New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2016218258

Partial sharing of a set of meromorphic functions and normality

Virender Singh, Banarsi Lal, K.S. Charak and Shittal Sharma

Department of Mathematics, University of Jammu, Jammu, India

Received: 22 April 2016, Revised: 26 April 2016, Accepted: 28 April 2016
Published online: 28 April 2016.

Abstract: By using the idea of partial sharing of a set of meromorphic functions by a member of a family of meromorphic functions and its kth derivative we obtain a normality criterion generalizing some of the earlier results on shared sets and normal families of meromorphic functions. Further we prove a normality criterion which improves Marty's theorem and its reverse counterpart.

Keywords: Normal families, meromorphic function, partial sharing of sets, spherical derivative, nevanlinna theory.

1 Introduction and Main Results

Let f be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notions of the Nevanlinna value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$ (see [6]). By $S(r,f)$, as usual, we shall mean a quantity that satisfies

$$S(r,f) = \circ(T(r,f)) \text{ as } r \to \infty,$$

possibly outside an exceptional set of finite logarithmic measure.

A family $\mathscr{F}$ of meromorphic functions defined on a domain $D \subseteq \overline{\mathbb{C}}$ is said to be normal in D if every sequence of elements of $\mathscr{F}$ contains a subsequence which converges locally uniformly in D with respect to the spherical metric, to a meromorphic function or ∞ (see [9]).

Two nonconstant meromorphic functions f and g defined on a domain D are said to share a set S of distinct meromorphic functions in D if $\bigcup_{\phi\in S} \overline{E}_f(\phi) = \bigcup_{\phi\in S} \overline{E}_g(\phi)$, where $\overline{E}_f(\phi) = \{z \in D : f(z) = \phi(z)\}$. However, if $\bigcup_{\phi\in S} \overline{E}_f(\phi) \subseteq \bigcup_{\phi\in S} \overline{E}_g(\phi)$, then we say that f share S partially with g and we write $f(z) \in S \Rightarrow g(z) \in S$.

Schwick [10] proved that if there exist three distinct finite value a_1, a_2, a_3 in $\mathbb{C}$ such that f and f' share a_i, $i = 1,2,3$ on D for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.

Fang [3] and Liu and Pang [7] extended the Schwick's result using the idea of shared sets. They precisely proved:

Theorem 1. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2 and a_3 be three distinct finite complex numbers. If for every $f \in \mathscr{F}$, f and f' share the set $S = \{a_1,a_2,a_3\}$, then $\mathscr{F}$ is normal in D.*

In 2010, Chen [2] proved the following three results concerning a shared set of values:

* Corresponding author e-mail: virendersingh2323@gmail.com

Theorem 2. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2 and a_3 be three nonzero distinct finite complex numbers and let $S = \{a_1, a_2, a_3\}$. If for every $f \in \mathscr{F}$, $f(z) \in S \Rightarrow f'(z) \in S$, then $\mathscr{F}$ is normal in D.*

Theorem 3. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, all of whose poles are of multiplicity at least 3, let a_1, a_2 and a_3 be three distinct finite complex numbers, let $S = \{a_1, a_2, a_3\}$, and let M be a positive number. If for every $f \in \mathscr{F}$, $|f'(z)| \leq M$ whenever $f(z) \in S$, then $\mathscr{F}$ is normal in D.*

Theorem 4. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, all of whose zeros are multiple. Let a_1 and a_2 be two nonzero distinct finite complex numbers and let $S = \{a_1, a_2\}$. If for every $f \in \mathscr{F}$, $f(z) \in S \Rightarrow f'(z) \in S$, then $\mathscr{F}$ is normal in D.*

Chen [2] has given an example to show that the cardinality of S in Theorem 2 and Theorem 3 cannot be reduced. But in Theorem 4, as far as we know, whether the condition on the multiplicity of the zeros and that on the values in S, are essential. We give here following examples to establish that these conditions are essential.

Example 1. Consider the family

$$\mathscr{F} = \{f_n(z) = \tan nz \ : \ n = 1, 2, \cdots\},$$

on the unit disk $\mathbb{D}$, and the set $S = \{i, -i\}$. Then each $f \in \mathscr{F}$ has simple zeros, and for every $f \in \mathscr{F}$, $f(z) \in S \Rightarrow f'(z) \in S$. But $\mathscr{F}$ is not normal in $\mathbb{D}$. Thus the condition on the multiplicity of zeros is essential in Theorem 4.

Example 2. Consider the family

$$\mathscr{F} = \left\{ f_n(z) = \frac{e^{nz}}{n} \ : \ n = 2, 3, \cdots \right\}$$

on the unit disk $\mathbb{D}$, and the set $S = \{0, \infty\}$. Then for every $f \in \mathscr{F}$, $f(z) \in S \Rightarrow f'(z) \in S$. But $\mathscr{F}$ is not normal in $\mathbb{D}$. Thus the condition that S has nonzero finite values is essential in Theorem 4.

In this paper, we generalize these results by replacing the elements of the shared set S by distinct meromorphic functions as follows:

Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, all of whose poles are of multiplicity at least 3, and let $S := \{\phi_1, \phi_2, \cdots, \phi_n\}$ be a set of $n-$distinct meromorphic functions in D, where $n \geq 3$.

Theorem 5. *If*
(i) for a given $m \in \mathbb{N}$ and for each $f \in \mathscr{F}$, $f(z) \in S \Rightarrow f^{(k)}(z) \in S$, $1 \leq k \leq m$, and
(ii) $\forall\, z_0 \in D$, the cardinality of the set $\{\phi_1(z_0), \phi_2(z_0), \cdots, \phi_n(z_0)\}$ is at most 2 implies that $f(z_0) \neq \phi_i(z_0)$ for at least 2 functions ϕ_i (depending on f),
then $\mathscr{F}$ is normal in D.

Theorem 6. *If*
(i) there is a constant $M > 0$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) \in S \ \forall f \in \mathscr{F}$, $1 \leq k \leq m$, where m is a given positive integer, and
(ii) $\forall\, z_0 \in D$, the cardinality of the set $\{\phi_1(z_0), \phi_2(z_0), \cdots, \phi_n(z_0)\}$ is at most 2 implies that $f(z_0) \neq \phi_i(z_0)$ for at least 2 functions ϕ_i (depending on f),
then $\mathscr{F}$ is normal in D.

Example 3. [4] Consider the family

$$\mathscr{F} = \left\{ f_n(z) = \frac{n+1}{2n} e^{nz} + \frac{n-1}{2n} e^{-nz} \ : \ n = 2, 3, \cdots \right\}$$

on the unit disk $\mathbb{D}$ and set $S=\{-1,1\}$. Then for any $f_n\in\mathscr{F}$, we have $n^2[f_n^2(z)-1]=[f_n'(z)]^2-1$. Thus $f_n(z)\in S \Rightarrow f_n'(z)\in S$ and $|f_n'(z)|\le 1$, but $\mathscr{F}$ is not normal in $\mathbb{D}$. This shows that the cardinality of S in Theorem 5 and Theorem 6 cannot be reduced.

Example 4. Consider the family

$$\mathscr{F}=\{f_n(z)=nz\ :\ n=1,2,\cdots\}$$

on the unit disk $\mathbb{D}$, and set $S=\{0,-1,\infty\}$. Then $f_n(0)\in S$ but $f_n'(0)\notin S$ and $|f_n'(0)|\to\infty$ as $n\to\infty$. Note that $\mathscr{F}$ is not normal in $\mathbb{D}$. Thus condition (i) in Theorem 5 and as well as in Theorem 6 is essential.

Example 5. Consider the family

$$\mathscr{F}=\{f_n(z)=2nz^2\ :\ n=1,2,\cdots\}$$

on the unit disk $\mathbb{D}$. Let $S=\{\phi_1,\phi_2,\phi_3\}$, where $\phi_1(z)=z^2$, $\phi_2(z)=z^2/2$ and $\phi_3(z)=z^2/3$. Then for every $f\in\mathscr{F}$, $f(z)\in S\Rightarrow f'(z)\in S$ and $|f'(z)|\le M$, where M is a positive number. However, the family $\mathscr{F}$ is not normal in $\mathbb{D}$. Note that $f_n(0)=\phi_1(0)=\phi_2(0)=\phi_3(0)$. Therefore, the condition (ii) cannot be dropped in Theorem 5 and Theorem 6.

Remark.

1. If $m\ge 3$, then the conclusion of Theorem5 and Theorem 6 hold without the condition on the multiplicity of the poles.
2. Since $|f'(z)|\le M$ implies $f^{\#}(z)\le M$, Theorem 6 generalizes Marty's theorem by taking $m=1$.

Recently, Grahl and Nevo [5] gave the following reverse counterpart to Marty's theorem:

Theorem 7. *Let some $M>0$ be given and set*

$$\mathscr{G}:=\left\{f\in\mathscr{M}(\mathbb{D}):f^{\#}(z)\ge M \text{ for all } z\in\mathbb{D}\right\}.$$

Then $\mathscr{G}$ is normal in $\mathbb{D}$.

Here, we prove a generalization of Theorem 7 as:

Theorem 8. *Let k and n be two positive integers with $k\ge 2$ and $n\ge 3$. Let $\mathscr{H}$ be a family of meromorphic functions in a domain D, all of whose zeros are of multiplicity at least $k+1$, and let the set $S=\{\phi_1,\phi_2,\cdots,\phi_n\}$, where ϕ_i $(i=1,2,\cdots,n)$ are meromorphic functions on D such that $\phi_i(z)\ne\phi_j(z)$ for $i\ne j$, $z\in D$. If, for every $f\in\mathscr{H}$,*

$$f^{(k)}(z)\in S\Rightarrow f^{\#}(z)\ge M,$$

where $M>0$ is a constant, then $\mathscr{H}$ is normal in D.

The following examples show that various conditions in Theorem 8 cannot be dropped:

Example 6. Consider the family

$$\mathscr{H}=\left\{f_n(z)=\frac{1}{nz}:n=1,2,\cdots\right\}$$

on the open unit disk $\mathbb{D}$, and let $S=\{0,\infty\}$. Clearly, for every n, $f_n^{(k)}(0)\in S\Rightarrow f_n^{\#}(0)=n\to\infty$ as $n\to\infty$. However, the family $\mathscr{H}$ is not normal in $\mathbb{D}$. Thus the cardinality of S cannot be reduced.

Example 7. Consider the family

$$\mathscr{H}=\{f_n(z)=nz^k:n=1,2,\cdots\}$$

on the open unit disk $\mathbb{D}$, and let $S=\{0,1,\infty\}$. Clearly, for every $f\in\mathscr{H}$, $f^{(k)}(z)\in S\Rightarrow f^{\#}(z)\ge M$, for some positive constant M . However, the family $\mathscr{H}$ is not normal in $\mathbb{D}$. This shows that the condition on the multiplicity of zeros in Theorem 8 is essential.

Example 8. Consider the family

$$\mathscr{H} = \{f_n(z) = nz^3 : n = 1, 2, \cdots\}$$

on the open unit disk $\mathbb{D}$, and let $S = \{0, 1, \infty\}$. Clearly, for every $f \in \mathscr{H}$, $f_n''(0) \in S \Rightarrow f^{\#}(0) = 0$. However, the family $\mathscr{H}$ is not normal in $\mathbb{D}$. Therefore the condition "$f^{(k)}(z) \in S \Rightarrow f^{\#}(z) \geq M$" is essential.

Throughout the paper, we shall denote the open disk with center at z_0 and radius r by $D(z_0, r)$ and the punctured disk by $D^*(z_0, r)$.

2 Proof of the main results

We need the following results for the proof of our main results:

Lemma 1. [8] *Let $\mathscr{F}$ be a family of functions meromorphic in $\mathbb{D}$ all of whose zeros have multiplicity at least m and all of whose poles have multiplicity at least p. Then, if $\mathscr{F}$ is not normal at a point $z_0 \in \mathbb{D}$, there exist, for each $\alpha : -p < \alpha < m$,*
(i) a real number r: $r < 1$,
(ii) points z_n: $|z_n| < r$,
(iii) positive numbers ρ_n: $\rho_n \to 0$,
(iv) functions $f_n \in F$ such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta)$ converges locally uniformly with respect to the spherical metric to $g(\zeta)$, where $g(\zeta)$ is a non constant meromorphic function on $\mathbb{C}$ and $g^{\#}(\zeta) \leq g^{\#}(0) = 1$.

Lemma 2. [1] *Let $\mathscr{F}$ be a family of meromorphic functions in a domain $\mathbb{D}$ and let a and b be distinct functions holomorphic on $\mathbb{D}$. Suppose that, for any $f \in \mathscr{F}$ and any $z \in \mathbb{D}$, $f(z) \neq a(z)$ and $f(z) \neq b(z)$. If $\mathscr{F}$ is normal in $\mathbb{D} - \{0\}$, then $\mathscr{F}$ is normal in $\mathbb{D}$.*

Proof. [**Proof of Theorem 5.**] Since normality is a local property, it is enough to show that $\mathscr{F}$ is normal at each $z_0 \in D$. Let $S_1 = \{\phi_1(z_0), \phi_2(z_0), \cdots, \phi_n(z_0)\}$. We distinguish the following cases:
Case 1. Suppose that all the values in S_1 are finite.
Here the following subcases arise:
Subcase 1.1. When cardinality of S_1 is at least three.

Suppose that $\mathscr{F}$ is not normal at z_0. Then by Lemma 1, we can find a sequence $\{f_j\}$ in $\mathscr{F}$, a sequence $\{z_j\}$ of complex numbers with $z_j \to z_0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \to 0$ such that

$$g_j(\zeta) = f_j(z_j + \rho_j\zeta)$$

converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function $g(\zeta)$ on $\mathbb{C}$, all of whose poles are of multiplicity at least 3, such that $g^{\#}(\zeta) \leq g^{\#}(0) = 1$ for all $\zeta \in \mathbb{C}$.

Clearly g assume at least one of the values of S_1, otherwise g becomes constant by Picard's theorem. Let $\zeta_0 \in \mathbb{C}$ be such that $g(\zeta_0) - \phi_i(z_0) = 0$, for some $i = 1, 2, \cdots, n$. Since $g(\zeta) \not\equiv \phi_i(z_0)$, by Hurwitz's theorem there exist a sequence of points $\zeta_j \to \zeta$ such that for sufficiently large j,

$$g_j(\zeta_j) = f_j(z_j + \rho_j\zeta_j) = \phi_i(z_j + \rho_j\zeta_j) \in S.$$

By hypothesis, for every $f \in \mathscr{F}$, $f(z) \in S \Rightarrow f^{(k)}(z) \in S$ ($k = 1, 2, \cdots, m$), it follows that

$$f_j^{(k)}(z_j + \rho_j\zeta_j) \in S,$$

and hence

$$g_j^{(k)}(\zeta_j) = \rho_j^k f_j^{(k)}(z_j + \rho_j \zeta_j) = \rho_j^k \phi_i(z_j + \rho_j \zeta_j),$$

for some $i = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, m$. Therefore

$$g^{(k)}(\zeta_0) = \lim_{j \to \infty} g_j^{(k)}(\zeta_j) = 0$$

for $k = 1, 2, \cdots, m$, and so ζ_0 is a zero of multiplicity at least $m+1$ for $g(\zeta) - \phi_i(z_0)$, $(i = 1, 2, \cdots, n)$. Since poles of g have multiplicity at least 3, by Second fundamental theorem of Nevanlinna, we have

$$\begin{aligned}
(n-1)T(r,g) &\le \overline{N}\left(r, \frac{1}{g - \phi_1(z_0)}\right) + \overline{N}\left(r, \frac{1}{g - \phi_2(z_0)}\right) + \cdots + \overline{N}\left(r, \frac{1}{g - \phi_n(z_0)}\right) + \overline{N}(r,g) + S(r,g). \\
&\le \frac{1}{m+1}\left[N\left(r, \frac{1}{g - \phi_1(z_0)}\right) + N\left(r, \frac{1}{g - \phi_2(z_0)}\right) + \cdots + N\left(r, \frac{1}{g - \phi_n(z_0)}\right)\right] + \frac{1}{3}N(r,g) + S(r,g) \\
&\le \frac{n}{m+1}T(r,g) + \frac{1}{3}T(r,g) + S(r,g) \\
&= \frac{3n+m+1}{3m+3}T(r,g) + S(r,g),
\end{aligned}$$

which is a contradiction as $n \ge 3$. Thus $\mathscr{F}$ is normal at z_0.

Subcase 1.2. When cardinality of S_1 is at most two.

By hypothesis (ii), $f(z_0) \ne \phi_i(z_0)$ for at least two functions ϕ_i, $(i = 1, 2, \cdots, n)$. So we can find a small neighbourhood, say $D(z_0, r)$ such that $\phi_i(z) \ne \phi_j(z)$ $(1 \le i, j \le n)$ in $D^*(z_0, r)$. Thus by subcase 1.1, $\mathscr{F}$ is normal in $D^*(z_0, r)$. Now we show that $\mathscr{F}$ is normal at z_0.

Since $f(z_0) \ne \phi_i(z_0)$ for at least two functions ϕ_i and each $\phi_i(z_0)$ is finite, we find that for every $f \in \mathscr{F}$, $f(z) \ne \phi_i(z)$ for at least two functions ϕ_i which are holomorphic in $D(z_0, r)$. Thus by Lemma 2, $\mathscr{F}$ is normal at z_0.

Case 2. Suppose one of the value in S_1 is infinite.

Without loss of generality, assume that $\phi_1(z_0) = \infty$. We take $h \notin S_1$ and consider the family

$$\mathscr{G} = \left\{ g = \frac{1}{f - h} : f \in \mathscr{F} \right\}.$$

Clearly for every $f \in \mathscr{F}$,

$$f(z_0) \in S_1 \text{ implies } g(z_0) \in S_2 = \left\{ 0, \frac{1}{\phi_1(z_0) - h}, \frac{1}{\phi_2(z_0) - h}, \cdots, \frac{1}{\phi_n(z_0) - h} \right\},$$

with all the values in S_2 finite. So we can find a small neighbourhood $D(z_0, r)$ of z_0 such that

$$f(z) \in S \text{ implies } g(z) \in T = \left\{ 0, \frac{1}{\phi_1(z) - h}, \frac{1}{\phi_2(z) - h}, \cdots, \frac{1}{\phi_n(z) - h} \right\}.$$

Thus by Case 1, $\mathscr{G}$ is normal at z_0 and which in turn implies that the family $\mathscr{F}$ is normal at z_0.

Proof. [**Proof of Theorem 6.**] Since normality is a local property, it is enough to show that $\mathscr{F}$ is normal at each $z_0 \in D$. Let $S_1 = \{\phi_1(z_0), \phi_2(z_0), \cdots, \phi_n(z_0)\}$.

The proof is similar to that of Theorem 5 except the case when all the values in S_1 are finite, and cardinality of S_1 is at least three. So here we consider that case only.

Suppose that $\mathscr{F}$ is not normal at z_0. Then by Lemma 1, we can find a sequence $\{f_j\}$ in $\mathscr{F}$, a sequence $\{z_j\}$ of complex numbers with $z_j \to z_0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \to 0$ such that

$$g_j(\zeta) = f_j(z_j + \rho_j\zeta)$$

converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function $g(\zeta)$ on $\mathbb{C}$, all of whose poles are of multiplicity at least 3, such that $g^{\#}(\zeta) \le g^{\#}(0) = 1$ for all $\zeta \in \mathbb{C}$.

Clearly g assume at least one of the value in set S_1, otherwise g becomes constant by Picard's theorem. Let $\zeta_0 \in \mathbb{C}$ be such that $g(\zeta_0) - \phi_i(z_0) = 0$, for some $i = 1, 2, \cdots, n$. Since $g(\zeta) \not\equiv \phi_i(z_0)$, by Hurwitz's theorem there exist a sequence of points $\zeta_j \to \zeta$ such that for sufficiently large j,

$$g_j(\zeta_j) = f_j(z_j + \rho_j\zeta_j) = \phi_i(z_j + \rho_j\zeta_j) \in S.$$

By hypothesis, for every $f \in \mathscr{F}$, $|f^{(k)}(z)| \le M$ whenever $f(z) \in S$ ($k = 1, 2, \cdots, m$), it follows that

$$|f_j^{(k)}(z_j + \rho_j\zeta_j)| \le M,$$

and hence

$$|g_j^{(k)}(\zeta_j)| = |\rho_j^k f_j^{(k)}(z_j + \rho_j\zeta_j)| \le \rho_j^k M,$$

for $k = 1, 2, \cdots, m$. Therefore

$$g^{(k)}(\zeta_0) = \lim_{j\to\infty} g_j^{(k)}(\zeta_j) = 0$$

for $k = 1, 2, \cdots, m$, and so ζ_0 is a zero of multiplicity at least $m+1$ for $g(\zeta) - \phi_i(z_0)$, $(i = 1, 2, \cdots, n)$.
Since poles of g have multiplicity at least 3, using the Second fundamental theorem of Nevanlinna we arrive at a contradiction (as obtained in the proof of Theorem 5) showing $\mathscr{F}$ is normal at z_0.

Proof. [**Proof of Theorem 8.**]Since normality is a local property, it is enough to show that $\mathscr{H}$ is normal at each $z_0 \in D$. Suppose that $\mathscr{H}$ is not normal at some point $z_0 \in D$. Then by Lemma 1, we can find a sequence $\{f_j\}$ in $\mathscr{H}$, a sequence $\{z_j\}$ of complex numbers with $z_j \to z_0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \to 0$ such that

$$g_j(\zeta) = \rho_j^{-k} f_j(z_j + \rho_j\zeta)$$

converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function $g(\zeta)$ on $\mathbb{C}$, all of whose zeros have multiplicity at least $k+1$, such that $g^{\#}(\zeta) \le g^{\#}(0) = 1$ for all $\zeta \in \mathbb{C}$.

Clearly $g^{(k)}$ assume at least one of the value $\phi_i(z_0)$, $(i = 1, 2, \cdots, n)$, otherwise $g^{(k)}$ becomes constant by Picard's theorem. Let $\zeta_0 \in \mathbb{C}$ be such that $g^{(k)}(\zeta_0) - \phi_i(z_0) = 0$ for some $i = 1, 2, \cdots, n$. Clearly, $g^{(k)}(\zeta) \not\equiv \phi_i(z_0)$, for otherwise g would be a polynomial of degree at most k, which is a contradiction. By Hurwitz theorem, there exist a sequence of points $\zeta_j \to \zeta_0$ such that for sufficiently large j, we have

$$g_j^{(k)}(\zeta_j) = f_j^{(k)}(\zeta_j + \rho_j \zeta_j) = \phi_i(z_j + \rho_j \zeta_j) \in S.$$

By hypothesis, for every $f \in \mathscr{H}$, $f^{\#}(z) \geq M$ whenever $f^{(k)}(z) \in S$, it follows that

$$f_j^{\#}(\zeta_j + \rho_j \zeta_j) \geq M,$$

and hence,

$$\begin{aligned} g^{\#}(\zeta_0) &= \lim_{j\to\infty} g_j^{\#}(\zeta_j) \\ &= \lim_{j\to\infty} \rho_j^{-k+1} (f_j)^{\#}(\zeta_j + \rho_j \zeta_j) \\ &\geq \lim_{j\to\infty} \rho_j^{-k+1} M \to \infty, \end{aligned}$$

which is a contradiction to the fact that g has bounded spherical derivative. Hence $\mathscr{H}$ is normal at z_0.

Acknowledgment. The work of first and second author is supported by CSIR India, and that of the fourth author is supported by the UGC India.

References

[1] Chang, J.M., Fang, M.L., Zalcman, L.Composite meromorphic functions and normal families. Proc. Roy. Soc. Edinburgh, Ser. A, **139**, 57-72(2009).
[2] Chen, J.F.Shared sets and normal families of meromorphhic fuunctions. Rocky Mountain Journal of Mathematics (5) **40**, 517-524 (2010).
[3] Fang, M.L.A note on sharing values and normality. Chinese J. Math. Study **29**, 29-32 (1996).
[4] Fang, M.L., Zalcman, L.Normal families and uniqueness theorems for entire functions. J. Math. Anal. Appl. **280**, 273-283 (2003).
[5] Grahl J., Nevo, S.Spherical derivatives and normal families. J. Anal. Math., **117**, 119-128 (2012).
[6] Hayman, W.K.Meromorphic Functions Oxford, UK: Clarendon Press. (1964).
[7] Liu, X.J., Pang, X.C.Shared values and normal families. Acta Math. Sinica. **50**, 409-412 (2007).
[8] Pang, X., Zalcman, L.Normal families and shared values. Bull. Lond. Math. Soc. **32**, 325-331 (2000).
[9] Schiff, J.L.Normal Families. Springer-Verlag, New York, 1993.
[10] Schwick, W.Sharing values and normality. Arch. Math. (Basel)**59**, 50-54 (1992).