} && \psi_{\widetilde{J}X^H}Y^v = ((\hat{\nabla}_Y\varphi)X)^v - ((L_Y\varphi)X)^v + (\eta\hat{\nabla}_Y X)^v\xi^H - (\eta L_Y X)^v\xi^H,\\
&\text{(iii)} && \psi_{\widetilde{J}X^v}Y^v = -(\eta(X)\nabla_\xi Y)^v,\\
&\text{(iv)} && \psi_{\widetilde{J}X^H}Y^H = ((\hat{\nabla}_Y\varphi)X)^H - ((L_Y\varphi)X)^H + (\eta\hat{\nabla}_Y X)^v\xi^v - (\eta L_Y X)^v\xi^v,
\end{aligned}$$

where $X,Y \in \Im_0^1(M_n)$, a tensor field $\varphi \in \Im_1^1(M_n)$, a vector field ξ and a $1-$form $\eta \in \Im_1^0(M_n)$.

Proof. For $\widetilde{J} = \varphi^H - \xi^v\otimes\eta^v - \xi^H\otimes\eta^H$, we get

(i) $\psi_{\widetilde{J}X^v}Y^H = \nabla^H_{\widetilde{J}X^v}Y^H - \widetilde{J}\nabla^H_{X^v}Y^H$
$= \nabla^H_{(\varphi X)^v - (\eta(X)\xi)^H}Y^H - (\varphi^H - \xi^v\otimes\eta^v - \xi^H\otimes\eta^H)\nabla^H_{X^v}Y^H$
$= \nabla^H_{(\varphi X)^v}Y^H - (\eta(X))^v\nabla^H_{\xi^H}Y^H$
$= -(\eta(X))^v(\nabla_\xi Y)^H$
$= -(\eta(X)\nabla_\xi Y)^H,$

(ii) $$\begin{aligned}\psi_{\tilde{J}X^H}Y^v &= \nabla^H_{\tilde{J}X^H}Y^v - \tilde{J}\nabla^H_{X^H}Y^v\\ &= \nabla^H_{(\varphi X)^H-(\eta(X)\xi)^v}Y^v - (\varphi^H - \xi^v\otimes\eta^v - \xi^H\otimes\eta^H)\nabla^H_{X^H}Y^v\\ &= \nabla^H_{(\varphi X)^H}Y^v - (\eta(X))^v\nabla^H_{\xi^v}Y^v - \varphi^H(\nabla_X Y)^v + \eta^v(\nabla_X Y)^v\xi^v\\ &\quad +\eta^H(\nabla_X Y)^v\xi^H\\ &= (\hat{\nabla}_Y\varphi X)^v + [\varphi X, Y]^v - \varphi^H((\hat{\nabla}_Y X)^v + [X,Y]^v) + (\eta\nabla_X Y)^v\xi^H\\ &= ((\hat{\nabla}_Y\varphi)X)^v + (\varphi\hat{\nabla}_Y X)^v - ((L_Y\varphi)X)^v - (\varphi(L_Y X))^v\\ &\quad -(\varphi\hat{\nabla}_Y X)^v + (\varphi(L_Y X))^v + (\eta\nabla_X Y)^v\xi^H\\ &= ((\hat{\nabla}_Y\varphi)X)^v - ((L_Y\varphi)X)^v + (\eta\nabla_X Y)^v\xi^H\\ &= ((\hat{\nabla}_Y\varphi)X)^v - ((L_Y\varphi)X)^v + \eta^H((\hat{\nabla}_Y X)^v + [X,Y]^v)\xi^H\\ &= ((\hat{\nabla}_Y\varphi)X)^v - ((L_Y\varphi)X)^v + (\eta\hat{\nabla}_Y X)^v\xi^H - (\eta L_Y X)^v\xi^H,\end{aligned}$$

(iii) $$\begin{aligned}\psi_{\tilde{J}X^v}Y^v &= \nabla^H_{\tilde{J}X^v}Y^v - \tilde{J}\nabla^H_{X^v}Y^v\\ &= \nabla^H_{(\varphi X)^v-(\eta(X)\xi)^H}Y^v - (\varphi^H - \xi^v\otimes\eta^v - \xi^H\otimes\eta^H)\nabla^H_{X^v}Y^v\\ &= \nabla^H_{(\varphi X)^v}Y^v - (\eta(X))^v\nabla^H_{\xi^H}Y^v\\ &= -(\eta(X))^v(\nabla_\xi Y)^v\\ &= -(\eta(X)\nabla_\xi Y)^v,\end{aligned}$$

(iv) $$\begin{aligned}\psi_{\tilde{J}X^H}Y^H &= \nabla^H_{\tilde{J}X^H}Y^H - \tilde{J}\nabla^H_{X^H}Y^H\\ &= \nabla^H_{(\varphi X)^H-(\eta(X)\xi)^v}Y^H - (\varphi^H - \xi^v\otimes\eta^v - \xi^H\otimes\eta^H)\nabla^H_{X^H}Y^H\\ &= \nabla^H_{(\varphi X)^H}Y^H - (\eta(X))^v\nabla^H_{\xi^v}Y^H - \varphi^H(\nabla_X Y)^H + \eta^v(\nabla_X Y)^H\xi^v + \eta^H(\nabla_X Y)^H\xi^H\\ &= (\nabla_{\varphi X}Y)^H - (\varphi\nabla_X Y)^H + (\eta\nabla_X Y)^v\xi^v\\ &= ((\hat{\nabla}_Y\varphi X) + [\varphi X, Y])^H - \varphi^H(\hat{\nabla}_Y X + [X,Y])^H + \eta^v(\hat{\nabla}_Y X + [X,Y])^H\xi^v\\ &= ((\hat{\nabla}_Y\varphi)X)^H + (\varphi\hat{\nabla}_Y X)^H - ((L_Y\varphi)X)^H - (\varphi(L_Y X))^H\\ &\quad -(\varphi\hat{\nabla}_Y X)^H + (\varphi(L_Y X))^H + (\eta\hat{\nabla}_Y X)^v\xi^v - (\eta L_Y X)^v\xi^v\\ &= ((\hat{\nabla}_Y\varphi)X)^H - ((L_Y\varphi)X)^H + (\eta\hat{\nabla}_Y X)^v\xi^v - (\eta L_Y X)^v\xi^v.\end{aligned}$$

Corollary 2. *If we put $X = \xi$, i.e. $\eta(\xi) = 1$ and ξ has the conditions of (38), then we have*

(i) $\psi_{\tilde{J}\xi^v}Y^H = -(\nabla_\xi Y)^H$,

(ii) $\psi_{\tilde{J}\xi^H}Y^v = ((\hat{\nabla}_Y\varphi)\xi)^v - ((L_Y\varphi)\xi)^v - ((\hat{\nabla}_Y\eta)\xi)^v\xi^H + ((L_Y\eta)X)^v\xi^H$,

(iii) $\psi_{\tilde{J}\xi^v}Y^v = -(\nabla_\xi Y)^v$,

(iv) $\psi_{\tilde{J}\xi^H}Y^H = ((\hat{\nabla}_Y\varphi)\xi)^H - ((L_Y\varphi)\xi)^H - ((\hat{\nabla}_Y\eta)\xi)^v\xi^v + ((L_Y\eta)\xi)^v\xi^v$.

3 Conclusion

The paper deals with Tachibana and Vishnevskii operators applied to X^V and X^H in almost paracontact structure on tangent bundle $T(M)$. Firstly, we give some properties about vertical lifts, complete lifts, horizontal lifts and almost paracontact structure on tangent bundle and we get some general conclusions on M after the Tachibana and Vishnevskii operators applied on almost parakontakt structure. Later, by using features of almost parakontakt structure, we obtain several new results in almost paracontact structure on $T(M)$.

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