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Legendre collocation method for solving a class of the second order nonlinear differential equations with the mixed non-linear conditions

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Abstract: In this paper, a matrix method based on Legendre collocation points on interval [-1, 1] is proposed for the approximate solution of some second order nonlinear ordinary differential equations with the mixed nonlinear conditions in terms of Legendre polynomials. The method, by means of collocation points, transforms the differential equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown Legendre coefficients. The numerical results show the effectiveness of the method for this type of equation. When this method is compared with the other usual techniques, results would be easier and have higher accuracy.

Keywords: Nonlinear ordinary differential equations, Legendre polynomials, Collocation points.

1 Introduction

Nonlinear ordinary differential equations are frequently used to model a wide class of problem in many areas of scientific fields; chemical reactions, spring-mas systems bending of beams, resistor-capacitor-inductance circuits, pendulums, the motion of a rotating mass around another body and so forth [1,2]. These equations have also demonstrated their usefulness in ecology and economics. Thus, methods of solution for these equations are of great importance to engineers and scientist. In spite of the fact that many important differential equations can be solved by well known analytical techniques, a greater number of physically significant differential equations can not be solved [2,5].

In this research, we consider the second order nonlinear ordinary differential equation of the form

$$\begin{aligned}&A_0(x)y(x)+A_1(x)y'(x)+A_2(x)y''(x)+A_3(x)y^2(x)+A_4(x)y(x)y'(x)+\\&A_5(x)[(y'(x))]^2+A_6(x)y(x)y''(x)+A_7(x)y'(x)y''(x)+A_8(x)[(y''(x))]^2=g(x)\ ,\quad -1\le x\le 1\end{aligned} \tag{1}$$

under the mixed nonlinear conditions

$$\left[\sum_{k=0}^{1}\alpha_{ik}y^{(k)}(a_i)+\sum_{l=0}^{1}\beta_{ik}y^{(k)}(a_i)y^{(l)}(a_i)\right]=\lambda j\ ;\quad i,j=0,1 \tag{2}$$

and look for the approximate solution of (1) in the Legendre polynomial form

$$y(x)=\sum_{n=0}^{N}y_nP_n(x)\ ,\quad -1\le x\le 1 \tag{3}$$

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where y_n, $(n = 0, 1, 2, ..., N)$ are unknown Legendre coefficients. Here $P_n(x)$, $n = 0, 1, 2, ..., N$ are unknown Legendre polynomials defined by

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k}, \quad n = 0, 1, 2, \dots, \left[\frac{n}{2}\right] = \begin{cases}\frac{n}{2}, n \text{ even} \\ \frac{n-1}{2}, n \text{ odd}\end{cases}$$

2 Fundamental matrix relations

Let us consider the nonlinear differential equation (1) and find the matrix forms of each term in the equation. Firstly, we consider the solution $y(x)$ defined by a truncates series (3) and then we can convert it to the matrix form

$$y(x) = P(x)Y \tag{4}$$

where

$$P(x) = [P_0(x)\, P_1(x) \dots P_N(x)]$$

$$Y = [y_0\, y_1 \dots y_N]^T$$

If we differentiate Eq. (5) with respect to x, we obtain

$$y'(x) = P'(x)Y = P(x)\Pi^T Y \tag{5}$$

$$y''(x) = P'(x)\Pi^T Y = P(x)\left(\Pi^T\right)^2 Y$$

where if n is even

$$\Pi = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 3 & 0 & 7 & \cdots & 2N-3 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 2N-1 & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

if n is odd

$$\Pi = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 & \cdots & 2N-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & \cdots & 0 & 2N-1 & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

By using (5) the matrix form of expressions $y^2(x)$ is obtained as

$$y^2(x) = \left[1\ x\ \tfrac{1}{2}\left(3x^2-1\right)\ \cdots\ \cdots\right]\begin{bmatrix} P(x) & 0 & \cdots & 0 \\ 0 & P(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P(x) \end{bmatrix}\begin{bmatrix} y_0 Y \\ y_1 Y \\ \vdots \\ y_N Y \end{bmatrix} \tag{6}$$

or shortly

$$y^2(x) = P(x)\bar{P}(x)\bar{Y}$$

where

$$\bar{P}(x) = \begin{bmatrix} P(x) & 0 & \cdots & 0 \\ 0 & P(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P(x) \end{bmatrix}, \bar{Y} = \begin{bmatrix} y_0Y \; y_1Y \; \cdots \; y_NY \end{bmatrix}^T.$$

By using the equ. (4), (5), and (6) we obtain

$$y(x)y'(x) = P(x)\bar{P}(x)\left(\bar{\Pi}^T\right)\bar{Y} \tag{7}$$

Following a similar way to (6) we obtain

$$\left[(y'(x))\right]^2 = P(x)\Pi^T\bar{P}(x)\left(\bar{\Pi}^T\right)\bar{Y} \tag{8}$$

where

$$\left(\bar{\Pi}^T\right) = \begin{bmatrix} \left(\Pi^T\right) & 0 & \cdots & 0 \\ 0 & \left(\Pi^T\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\Pi^T\right) \end{bmatrix}.$$

Besides, applying a similar way to (4), (5) and (6), can be written, respectively

$$y(x)y''(x) = P(x)\bar{P}(x)\left(\bar{\bar{\Pi}}^T\right)\bar{Y} \tag{9}$$

$$y'(x)y''(x) = P(x)\Pi^T\bar{P}(x)\left(\bar{\bar{\Pi}}^T\right)\bar{Y} \tag{10}$$

$$\left[(y''(x))\right]^2 = P(x)\left(\Pi^T\right)^2\bar{P}(x)\left(\bar{\bar{\Pi}}^T\right)\bar{Y} \tag{11}$$

where

$$\left(\bar{\bar{\Pi}}^T\right) = \begin{bmatrix} \left(\Pi^T\right)^2 & 0 & \cdots & 0 \\ 0 & \left(\Pi^T\right)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\Pi^T\right)^2 \end{bmatrix}.$$

3 Matrix relations based on collocation points

Let us use the collocation points defined by

$$x_i = a - \frac{b-a}{N}i \;\;,\;\; i = 0, 1, ..., N \tag{12}$$

in order to

$$a = x_0 \langle x_1 \langle ... \langle x_n = b.$$

By using the collocation points (9) into Eq. (1), we get the equation

$$A_0(x_i)y(x_i)+A_1(x_i)y'(x_i)+A_2(x_i)y''(x_i)+A_3(x_i)y^2(x_i)+A_4(x_i)y(x_i)y'(x_i)+A_5(x_i)\left[\left(y'(x_i)\right)\right]^2$$
$$+A_6(x_i)y(x_i)y''(x_i)+A_7(x_i)y'(x_i)y''(x_i)+A_8(x_i)\left[\left(y''(x_i)\right)\right]^2=g(x_i),\ \ (i=0,1,...,N);\ \ -1\le x_i\le 1. \tag{13}$$

By using the relations (4), (5), (6), (7), (8), (9), (10) and (11); the system (13) can be written in the matrix form

$$\begin{aligned}&\left[A_0(x_i)P(x_i)I+A_1(x_i)P(x_i)\Pi^T+A_2(x_i)P(x_i)\left(\Pi^T\right)^2\right]Y+\\&+\left[A_3(x_i)P(x_i)\overline{P}(x_i)+A_4(x_i)P(x_i)\overline{P}(x_i)\left(\overline{\Pi}^T\right)+A_5(x_i)P(x_i)\Pi^T\overline{P}(x_i)\left(\overline{\Pi}^T\right)+\right.\\&\left.+A_6(x_i)P(x_i)\overline{P}(x_i)\left(\overline{\overline{\Pi}}^T\right)+A_7(x_i)P(x_i)\Pi^T\overline{P}(x_i)\left(\overline{\overline{\Pi}}^T\right)+A_8(x_i)P(x_i)\left(\Pi^T\right)^2\overline{P}(x_i)\left(\overline{\overline{\Pi}}^T\right)\right]\overline{Y}=g(x_i).\end{aligned} \tag{14}$$

Consequently, the fundamental matrix equations of (14) can be written in the following compact form

$$W(x_i)Y+V(x_i)\overline{Y}=g(x_i)$$

where

$$W(x_i)=A_0(x_i)P(x_i)I+A_1(x_i)P(x_i)\Pi^T+A_2(x_i)P(x_i)\left(\Pi^T\right)^2$$

and

$$\begin{aligned}V(x_i)=&A_3(x_i)P(x_i)\overline{P}(x_i)+A_4(x_i)P(x_i)\overline{P}(x_i)\left(\overline{\Pi}^T\right)+A_5(x_i)P(x_i)\Pi^T\overline{P}(x_i)\left(\overline{\Pi}^T\right)\\&+A_6(x_i)P(x_i)\overline{P}(x_i)\left(\overline{\overline{\Pi}}^T\right)+A_7(x_i)P(x_i)\Pi^T\overline{P}(x_i)\left(\overline{\overline{\Pi}}^T\right)+A_8(x_i)P(x_i)\left(\Pi^T\right)^2\overline{P}(x_i)\left(\overline{\overline{\Pi}}^T\right).\end{aligned}$$

Above expression can be rewritten shortly as

$$WY*+V\overline{Y}*=G \tag{15}$$

where

$$W=\begin{bmatrix}W(x_0)&0&\cdots&0\\0&W(x_1)&\cdots&0\\\vdots&\vdots&\ddots&\\0&0&\cdots&W(x_N)\end{bmatrix}_{(N+1)\times(N+1)^2},V=\begin{bmatrix}V(x_0)&0&\cdots&0\\0&V(x_1)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&V(x_N)\end{bmatrix}_{(N+1)\times(N+1)^2},$$

$$G=\begin{bmatrix}g(x_0)\\g(x_1)\\\vdots\\g(x_N)\end{bmatrix}_{(N+1)\times 1},I=\begin{bmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{bmatrix}_{(N+1)\times(N+1)},Y*=\begin{bmatrix}Y\\Y\\\vdots\\Y\end{bmatrix}_{(N+1)^2\times 1},\overline{Y}*=\begin{bmatrix}Y*\\Y*\\\vdots\\Y*\end{bmatrix}_{(N+1)^3\times 1}.$$

4 Method of solution

The fundamental matrix equation (15) corresponding to Eq. (1) can be written as,

$$WY*+V\overline{Y}*=G$$

or

$$[W;V;G \tag{16}$$

We can find to corresponding matrix equation for the conditions (2), using the relation (4),(5),(6),(7) and (8) as follows:

$$\{\alpha_{00}P(a_0)+\alpha_{10}P(a_1)+\alpha_{01}\bar{P}(a_0)+\alpha_{11}\bar{P}(a_1)\}Y+$$
$$\{\beta_{00}P(a_0)\bar{P}(a_0)+\beta_{10}P(a_1)\bar{P}(a_1)\left(\bar{\Pi}^T\right)+\beta_{01}P(a_0)\bar{P}(a_0)\left(\bar{\Pi}^T\right)+\beta_{11}P(a_1)\Pi^T\bar{P}(a_1)\left(\bar{\Pi}^T\right)\}\bar{Y}=\lambda_j\ ,\ j=0,1. \quad (17)$$

or shortly

$$K_iY+L_i\bar{Y}=\lambda_i,\ i=0,1,$$

so that

$$K_i=\sum_{k=0}^{1}[\alpha_{ik}P(a_i)]\left(\Pi^T\right)^kY,\ i=0,1,$$

$$L_i=\sum_{k=0}^{1}\beta_{ik}\left[P(a_i)\bar{P}(a_i)+P(a_i)\bar{P}(a_i)\left(\bar{\Pi}^T\right)+P(a_i)\Pi^T\bar{P}(a_i)\left(\bar{\Pi}^T\right)\right]\bar{Y}\ ,\ \ i=0,1,$$

where

$$P(a_i)=\left[P_0(a_i)\ P_1(a_i)\ \cdots\ P_N(a_i)\right],\bar{P}(a_i)=\begin{bmatrix}P(a_i)&0&\cdots&0\\0&P(a_i)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&P(a_i)\end{bmatrix}_{(N+1)\times(N+1)^2},i=0,1.$$

We can write the corresponding matrix form (17) for the mixed non-linear condition (2) in the augmented matrix form as

$$[K;\,L;\,\lambda] \quad (18)$$

where

$$K=\begin{bmatrix}k_{00}\ \ k_{01}\ \ \ldots k_{0N} & 0\\ 0 & k_{10}\ \ k_{11}\ \ \ldots k_{1N}\end{bmatrix}=\sum_{k=0}^{1}\alpha_{ik}\left[P(a_i)+P(a_i)\left(\Pi^T\right)^k\right]$$

$$L=\begin{bmatrix}l_{00}\ \ l_{01}\ \ \ldots l_{oN} & 0\\ 0 & l_{10}\ \ l_{11}\ \ \ldots l_{1N}\end{bmatrix}=\sum_{l=0}^{1}\beta_{ik}\left[P(a_i)\bar{P}(a_i)+P(a_i)\bar{P}(a_i)\left(\bar{\Pi}^T\right)+P(a_i)\Pi^T\bar{P}(a_i)\bar{\Pi}^T\right]$$

$$\lambda=\begin{bmatrix}\lambda_0\\\lambda_1\end{bmatrix}\ ,\quad 0=\left[0\ 0\cdots\ 0\right]_{1\times(N+1)}.$$

To obtain the approximate solution of Eq. (1) with the mixed non-linear condition (2) in the terms of Legendre polynomials, by replacing the row matrix (18) by the last row of the matrix (15), we obtain the required augmented matrix:

$$[\tilde{W};\tilde{V};\tilde{G}]=\begin{bmatrix}W(x_0)&0&\cdots&0\,;&V(x_0)&0&\cdots&0\,:&g(x_0)\\0&W(x_1)&\cdots&0\,;&0&V(x_1)&\cdots&0\,:&g(x_1)\\\vdots&\vdots&\ddots&\vdots\,;&\vdots&0&\ddots&\vdots\,:&\vdots\\0&0&\cdots&K\,;&0&0&\cdots&L\,:&\lambda\end{bmatrix}$$

or the corresponding matrix equation

$$\tilde{W}Y*+\tilde{V}\bar{Y}*=\tilde{G} \quad (19)$$

where

$$\tilde{W}=\begin{bmatrix} W(x_0) & 0 & \cdots & 0 \\ 0 & W(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K \end{bmatrix},\tilde{V}=\begin{bmatrix} V(x_0) & 0 & \cdots & 0 \\ 0 & V(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L \end{bmatrix},\tilde{G}=\begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ \lambda \end{bmatrix}.$$

The unknown coefficients set $\{y_0, y_1, ..., y_N\}$ can be determined from the nonlinear system (19). As a result, we can obtain approximate solution in the truncated series form (3).

5 Accuracy of solution

We can check the accuracy of the solution by following procedure [6 - 16]: The truncated Legendre series in (3) have to be approximately satisfying Eq. (1); that is for each $x = x_i \in [a,b]$, $i = 1, 2, ...$

$$E(x_i)=\left|\begin{array}{l} A_0(x_i)y(x_i)+A_1(x_i)y'(x_i)+a_2(x_i)y''(x_i)+A_3(x_i)y^2(x_i)+A_4(x_i)y(x_i)y'(x_i)+ \\ A_5(x_i)[(y'(x_i))]^2+A_6(x_i)y(x_i)y''(x_i)+A_7(x_i)y'(x_i)y''(x_i)+A_8(x_i)[(y''(x_i))]^2-g(x_i) \end{array}\right|\cong 0$$

and $E(x_i) \leq 10^{k_i}$ $(k\, is\, any\, positive\, integer)$ is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points$x_i$becomes smaller than the prescribed10^{-k}.

6 Numerical examples

The method of this research is useful in finding the solutions of second-order nonlinear ordinary differential equations in terms of Legendre polynomials. We illustrate it by following examples.

Example 1. Let us first consider the second-order nonlinear differential equation

$$(x-1)y''y - xy'y - 2xy = -2x^4 + 2 \tag{20}$$

with conditions

$$y(0) = -1\ ,\ y'(0) = 0\ ,\ -1 \leq x \leq 1$$

and the approximate solution $y(x)$ by the truncated Legendre polynomial

$$y(x) = \sum_{n=0}^{3} y_n P_n\ , \quad -1 \leq x \leq 1$$

where

$$A_0(x) = -2x,\ A_4(x) = -x,\ A_6(x) = (x-1),\ g(x) = -2x^4 + 2.$$

For $N = 3$ the collocation points become

$$x_0 = -1\ ,\ x_1 = \frac{-1}{3}\ ,\ x_2 = \frac{1}{3}\ ,\ x_3 = 1.$$

The augmented matrix for the fundamental matrix equation is calculated as

$$[\bar{W};\bar{V};\bar{G}] = \left[\begin{array}{cccc;cccccccccccccccc:c} 2 & -2 & 2 & -2 \,; & 0 & 1 & -9 & 35 & 0 & -1 & 9 & -35 & 0 & 1 & -9 & 35 & 0 & -1 & 9 & -35 & : & 0 \\ \frac{2}{3} & \frac{-2}{9} & \frac{-2}{9} & \frac{22}{81} \,; & 0 & \frac{1}{3} & \frac{-13}{3} & \frac{55}{9} & 0 & \frac{-1}{9} & \frac{13}{9} & \frac{-55}{27} & 0 & \frac{-1}{9} & \frac{13}{9} & \frac{-55}{27} & 0 & \frac{11}{81} & \frac{-143}{81} & \frac{605}{243} & : & \frac{160}{81} \\ 1 & 0 & \frac{-1}{2} & 0 \,; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & -1 \\ 0 & 1 & 0 & \frac{-5}{2} \,; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & 0 \end{array}\right]$$

From the obtained system, the coefficients $y_{0,}y_{1,}y_2$ and y_3 are found as

$$y_{0,} = -\frac{2}{3}\,,\ y_{1,} = 0\,,\ y_2 = \frac{2}{3}\,,\ y_3 = 0.$$

Hence we have the Legendre polynomial solution

$$y(x) = x^2 - 1$$

which is the exact solution.

Example 2. Our last example is nonlinear differential equation

$$y'' + xy - e^{-x}y^2 = xe^x \tag{21}$$

with nonlinear conditions

$$y(0) + 3y^2(0) = 4,\ \ y'(0) - y^2(0) = 0\,,\ \ -1 \le x \le\ 1,$$

Following the procedure in Section 4, we find the approximate solution of problem (21) for$N = 3$as

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

The solutions obtained for $N = 3, 7$ are compared with the exact solution is e^x, which are given in Figures 1, 2. We compare the numerical solution and absolute errors for $N = 3, 7$ in Table 1.

Present method					
x_i	Exact solution	$N = 3, y(x)$	Absolute errors for $N = 3$	$N = 7, y(x)$	Absolute errors for $N = 7$
-1	0.3678794412	0.33333333	3.454661 E-1	0.3678571429	2.22983 E-5
-0.8	0.4493289641	0.43466666	1.466229 E-2	0.4493251454	3.8187 E-6
-0.6	0.5488116361	0.544000	4.811636 E-3	0.5488112457	3.904 E-7
-0.4	0.670320046	0.66933333	9.867127 E-4	0.6703200305	1.55 E-8
-0.2	0.8187307531	0.81866666	6.40864 E-5	0.818730753	1.000 E-10
0	1.0	1.0	0	1.0	0
0.2	1.221402758	1.22133333	6.9425 E-5	1.221401758	1.000 E-12
0.4	1.491824698	1.49666666	1.15831 E-3	1.491824681	1.700 E-8
0.6	1.8221188	1.816000	6.1188 E-3	1.822118354	4.46 E-7
0.8	2.225540928	2.205333	2.0207595 E-2	2.225536366	4.562 E-6
1.0	2.718281828	2.66666666	5.1615161 E-2	2.718225396	2.786 E-5

Table 1: Comparison of the absolute errors of Example 2.

7 Conclusion

A new technique, using the Legendre polynomial, to numerically solve the second order nonlinear differential equations is presented. Nonlinear differential equations are usually difficult to solve analyticaly. Then it is required to obtain the approximate solutions. For this reason, the present method has been proposed for approximate solution and also analytical solution.

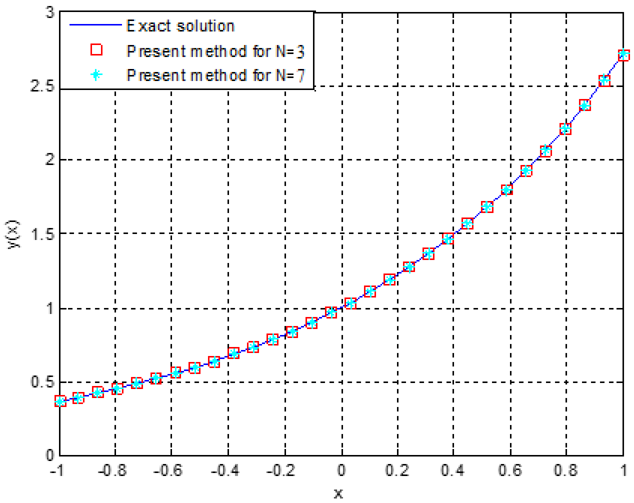

Fig. 1: Numerical and exact solution of Example 2 for $N = 3,7$.

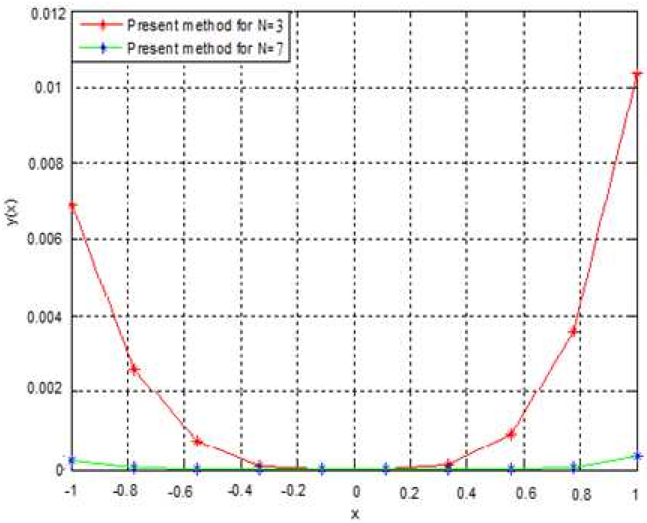

Fig. 2: Comparison of the absolute errors of Example 2 for $N = 3,7$.

On the other hand, from Table 1, it may be observed that the errors found for different N show close agreement for various values of x_i. Table and Figures indicate that as N increases, the errors decrease more rapidly; hence for better results, using large number N is recommended. Another considerable advantage of the method is that Legendre coefficients of the solution are found very easily by using the computer programs. On the other hand our Nth order approximation gives the exact solution when the solution is polynomial of degree equal to or less than N. If the solution is not polynomial, Legendre series approximation converges to the exact solution as N increases.

The method can also be extended to the high order nonlinear differential equations with variable coefficients, but some modifications are required.

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