Journal of Abstract and Computational Mathematics
http://www.ntmsci.com/jacm

Comparison of sinc methods for the solution of fractional boundary value problems

Sertan Alkan[1], *Mehmet Casim Serifoglu*[2], *Turgut Yeloglu*[3]

[1,3]Department of Mathematics, Mustafa Kemal University, Hatay, Turkey
[2]Institute of Natural and Applied Sciences, Mustafa Kemal University, Hatay, Turkey

Received: 15 May 2016, Accepted: 1 June 2016
Published online: 9 June 2016.

Abstract: In this study, sinc-Galerkin and sinc-collocation methods are presented to solve numerically some well-known class of fractional differential equations (FDEs) utilizing Mathematica. By using these two methods, FDEs with the variable coefficient and boundary values are examined. To obtain an approximate solution of the given class of differential equations with sinc methods is reduced a system of algebraic equations which is simpler form via theorems. Obtained numerical results and approximate solution functions are presented in the table and graphical forms, respectively. It can be concluded from tables and graphs that sinc-collocation method has the more accurate and effective results than sinc-Galerkin method.

Keywords: Fractional boundary value problems, collocation method, Galerkin method, sinc function, Caputo derivative.

1 Introduction

Many systems in applied sciences, such as, signal and image processing, earthquake engineering, electrochemistry and biomedical engineering, can be modeled by using fractional calculus in the form of fractional differential equations. In order to better analyze these systems, it is required to know the approximate solutions of these systems. For this aim, several solution methods are developed to get the approximate solutions of fractional differential equations. Some well-known numerical methods for obtaining the approximate solutions of FBVP are summarized as follows, but not limited to: Homotopy perturbation method [1,2], Differential transform method[3,4], Adomian decomposition method[5,6,7], Variational iteration method [8,9], Cubic spline method [10], Haar wavelet method[11] and Homotopy analysis method[12].

The sinc methods were introduced in [13] and expanded in [14] by Frank Stenger. The sinc functions were first analyzed in [15,16]. Later, sinc methods are studied by several authors in [17-24].

Particularly, in this paper, to compare the performance of sinc-collocation method and sinc-Galerkin method are applied ones to a class of fractional order boundary value problem with variable coefficients in the following form

$$y'' + p(x)y' + q(x)y^{(\alpha)} + r(x)y = f(x), \qquad 0 < \alpha < 1 \tag{1}$$
$$y(a) = 0, y(b) = 0$$

where $y^{(\alpha)}$ is the left Caputo fractional derivative of order α of $y(x)$

* Corresponding author e-mail: sertanalkan@mku.edu.tr

The rest of this paper is organized as follows. In section 2, we give some definitions and theorems for fractional calculus and sinc methods. In section 3, we use sinc methods to obtain an approximate solution of a general fractional differential equation and obtained results are given as some theorems.In section 4, some test problems are given to compare the ability of present methods by using tables and graphics. Finally, in section 5, the paper is completed with a conclusion.

2 Preliminaries and Notations

In this section, we recall notations and definitions of the sinc function and Caputo fractional derivative and derive useful formulas that are important for this paper. For more details see[24, 25].

Definition 1. *Let $f:[a,b]\to\mathbb{R}$ be a function, α a positive real number, n the integer satisfying $n-1\leq\alpha<n$, and Γ the Euler gamma function. Then, the left Caputo fractional derivative of order α of $f(x)$ is given as*

$$f^{(\alpha)}(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt. \tag{2}$$

Definition 2. *The Sinc function is defined on the whole real line $-\infty<x<\infty$ by*

$$sinc(x)=\begin{cases}\frac{\sin(\pi x)}{\pi x} & x\neq 0\\ 1 & x=0.\end{cases}$$

Definition 3. *For $h>0$ and $k=0,\pm1,\pm2,\ldots$ the translated sinc function with space node are given by:*

$$S(k,h)(x)=sinc\left(\frac{x-kh}{h}\right)=\begin{cases}\frac{\sin\left(\pi\frac{x-kh}{h}\right)}{\pi\frac{x-kh}{h}} & x\neq kh\\ 1 & x=kh.\end{cases}$$

Definition 4. *If $f(x)$ is defined on the real line, then for $h>0$ the series*

$$C(f,h)(x)=\sum_{k=-\infty}^{\infty}f(kh)sinc\left(\frac{x-kh}{h}\right)$$

is called the Whittaker cardinal expansion of f whenever this series converges.

In general, approximations can be constructed for infinite, semi-infinite and finite intervals. To construct an approximation on the interval (a,b) the conformal map

$$\phi(z)=\ln\left(\frac{z-a}{b-z}\right)$$

is employed. The basis functions on the interval (a,b) are derived from the composite translated sinc functions

$$S_k(z)=S(k,h)(z)o\phi(z)=sinc\left(\frac{\phi(z)-kh}{h}\right).$$

The inverse map of $w=\phi(z)$ is

$$z=\phi^{-1}(w)=\frac{a+be^w}{1+e^w}.$$

The sinc grid points $z_k \in (a,b)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

Theorem 1. *Let Γ be $(0,1), F \in B(D_E)$, then for $h > 0$ sufficiently small,*

$$\int_\Gamma F(z)dz - h\sum_{j=-\infty}^{\infty}\frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2}\int_{\partial D}\frac{F(z)k(\phi,h)(z)}{\sin(\pi\phi(z)/h)}dz \equiv I_F \tag{3}$$

where

$$|k(\phi,h)|_{z\in\partial D} = \left|e^{\left[\frac{i\pi\phi(z)}{h}sgn(Im\phi(z))\right]}\right|_{z\in\partial D} = e^{\frac{-\pi d}{h}}.$$

For the sinc methods, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 2. *If there exist positive constants α, β and C such that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C\begin{cases} e^{-\alpha|\phi(x)|} & x \in \psi((-\infty,\infty)) \\ e^{-\beta|\phi(x)|} & x \in \psi((0,\infty)). \end{cases} \tag{4}$$

then the error bound for the quadrature rule (3) is

$$\left|\int_\Gamma F(x)dx - h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)}\right| \le C\left(\frac{e^{-\alpha Mh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta}\right) + |I_F| \tag{5}$$

The infinite sum in (3) is truncated with the use of (4) to arrive at the inequality (5). Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha M}}$$

and

$$N \equiv \left[\left\lfloor\frac{\alpha M}{\beta} + 1\right\rfloor\right]$$

where $[\lfloor . \rfloor]$ is an integer part of the statement and M is the integer value which specifies the grid size, then

$$\int_\Gamma F(x)dx = h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi\alpha dM)^{1/2}}\right). \tag{6}$$

3 Numerical Methods

3.1 The sinc-Galerkin method

An approximate solution of $y(x)$ in (1) is represented by the formula

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x), \qquad n = M+N+1 \tag{7}$$

where S_k is function $S(k,h)\circ\phi(x)$ for some fixed step size h. The unknown coefficients c_k in (7) are determined by orthogonalizing the residual with respect to the basis functions, i.e.

$$\langle y'',S_k\rangle+\langle p(x)y',S_k\rangle+\langle q(x)y^{(\alpha)},S_k\rangle+\langle r(x)y,S_k\rangle=\langle f(x),S_k\rangle,\qquad k=-M,\dots,N \tag{8}$$

The inner product used for the sinc-Galerkin method is defined by

$$\langle f,g\rangle=\int_a^b f(x)g(x)w(x)dx$$

where $w(x)$ a weight function which is taken for second-order boundary value problems in the following form

$$w(x)=\frac{1}{\phi'(x)}.$$

We need the following theorems for the approximation of inner products in (8).

Theorem 3. *The following relations hold:*

$$\langle y'',S_k\rangle\approx h\sum_{j=-M}^{N}\sum_{i=0}^{2}\frac{y(x_j)}{\phi'(x_j)h^i}\delta_{kj}^{(i)}g_{2,i}(x_j) \tag{9}$$

$$\langle p(x)y',S_k\rangle\approx -h\sum_{j=-M}^{N}\sum_{i=0}^{1}\frac{y(x_j)}{\phi'(x_j)h^i}\delta_{kj}^{(i)}g_{1,i}(x_j) \tag{10}$$

and for $G(x)=r(x)y(x)$ and $G(x)=f(x)$

$$\langle G,S_k\rangle\approx h\frac{G(x_k)w(x_k)}{\phi'(x_k)}. \tag{11}$$

Theorem 4. *The following relation holds for $0<\alpha<1$:*

$$\langle q(x)y^{(\alpha)},S_k\rangle\approx -\frac{h}{\Gamma(1-\alpha)}\sum_{j=-M}^{N}\frac{y(x_j)}{\phi'(x_j)}\frac{d}{dx}\left[h_L\sum_{r=-L}^{L}\frac{(x_r-x)^{\alpha}K(x_r)}{\xi'(x_r)}\right]_{x=x_j} \tag{12}$$

where $K(x)=q(x)S_k(x)w(x)$, $\xi(t)=\ln\left(\frac{t-x}{1-t}\right)$ and $h_L=\pi/\sqrt{L}$.

The proofs of these theorems and values of $g_{k,i}(x)$ can be found in [24]. Replacing each term of (8) with the approximation defined in (9)-(12), respectively, and replacing $y(x_j)$ by c_j, and dividing by h, we obtain the following theorem.

Theorem 5. *If the assumed approximate solution of the boundary-value problem (1) is (7), then the discrete sinc-Galerkin system for the determination of the unknown coefficients $\{c_j\}_{j=-M}^{N}$ is given, for $k=-M,\dots,N$, by*

$$\sum_{j=-M}^{N}\left\{\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}c_j-\sum_{i=0}^{1}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{1,i}(x_j)}{\phi'(x_j)}c_j-\frac{1}{\Gamma(1-\alpha)}\frac{c_j}{\phi'(x_j)}\frac{d}{dx}\left[h_L\sum_{r=-L}^{L}\frac{(x_r-x)^{\alpha}K(x_r)}{\xi'(x_r)}\right]_{x=x_j}\right\}$$
$$+\frac{r(x_k)w(x_k)}{\phi'(x_k)}c_k=\frac{f(x_k)w(x_k)}{\phi'(x_k)}. \tag{13}$$

Now we define some notations to represent in the matrix-vector form for system (13). Let $\mathbf{D}(y)$ denotes a diagonal matrix whose diagonal elements are $y(x_{-M}),y(x_{-M+1}),\dots,y(x_N)$ and non-diagonal elements are zero, also let $\mathbf{I}^{(i)}$ denotes the matrices for $0\le i\le 2$ by

$$\mathbf{I}^{(i)}=[\delta_{jk}^{(i)}],\qquad j,k=-M,\dots,N.$$

and

$$\mathbf{F}=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left[h_L\sum_{r=-L}^{L}\frac{(x_r-x)^{\alpha}K(x_r)}{\xi'(x_r)}\right]_{x=x_j}$$

where $\mathbf{D},\mathbf{F},\mathbf{I}^{(0)},\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ are square matrices of order $n\times n$. In order to calculate unknown coefficients c_k in linear system (13), we rewrite this system by using the above notations in matrix-vector form as

$$\mathbf{Ac}=\mathbf{B} \tag{14}$$

where

$$\mathbf{A}=\sum_{j=0}^{2}\frac{1}{h^j}\mathbf{I}^{(2)}\mathbf{D}\left(\frac{g_{2,j}}{\phi'}\right)-\sum_{j=0}^{1}\frac{1}{h^j}\mathbf{I}^{(1)}\mathbf{D}\left(\frac{g_{1,j}}{\phi'}\right)+\mathbf{D}\left(\frac{1}{\phi'}\right)\mathbf{F}+\mathbf{I}^{(0)}\mathbf{D}\left(\frac{g_{0,0}}{\phi'}\right)$$
$$\mathbf{B}=\mathbf{D}\left(\frac{wf}{\phi'}\right)\mathbf{1}$$
$$\mathbf{c}=\left(c_{-M},c_{-M+1},\ldots,c_{N-1},c_N\right)^T$$

Now we have linear system of n equations in the n unknown coefficients given by (14). Solving it, we can obtain the unknown coefficients c_k that are necessary for approximate solution in (7).

3.2 The sinc-collocation method

We assume an approximate solution for $y(x)$ in problem (1) by the finite expansion of sinc basis functions

$$y_n(x)=\sum_{k=-M}^{N}c_kS_k(x),\qquad n=M+N+1 \tag{15}$$

where $S_k(x)$ is the function $S(k,h)\circ\phi(x)$. The unknown coefficients c_k in (15) are determined by sinc-collocation method. For this purpose, the first and second derivatives of $y_n(x)$ are given by

$$\frac{d}{dx}y_n(x)=\sum_{k=-M}^{N}c_k\phi'(x)\frac{d}{d\phi}S_k(x) \tag{16}$$

$$\frac{d^2}{dx^2}y_n(x)=\sum_{k=-M}^{N}c_k\left(\phi''(x)\frac{d}{d\phi}S_k(x)+(\phi')^2\frac{d^2}{d\phi^2}S_k(x)\right). \tag{17}$$

Similarly, α order derivative of $y_n(x)$ for $0<\alpha<1$ is given by the following theorem.

Theorem 6. *If ξ is a conformal map for the interval $[a,x]$, then α order derivative of $y_n(x)$ for $0<\alpha<1$ is given by*

$$y_n^{(\alpha)}(x)=\sum_{k=-M}^{N}c_kR(x) \tag{18}$$

where

$$R(x)=S_k^{(\alpha)}(x)\approx\frac{h_L}{\Gamma(1-\alpha)}\sum_{r=-L}^{L}\frac{(x-x_r)S_k'(x_r)}{\xi'(x_r)}.$$

Proof. see [26]. Replacing each term of (1) with the approximation given in (15)-(18), multiplying the resulting equation by $\{(1/\phi')^2\}$ and setting $x = x_j$, we obtain the following linear system

$$\sum_{k=-M}^{N} c_k \left\{ \frac{d^2}{d\phi^2} S_k + \left[p\left(\frac{1}{\phi'}\right) - \left(\frac{1}{\phi'}\right)' \right] \frac{d}{d\phi} S_k + q\left(\frac{1}{\phi'}\right)^2 R + r\left(\frac{1}{\phi'}\right)^2 S_k \right\}(x_j) \\ = \left(\hat{f}\left(\frac{1}{\phi'}\right)^2 \right)(x_j), \qquad j = -M, \ldots, N.$$

By using Lemma 1 in [27], we know that

$$\delta_{jk}^{(0)} = \delta_{kj}^{(0)}, \qquad \delta_{jk}^{(1)} = -\delta_{kj}^{(1)}, \qquad \delta_{jk}^{(2)} = \delta_{kj}^{(2)}$$

then we obtain the following theorem.

Theorem 7. *If the assumed approximate solution of boundary value problem (1) is (15), then the discrete sinc-collocation system for the determination of the unknown coefficients* $\{c_k\}_{k=-M}^{N}$ *is given by*

$$\sum_{k=-M}^{N} c_k \left\{ \frac{1}{h^2}\delta_{jk}^{(2)} + \frac{1}{h}\left[\left(\frac{1}{\phi'}\right)' - p\left(\frac{1}{\phi'}\right) \right](x_j)\delta_{jk}^{(1)} + \left(q\left(\frac{1}{\phi'}\right)^2 R \right)(x_j) + \left(r\left(\frac{1}{\phi'}\right)^2 \right)(x_j)\delta_{jk}^{(0)} \right\} \\ = \left(\hat{f}\left(\frac{1}{\phi'}\right)^2 \right)(x_j), \qquad j = -M, \ldots, N. \tag{19}$$

Now we define some notations to represent in the matrix-vector form for system (9). Let $\mathbf{D}(y)$ denotes a diagonal matrix whose diagonal elements are $y(x_{-M}), y(x_{-M+1}), \ldots, y(x_N)$ and non-diagonal elements are zero, let $\mathbf{F} = R(x_j)$ denote a matrix and also let $\mathbf{I}^{(i)}$ denotes the matrices

$$\mathbf{I}^{(i)} = [\delta_{jk}^{(i)}], \qquad i = 0, 1, 2$$

where $\mathbf{D}, \mathbf{F}, \mathbf{I}^{(0)}, \mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ are square matrices of order $n \times n$. In order to calculate unknown coefficients c_k in linear system (19), we rewrite this system by using the above notations in matrix-vector form as

$$\mathbf{Ac} = \mathbf{B} \tag{20}$$

where

$$\mathbf{A} = \frac{1}{h^2}\mathbf{I}^{(2)} + \frac{1}{h}\mathbf{D}\left(\left(\frac{1}{\phi'}\right)' - p\left(\frac{1}{\phi'}\right) \right)\mathbf{I}^{(1)} + \mathbf{D}\left(q\left(\frac{1}{\phi'}\right)^2 \right)\mathbf{F} + \mathbf{D}\left(r\left(\frac{1}{\phi'}\right)^2 \right)\mathbf{I}^{(0)}$$

$$\mathbf{B} = \mathbf{D}\left(\frac{f}{(\phi')^2} \right)\mathbf{1}$$

$$\mathbf{c} = (c_{-M}, c_{-M+1}, \ldots, c_N)^T.$$

Now we have linear system of n equations in the n unknown coefficients given by (20). When it is solved, we can obtain the unknown coefficients c_k that are necessary for approximate solution in (7).

Table 1: Maximum errors of the present methods for *Example 1*.

N	E_{SC}	E_{SG}
5	1.604×10^{-3}	2.633×10^{-3}
10	7.453×10^{-5}	2.345×10^{-4}
20	1.073×10^{-6}	4.289×10^{-6}
40	3.962×10^{-9}	1.252×10^{-7}

Table 2: Numerical results for *Example 1* when $N = 20$.

x	Exact	SC Error	SG Error
0	0	0	0
0.1	0.009	3.258×10^{-6}	3.144×10^{-6}
0.2	0.032	2.270×10^{-7}	2.793×10^{-6}
0.3	0.063	6.981×10^{-6}	6.967×10^{-6}
0.4	0.096	1.166×10^{-5}	9.308×10^{-6}
0.5	0.125	1.073×10^{-6}	2.933×10^{-6}
0.6	0.144	1.065×10^{-5}	8.611×10^{-6}
0.7	0.147	5.127×10^{-6}	1.022×10^{-6}
0.8	0.128	1.434×10^{-6}	1.066×10^{-8}
0.9	0.081	1.778×10^{-6}	8.207×10^{-7}
1	0	0	0

4 Computational examples

In this section, some numerical examples are presented to show the accuracy of the introduced methods by `MATHEMATICA 10`. In all examples, $h = \pi/\sqrt{N}, N = M = L$ are taken into account. In the examples, the maximum absolute error at sinc grid points is taken as like [18]

$$E_{SC} = \max_{-N \le i \le N} |y_{exact}(x_i) - y_{n,SC}(x_i)|$$

and

$$E_{SG} = \max_{-N \le i \le N} |y_{exact}(x_i) - y_{n,SG}(x_i)|.$$

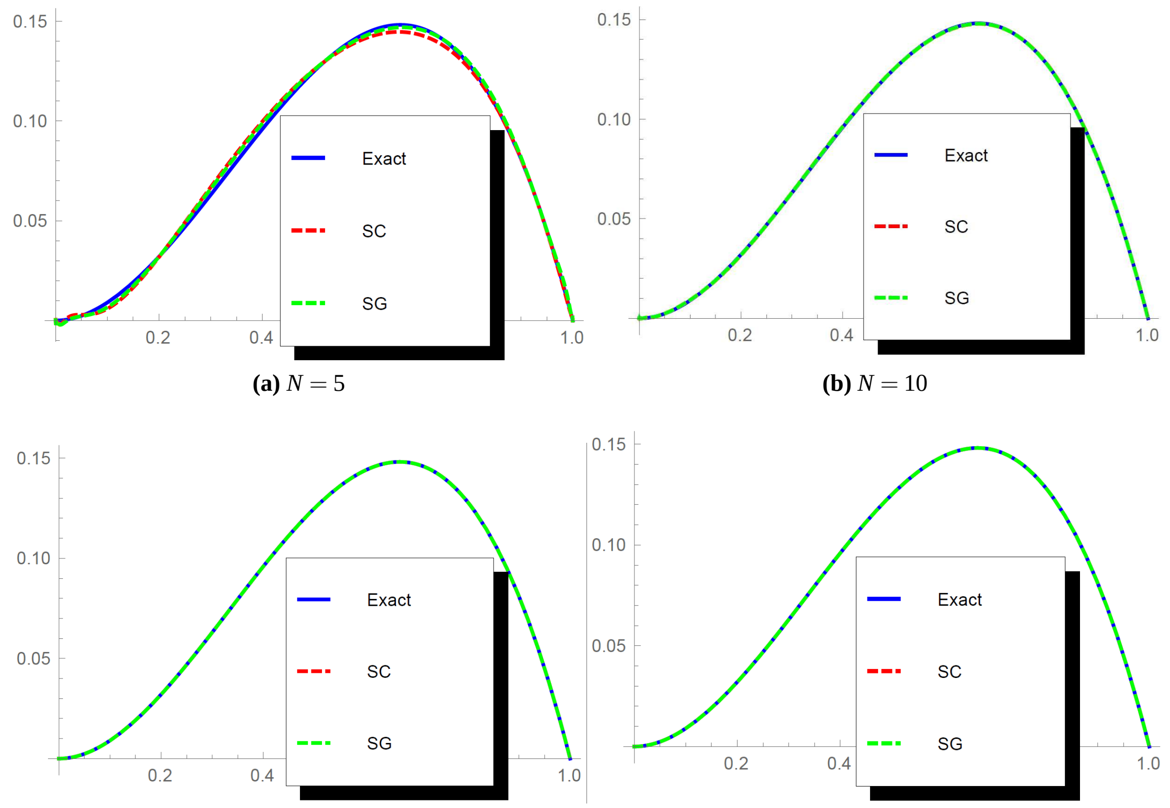
Example 1. Consider linear fractional boundary value problem

$$y''(x) - xy'(x) + x^2 y^{(0.3)}(x) = f(x)$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \qquad y(1) = 0$$

where $f(x) = -3x^3 + 2x^2 - 6x + 2 - \frac{6}{\Gamma(3.7)}x^{4.7} + \frac{2}{\Gamma(2.7)}x^{3.7}$. The exact solution of this problem is $y(x) = x^2(1-x)$. The numerical solutions which are obtained by using the present method for this problem are presented in Table 1 and Table 2. Additionally, the graphics of the exact and approximate solutions for different values of N are given in Figure 1.



(a) $N = 5$ **(b)** $N = 10$

(c) $N = 20$ **(d)** $N = 40$

Fig. 1: Graphs of exact and approximate solutions for Example 1

Example 2. Consider the following singular linear fractional boundary value problem

$$y''(x) + \frac{1}{x} y^{(0.7)}(x) + \frac{1}{x-1} y(x) = f(x)$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \qquad y(1) = 0$$

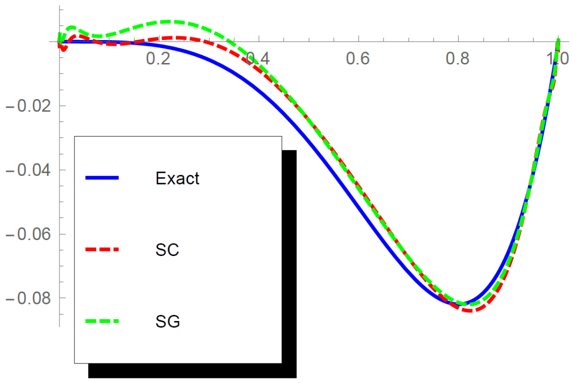
where $f(x) = x^4 + 20x^3 - 12x^2 + \frac{120}{\Gamma(5.3)} x^{3.3} - \frac{24}{\Gamma(4.3)} x^{2.3}$. The exact solution of this problem is $y(x) = x^4(x-1)$. The numerical solutions which are obtained by using the present method for this problem are presented in Table 3 and Table 4. Additionally, the graphics of the exact and approximate solutions for different values of N are given in Figure 2.

Table 3: Maximum errors of the present methods for *Example 2*.

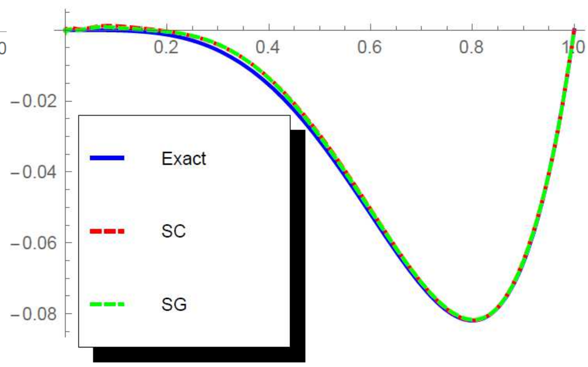
N	E_{SC}	E_{SG}
5	6.760×10^{-3}	7.230×10^{-3}
10	1.690×10^{-3}	1.519×10^{-3}
20	4.118×10^{-4}	4.550×10^{-4}
40	7.567×10^{-5}	2.043×10^{-4}

Table 4: Numerical results for *Example 2* when $N = 20$.

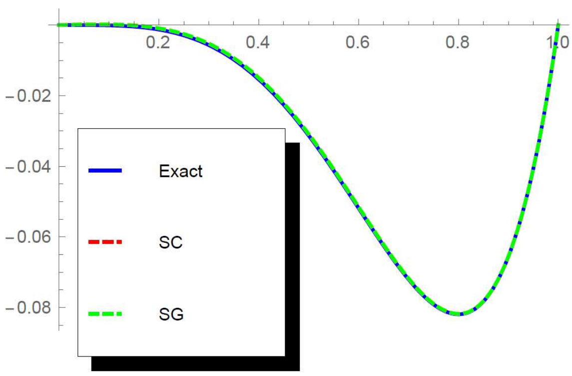
x	Exact	SC Error	SG Error
0	0	0	0
0.1	0.009	2.470×10^{-4}	2.754×10^{-4}
0.2	0.032	3.550×10^{-4}	3.777×10^{-4}
0.3	0.063	3.912×10^{-4}	4.340×10^{-4}
0.4	0.096	4.380×10^{-4}	4.697×10^{-4}
0.5	0.125	4.013×10^{-4}	4.125×10^{-4}
0.6	0.144	3.235×10^{-4}	3.380×10^{-4}
0.7	0.147	3.344×10^{-4}	2.558×10^{-4}
0.8	0.128	1.301×10^{-4}	1.402×10^{-4}
0.9	0.081	2.215×10^{-5}	5.324×10^{-5}
1	0	0	0



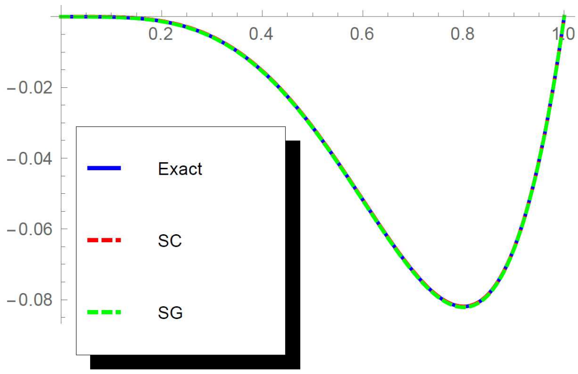
(a) $N = 5$



(b) $N = 10$



(c) $N = 20$



(d) $N = 40$

Fig. 2: Graphs of exact and approximate solutions for Example 2

5 Conclusion

In the present study, sinc-Galerkin and sinc-collocation methods are applied to find the approximate solutions of fractional order two-point boundary value problems. In order to compare the performance of the methods for FBVPs, they are applied to some special examples and obtained solutions are compared with exact solutions and each other. Then, differences are shown in tables and graphical forms. Observing these tables and graphical forms, it can be concluded that sinc-collocation method has the more accurate and effective results than sinc-Galerkin methods for obtaining the approximate solution of FBVPs. Also, numerical results can be obtained with less computation procedure by using sinc-collocation method than sinc-Galerkin method.

Acknowledgements

The authors express their sincere thanks to the referee(s) for the careful and details reading of the manuscript.

References

[1] Wang, Q., Homotopy perturbation method for fractional KdV equation, *Appl. Math. Comput.*, 2007, V.190, N.2, pp.1795-1802.

[2] Wang, Q., Homotopy perturbation method for fractional KdV-Burgers equation, *Chaos Soliton Fract.*, 2008, V.35, N.5, pp.843-850.

[3] Arikoglu, A.,Ozkol, I., Solution of fractional differential equations by using differential transform method, *Chaos Soliton Fract.*, 2007, V.34, N.5, pp.1473-1481.

[4] Secer, A., Akinlar, M.A., Cevikel, A., Efficient solutions of systems of fractional PDEs by differential transform method, *Adv. Differ. Equ-Ny.*, 2012, V.2012, N.1,pp.1-7.

[5] Jafari, H., Daftardar-Gejji, V., Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method, *Appl. Math. Comput.*,2006,V.180, N.2, pp.700-706.

[6] Wang, Q., Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method, *Appl. Math. Comput.*, 2006, V.182, N.2, pp.1048-1055.

[7] Daftardar-Gejji, V., Bhalekar, S., Solving multi-term linear and non-linear diffusion-wave equations of fractional order by Adomian decomposition method, *Appl. Math. Comput.*, 2008, V.202, N.1, pp.113-120.

[8] Odibat, Z. M., Momani, S., Application of variational iteration method to nonlinear differential equations of fractional order, *Int. J. Nonlinear Sci. Num.*,2006, V.7, N.1, pp.27-34.

[9] Momani, S.,Odibat, Z., Numerical comparison of methods for solving linear differential equations of fractional order, *Chaos Soliton Fract.*, 2007, V.31, N.5, pp.1248-1255.

[10] Zahra, W. K., Elkholy, S. M., Cubic Spline Solution Of Fractional Bagley-Torvik Equation, *Electron. J. Math. Anal. Appl.*, 2013, V.1, N.2, pp.230-241.

[11] Rehman, M.U., Khan, R.A., A numerical method for solving boundary value problems for fractional differential equations, *Appl. Math. Model.*, 2012, V.36, N.3, pp.894-907.

[12] Hashim, I., Abdulaziz, O., Momani, S., Homotopy analysis method for fractional IVPs, *Commun. Nonlinear Sci.*, 2009, V.14, N.3, pp.674-684.

[13] Stenger, F., Approximations via Whittaker's cardinal function, *J. Approx. Theory*, 1976, V.17, N.3, pp.222-240.

[14] Stenger, F., A sinc-Galerkin method of solution of boundary value problems, *Math. Comput.*, 1979, V.33, N.145, pp.85-109.

[15] Whittaker, E.T., On the functions which are represented by the expansions of the interpolation theory, *Proc. R. Soc. Edinb.*, 1915, V.35, pp.181-194.

[16] Whittaker, J.M., Interpolation Function Theory, *Cambridge Tracts in Mathematics and Mathematical Physics*, 1935, V.33, Cambridge University Press, London.

[17] Carlson, T., Dockery, J., Lund, J., A sinc-collocation method for initial value problems, *Math. Comput.*, 1997, V.66, N.217, pp.215-235.

[18] Mohsen, A., El-Gamel, M., On the Galerkin and collocation methods for two-point boundary value problems using sinc bases, *Comput. Math. Appl.*, 2008, V.56, N.4, pp.930-941.

[19] Rashidinia, J., Nabati, M., Sinc-Galerkin and Sinc-Collocation methods in the solution of nonlinear two-point boundary value problems, *Comput. Appl. Math.*, 2013, V.32, N.2, pp.315-330.

[20] Saadatmandi, A., Dehghan, M., The use of Sinc-collocation method for solving multi-point boundary value problems, *Commun. Nonlinear Sci.*, 2012, V.17, N.2, pp.593-601.

[21] Parand, K., Dehghan, M., Pirkhedri, A., Sinc-collocation method for solving the Blasius equation, *Phys. Lett. A*, 2009, V.373, N.44, pp.4060-4065.

[22] Dehghan, M., Saadatmandi, A., The numerical solution of a nonlinear system of second-order boundary value problems using the sinc-collocation method, *Math. Comput. Model.*, 2007, V.46, N.11, pp.1434-1441.

[23] El-Gamel, M., Sinc-collocation method for solving linear and nonlinear system of second-order boundary value problems, *Appl. Math.*, 2012, V.3, N.11, pp.1627-1633.

[24] Secer, A., Alkan, S., Akinlar, M. A., Bayram, M., Sinc-Galerkin method for approximate solutions of fractional order boundary value problems, *Bound. Value Probl.*, 2013, V.2013, N.1, pp.1-14.

[25] Almeida, R., Torres, D.F.M., Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives, *Commun. Nonlinear Sci*, 2011, V.16, N.3, pp.1490-1500.

[26] Alkan, S., A new solution method for nonlinear fractional integro-differential equations, *DCDS-S*, V.8, N.6, pp.1065-1077, 2015.

[27] Alkan, S., Secer, A., Solution of nonlinear fractional boundary value problems with nonhomogeneous boundary conditions, *Appl. Comput. Math.*,2015, V.14, N.3, pp.284-295.