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Hermite-Hadamard-Fejer type inequalities for GA-s convex functions via fractional integrals

Imdat Iscan[1] *and Mehmet Kunt*[2]

[1]Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Giresun, Turkey
[2]Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, Trabzon, Turkey

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Abstract: In this paper, some Hermite-Hadamard-Fejer type integral inequalities for GA-s convex functions in fractional integral forms are obtained.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, Hadamard fractional integrals, GA-s convex functions.

1 Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature as Hermite-Hadamard's inequality [2].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or their weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [1], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1):

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{2}$$

holds, where $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve, and extend the inequalities (1) and (2) see [3,14,15,17].

Definition 1. [11,12]. *A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if*

$$f(x^t y^{1-t}) \leq t f(x) + (1-t) f(y)$$

* Corresponding author e-mail: mkunt@ktu.edu.tr

for all $x,y \in I$ *and* $t \in [0,1]$.

Definition 2. [16]. *Let* $f : I \subset [0,\infty) \to [0,\infty)$ *and* $s \in (0,1]$. *A function* $f(x)$ *is said to be GA-s convex (geometric-arithmetically s-convex) on I if*

$$f(x^t y^{1-t}) \le t^s f(x) + (1-t)^s f(y)$$

for all $x,y \in I$ *and* $t \in [0,1]$.

In [10], Latif et al. established the following inequality which is the weighted generalization of Hermite-Hadamard inequality for GA-convex functions as follows:

Theorem 2. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex function and* $a,b \in I$ *with* $a < b$. *Let* $g:[a,b] \to [0,\infty)$ *be continuous positive mapping and geometrically symmetric to* $\sqrt{ab}$. *Then*

$$f\left(\sqrt{ab}\right) \int_a^b \frac{g(x)}{x} dx \le \int_a^b \frac{f(x)g(x)}{x} dx \le \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x} dx. \tag{3}$$

The following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 3. [9]. *Let* $f \in L[a,b]$. *The Hadamard integrals* $J^{\alpha}_{a+}f$ *and* $J^{\alpha}_{b-}f$ *of order* $\alpha > 0$ *with* $b > a \ge 0$ *are defined by*

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \ x > a$$

and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} f(t) \frac{dt}{t}, \ x < b$$

respectively, where $\Gamma(\alpha)$ *is the Gamma function defined by* $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [4,5,6,7,18,19].

In [5], Iscan presented Hermite–Hadamard's inequalities for GA-convex functions in fractional integral forms as follows:

Theorem 3. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a function such that* $f \in L[a,b]$ *where* $a,b \in I$ *with* $a < b$. *If* f *is a GA-convex function on* $[a,b]$, *then the following inequalities for fractional integrals holds:*

$$f\left(\sqrt{ab}\right) \le \frac{\Gamma(\alpha+1)}{2\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J^{\alpha}_{a+} f(b) + J^{\alpha}_{b-} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{4}$$

with $\alpha > 0$.

In [8], the authors presented Hermite–Hadamard-Fejer inequalities for GA-convex functions in fractional integral forms as follows:

Theorem 4. *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex function with* $a < b$ *and* $f \in L[a,b]$. *If* $g : [a,b] \to \mathbb{R}$ *is nonnegative, integrable and geometrically symmetric with respect to* $\sqrt{ab}$, *then the following inequalities for fractional integrals holds:*

$$f\left(\sqrt{ab}\right) \left[J^{\alpha}_{a+} g(b) + J^{\alpha}_{b-} g(a)\right] \le \left[J^{\alpha}_{a+}(fg)(b) + J^{\alpha}_{b-}(fg)(a)\right] \le \frac{f(a)+f(b)}{2} \left[J^{\alpha}_{a+} g(b) + J^{\alpha}_{b-} g(a)\right] \tag{5}$$

with $\alpha > 0$.

Lemma 1. [8] . *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$ *and* $f' \in L[a,b]$. *If* $g : [a,b] \to \mathbb{R}$ *is integrable and geometrically symmetric with respect to* $\sqrt{ab}$ *then the following equality for fractional integrals holds:*

$$\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]=\frac{1}{\Gamma(\alpha)}\int_a^b\left[\begin{array}{c}\int_a^t\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\\ -\int_t^b\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\end{array}\right]f'(t)dt \quad (6)$$

with $\alpha > 0$.

Lemma 2. [13,19]. *For* $0 < \alpha \le 1$ *and* $0 \le a < b$, *we have*

$$|a^{\alpha}-b^{\alpha}| \le (b-a)^{\alpha}.$$

In this paper, we obtain some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for GA-s convex function in fractional integral forms.

2 Main results

Throughout this section, let $\|g\|_{\infty} = \sup_{t\in[a,b]} |g(t)|$, for the continuous function $g : [a,b] \to \mathbb{R}$.

Theorem 5. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *where* $a,b \in I$ *with* $a < b$ *and* $f' \in L[a,b]$. *If* $|f'|$ *is GA-s convex on* $[a,b]$, $g : [a,b] \to \mathbb{R}$ *is continuous and geometrically symmetric with respect to* $\sqrt{ab}$, *then the following inequality for fractional integrals holds:*

$$\left|\begin{array}{c}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \le \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(b)\right|\right], \quad (7)$$

where

$$C_1(\alpha)=\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right](1-u)^s a^{1-u}b^u du\\ +\int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right](1-u)^s a^{1-u}b^u du\end{array}\right],$$

$$C_2(\alpha)=\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]u^s a^{1-u}b^u du\\ +\int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right]u^s a^{1-u}b^u du\end{array}\right],$$

with $\alpha > 0$.

Proof. From Lemma 1 we have

$$\left|\begin{array}{c}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \le \frac{1}{\Gamma(\alpha)}\int_a^b\left|\int_a^t\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}-\int_t^b\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(t)\right|dt.$$

Setting $t = a^{1-u}b^u$ and $dt = a^{1-u}b^u\ln\left(\frac{b}{a}\right)du$ gives

$$\left|\begin{array}{c}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \le \frac{1}{\Gamma(\alpha)}\int_0^1\left|\begin{array}{c}\int_a^{a^{1-u}b^u}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\\ -\int_{a^{1-u}b^u}^b\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\end{array}\right|\left|f'(a^{1-u}b^u)\right|a^{1-u}b^u\ln\left(\frac{b}{a}\right)du. \quad (8)$$

Since $g:[a,b]\to\mathbb{R}$ is geometrically symmetric with respect to $\sqrt{ab}$ we write

$$\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}=\int_{a}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(\frac{ab}{s})\frac{ds}{s}=\int_{a}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}.$$

Then we have

$$\begin{aligned}\left|\int_{a}^{a^{1-u}b^{u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}-\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right| &= \left|\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right| \\ &\le \begin{cases}\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s} & u\in\left[0,\frac{1}{2}\right]\\ \int_{a^{u}b^{1-u}}^{a^{1-u}b^{u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s} & u\in\left[\frac{1}{2},1\right]\end{cases} \\ &\le \|g\|_{\infty}\begin{cases}\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s} & u\in\left[0,\frac{1}{2}\right]\\ \int_{a^{u}b^{1-u}}^{a^{1-u}b^{u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s} & u\in\left[\frac{1}{2},1\right]\end{cases} \\ &= \|g\|_{\infty}\frac{\left(\ln\frac{b}{a}\right)^{\alpha}}{\alpha}\begin{cases}(1-u)^{\alpha}-u^{\alpha} & u\in\left[0,\frac{1}{2}\right]\\ u^{\alpha}-(1-u)^{\alpha} & u\in\left[\frac{1}{2},1\right]\end{cases}. \end{aligned}\tag{9}$$

Since $|f'|$ is GA-s convex on $[a,b]$, we have

$$\left|f'(a^{1-u}b^{u})\right|\le(1-u)^{s}\left|f'(a)\right|+u^{s}\left|f'(b)\right|,\tag{10}$$

A combination (8), (9) and (10)

$$\begin{aligned}&\left|\begin{array}{c}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \le \frac{1}{\Gamma(\alpha)}\int_{0}^{1}\left|\begin{array}{c}\int_{a}^{a^{1-u}b^{u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\\ -\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\end{array}\right|\left|f'(a^{1-u}b^{u})\right|a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du. \\ &\le \frac{1}{\Gamma(\alpha)}\int_{0}^{\frac{1}{2}}\left(\|g\|_{\infty}\frac{\left(\ln\frac{b}{a}\right)^{\alpha}}{\alpha}\left[(1-u)^{\alpha}-u^{\alpha}\right]\right)\left[\begin{array}{c}(1-u)^{s}|f'(a)|\\ +u^{s}|f'(b)|\end{array}\right]a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}\left(\|g\|_{\infty}\frac{\left(\ln\frac{b}{a}\right)^{\alpha}}{\alpha}\left[u^{\alpha}-(1-u)^{\alpha}\right]\right)\left[\begin{array}{c}(1-u)^{s}|f'(a)|\\ +u^{s}|f'(b)|\end{array}\right]a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du \\ &\le \frac{\ln^{\alpha+1}\left(\frac{b}{a}\right)\|g\|_{\infty}}{\Gamma(\alpha+1)}\left\{\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[(1-u)^{s}\left|f'(a)\right|+u^{s}\left|f'(b)\right|\right]a^{1-u}b^{u}du\right. \\ &\quad \left.+\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]\left[(1-u)^{s}\left|f'(a)\right|+u^{s}\left|f'(b)\right|\right]a^{1-u}b^{u}du\right\} \\ &= \frac{\ln^{\alpha+1}\left(\frac{b}{a}\right)\|g\|_{\infty}}{\Gamma(\alpha+1)} \\ &\quad \times\left\{\left[\begin{array}{c}\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right](1-u)^{s}a^{1-u}b^{u}du\\ +\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right](1-u)^{s}a^{1-u}b^{u}du\end{array}\right]\left|f'(a)\right|\right. \\ &\quad \left.+\left[\begin{array}{c}\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]u^{s}a^{1-u}b^{u}du\\ +\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]u^{s}a^{1-u}b^{u}du\end{array}\right]\left|f'(b)\right|\right\}.\end{aligned}$$

This completes the proof.

Corollary 1. *In Theorem 5;*
(1) *If we take $\alpha=1$ we have the following Hermite-Hadamard-Fejer inequality for GA-s convex functions which is related*

to the right-hand side of (3):

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x}dx-\int_a^b \frac{f(x)g(x)}{x}dx\right| \le \frac{\|g\|_\infty \ln^2\left(\frac{b}{a}\right)}{2}\left[C_1(1)\left|f'(a)\right|+C_2(1)\left|f'(b)\right|\right],$$

(2) *If we take* $g(x)=1$ *we have the following Hermite-Hadamard inequality for GA-s convex functions in fractional integral forms which is related to the right-hand side of (4):*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right| \le \frac{\ln\left(\frac{b}{a}\right)}{2}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(b)\right|\right],$$

(3) *If we take* $\alpha=1$ *and* $g(x)=1$ *we have the following Hermite-Hadamard inequality for GA-s convex functions:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(x)}{x}dx\right| \le \frac{\ln\left(\frac{b}{a}\right)}{2}\left[C_1(1)\left|f'(a)\right|+C_2(1)\left|f'(b)\right|\right].$$

Theorem 6. *Let* $f: I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I° *where* $a,b\in I$ *with* $a<b$ *and* $f'\in L[a,b]$. *If* $|f'|^q$, $q\ge 1$ *is GA-s convex on* $[a,b]$, $g:[a,b]\to\mathbb{R}$ *is continuous and geometrically symmetric with respect to* $\sqrt{ab}$, *then the following inequalities for fractional integrals holds:*

$$\left|\begin{array}{c}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^\alpha g(b)+J_{b-}^\alpha g(a)\right]\\ -\left[J_{a+}^\alpha (fg)(b)+J_{b-}^\alpha (fg)(a)\right]\end{array}\right| \le \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}$$
$$\times\left[\left(1-\frac{1}{2^\alpha}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}\left[C_3(\alpha)\left|f'(a)\right|^q+C_4(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$C_3(\alpha)=\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^\alpha-u^\alpha\right](1-u)^s\left(a^{1-u}b^u\right)^q du\\ +\int_{\frac{1}{2}}^1\left[u^\alpha-(1-u)^\alpha\right](1-u)^s\left(a^{1-u}b^u\right)^q du\end{array}\right],$$

$$C_4(\alpha)=\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^\alpha-u^\alpha\right]u^s\left(a^{1-u}b^u\right)^q du\\ +\int_{\frac{1}{2}}^1\left[u^\alpha-(1-u)^\alpha\right]u^s\left(a^{1-u}b^u\right)^q du\end{array}\right],$$

with $\alpha>0$.

Proof. Similarly the proof of Theorem 5, using Lemma 1, (8), (9) and power mean inequality we have

$$\left|\begin{array}{c}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^\alpha g(b)+J_{b-}^\alpha g(a)\right]\\ -\left[J_{a+}^\alpha (fg)(b)+J_{b-}^\alpha (fg)(a)\right]\end{array}\right| \le \frac{1}{\Gamma(\alpha)}\int_0^1\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(a^{1-u}b^u)\right|a^{1-u}b^u\ln\left(\frac{b}{a}\right)du$$
$$\le\frac{\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\int_0^1\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|du\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(a^{1-u}b^u)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}$$

Using GA-s convexity of $|f'|^q$

$$\begin{aligned}
&\left|\begin{array}{c}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \leq \frac{\|g\|_{\infty}^{1-\frac{1}{q}}\ln^{\alpha\left(1-\frac{1}{q}\right)+1}\left(\frac{b}{a}\right)}{\alpha^{1-\frac{1}{q}}\Gamma(\alpha)}\left[\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]du+\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]du\right]^{1-\frac{1}{q}}\\
&\times\left[\int_0^1\left|\int_{a^{1-u}b^u}^{a^u b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right|\left[(1-u)^s\left|f'(a)\right|^q+u^s\left|f'(b)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}\\
&\leq \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}\\
&\times\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[(1-u)^s|f'(a)|^q+u^s|f'(b)|^q\right]\left(a^{1-u}b^u\right)^q du\\ +\int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right]\left[(1-u)^s|f'(a)|^q+u^s|f'(b)|^q\right]\left(a^{1-u}b^u\right)^q du\end{array}\right]^{\frac{1}{q}}\\
&= \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}\\
&\times\left\{\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right](1-u)^s\left(a^{1-u}b^u\right)^q du\\ +\int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right](1-u)^s\left(a^{1-u}b^u\right)^q du\end{array}\right]|f'(a)|^q\right.\\
&\left.+\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]u^s\left(a^{1-u}b^u\right)^q du\\ +\int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right]u^s\left(a^{1-u}b^u\right)^q du\end{array}\right]|f'(b)|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof.

Corollary 2. *In Theorem 6;*
(1) *If we take $\alpha=1$ we have the following Hermite-Hadamard-Fejer inequality for GA-s convex functions which is related to the right-hand side of (3):*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right|\leq\|g\|_{\infty}\ln^2\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left[C_3(1)\left|f'(a)\right|^q+C_4(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

(2) *If we take $g(x)=1$ we have the following Hermite-Hadamard inequality for GA-s convex functions in fractional integral forms which is related to the right-hand side of (4):*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{2}\left[\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}\left[C_3(\alpha)\left|f'(a)\right|^q+C_4(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

(3) *If we take $\alpha=1$ and $g(x)=1$ we have the following Hermite-Hadamard inequality for GA-s convex functions:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(x)}{x}dx\right|\leq\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left[C_3(1)\left|f'(a)\right|^q+C_4(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}.$$

Theorem 7. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° where $a,b\in I$ with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$, $q>1$ is GA-s convex on $[a,b]$, $g:[a,b]\to\mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals holds:*

(i)

$$\left| \begin{array}{c} \left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] \\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right] \end{array} \right| \leq \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left[C_5\left|f'(a)\right|^q+C_6\left|f'(b)\right|^q\right]^{\frac{1}{q}} \tag{11}$$

with $\alpha > 0$.

(ii)

$$\left| \begin{array}{c} \left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] \\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right] \end{array} \right| \leq \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\frac{1}{\alpha p+1}\right]^{\frac{1}{p}}\left[C_5\left|f'(a)\right|^q+C_6\left|f'(b)\right|^q\right]^{\frac{1}{q}} \tag{12}$$

for $0 < \alpha \leq 1$. *Where*

$$C_5 = \int_0^1 (1-u)^s \left(a^{1-u}b^u\right)^q du, \;\; C_6 = \int_0^1 u^s \left(a^{1-u}b^u\right)^q du,$$

and $1/p+1/q=1$.

Proof. (i) Using Lemma 1, (8), (9), Hölder's inequality and GA-s convexity of $|f'|^q$ we have

$$\begin{aligned}
\left| \begin{array}{c} \left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] \\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right] \end{array} \right| &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 \left|\int_{a^{1-u}b^u}^{a^u b^{1-u}} \left(\ln\frac{b}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}\right| \left|f'(a^{1-u}b^u)\right| a^{1-u}b^u \ln\left(\frac{b}{a}\right) du \\
&\leq \frac{\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\int_0^1 \left|\int_{a^{1-u}b^u}^{a^u b^{1-u}} \left(\ln\frac{b}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}\right|^p du\right]^{\frac{1}{p}}\left[\int_0^1 \left|f'(a^{1-u}b^u)\right|^q \left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\
&\leq \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]^p du+\int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right]^p du\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_0^1\left[(1-u)^s\left|f'(a)\right|^q+u^s\left|f'(b)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\
&= \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]^p du+\int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right]^p du\right]^{\frac{1}{p}} \qquad (13) \\
&\quad\times\left[\left(\int_0^1 (1-u)^s\left(a^{1-u}b^u\right)^q du\right)\left|f'(a)\right|^q+\left(\int_0^1 u^s\left(a^{1-u}b^u\right)^q du\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \\
&\leq \frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{1}{q}}\left(\frac{b}{a}\right)}{q^{\frac{1}{q}}\Gamma(\alpha+1)}\left[\int_0^{\frac{1}{2}}(1-u)^{\alpha p}-u^{\alpha p}du+\int_{\frac{1}{2}}^1 u^{\alpha p}-(1-u)^{\alpha p}du\right]^{\frac{1}{p}} \\
&\quad\times\left[\left(\int_0^1 (1-u)^s\left(a^{1-u}b^u\right)^q du\right)\left|f'(a)\right|^q+\left(\int_0^1 u^s\left(a^{1-u}b^u\right)^q du\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \\
&= \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left[C_5\left|f'(a)\right|^q+C_6\left|f'(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Here we use

$$\left[(1-t)^{\alpha}-t^{\alpha}\right]^p \leq (1-t)^{\alpha p}-t^{\alpha p}$$

for $t\in[0,1/2]$ and

$$\left[t^{\alpha}-(1-t)^{\alpha}\right]^p \leq t^{\alpha p}-(1-t)^{\alpha p}$$

for $t\in[1/2,1]$, which follows from

$$(A-B)^q \leq A^q - B^q,$$

for any $A \geq B \geq 0$ and $q \geq 1$. Hence the inequality (11) is proved.
(ii) The inequality (12) is easily proved using (13) and Lemma 2.

Corollary 3. *In Theorem 7;*
(i) *In* (11); (1) *If we take* $\alpha = 1$ *we have the following Hermite-Hadamard-Fejer inequality for GA-s convex functions which is related to the right-hand side of* (3)*:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right| \leq \frac{\|g\|_\infty \ln^2\left(\frac{b}{a}\right)}{2}\left[\frac{2}{p+1}\left(1-\frac{1}{2^p}\right)\right]^{\frac{1}{p}}\left[C_5\left|f'(a)\right|^q + C_6\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

(2) *If we take* $g(x) = 1$ *we have the following Hermite-Hadamard inequality for GA-s convex functions in fractional integral forms which is related to the right-hand side of* (4)*:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left[C_5\left|f'(a)\right|^q + C_6\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

(3) *If we take* $\alpha = 1$ *and* $g(x) = 1$ *we have the following Hermite-Hadamard inequality for GA-s convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(x)}{x}dx\right| \leq \frac{\ln^1\left(\frac{b}{a}\right)}{2}\left[\frac{2}{p+1}\left(1-\frac{1}{2^p}\right)\right]^{\frac{1}{p}}\left[C_5\left|f'(a)\right|^q + C_6\left|f'(b)\right|^q\right]^{\frac{1}{q}}.$$

(ii) *In* (12);
(1) *If we take* $\alpha = 1$ *we have the following Hermite-Hadamard-Fejer inequality for GA-s convex functions which is related to the right-hand side of* (3)*:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right| \leq \frac{\|g\|_\infty \ln^2\left(\frac{b}{a}\right)}{2}\left[\frac{1}{p+1}\right]^{\frac{1}{p}}\left[C_5\left|f'(a)\right|^q + C_6\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

(2) *If we take* $g(x) = 1$ *we have the following Hermite-Hadamard inequality for GA-s convex functions in fractional integral forms which is related to the right-hand side of* (4)*:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left[\frac{1}{\alpha p+1}\right]^{\frac{1}{p}}\left[C_5\left|f'(a)\right|^q + C_6\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

(3) *If we take* $\alpha = 1$ *and* $g(x) = 1$ *we have the following Hermite-Hadamard inequality for GA-s convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(x)}{x}dx\right| \leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left[\frac{1}{p+1}\right]^{\frac{1}{p}}\left[C_5\left|f'(a)\right|^q + C_6\left|f'(b)\right|^q\right]^{\frac{1}{q}}.$$

3 Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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