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Strong convergence with a modified iterative projection method for hierarchical fixed point problems and variational inequalities

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Abstract: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}: C \to H$ be a sequence of nearly nonexpansive mappings such that $\mathfrak{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $V : C \to H$ be a γ-Lipschitzian mapping and $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator. This paper deals with a modified iterative projection method for approximating a solution of the hierarchical fixed point problem. It is shown that under certain approximate assumptions on the operators and parameters, the modified iterative sequence $\{x_n\}$ converges strongly to $x^* \in \mathfrak{F}$ which is also the unique solution of the following variational inequality:

$$\left\langle (\rho V - \mu F) x^*, x - x^* \right\rangle \leq 0, \ \forall x \in \mathfrak{F}.$$

As a special case, this projection method can be used to find the minimum norm solution of above variational inequality; namely, the unique solution x^* to the quadratic minimization problem: $x^* = argmin_{x\in\mathfrak{F}} \|x\|^2$. The results here improve and extend some recent corresponding results of other authors.

Keywords: Variational inequality, hierarchical fixed point, nearly nonexpansive mappings, strong convergence.

1 Introduction

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively, and C is a nonempty closed convex subset of H. The set of fixed points of a mapping T is denoted by $Fix(T)$, that is, $Fix(T) = \{x \in H : Tx = x\}$. Below we gather some basic definitions and results which are needed in the subsequent sections. Recall that a mapping $T : C \to H$ is called L-Lipschitzian if there exits a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$, $\forall x, y \in C$. In particular, if $L \in [0, 1)$, then T is said to be a contraction; if $L = 1$, then T is called a nonexpansive mapping. T is called nearly nonexpansive [1,2] with respect to a fixed sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$ if $\|T^n x - T^n y\| \leq \|x - y\| + a_n$, $\forall x, y \in C$ and $n \geq 1$.

A mapping $F : C \to H$ is called η-strongly monotone if there exists a constant $\eta \geq 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \ \forall x, y \in C.$$

In particular, if $\eta = 0$, then F is said to be monotone.

It is well known that for any $x \in H$, there exists a unique point $y_0 \in C$ such that

$$\|x - y_0\| = \inf\{\|x - y\| : y \in C\},$$

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where C is a nonempty closed convex subset of H. We denote y_0 by $P_C x$, where P_C is called the metric projection of H onto C. It is easy to see P_C is a nonexpansive mapping.

Let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: Find $x^* \in Fix(T)$ such that

$$\left\langle x^* - Sx^*, x - x^* \right\rangle \geq 0, \;\; x \in Fix(T). \tag{1}$$

The problem (1) is equivalent to the following fixed point problem: to find an $x^* \in C$ that satisfies $x^* = P_{Fix(T)} Sx^*$. We know that $Fix(T)$ is closed and convex, so the metric projection $P_{Fix(T)}$ is well defined.

It is known that the hierarchical fixed point problem (1) links with some monotone variational inequalities and convex programming problems; see [3,4,5,6,7,8]. Various methods have been proposed to solve the hierarchical fixed point problem; see Moudafi in [10], Mainge and Moudafi in [11], Yao and Liou in [12], Xu in [13], Marino and Xu in [14] and Bnouhachem and Noor in [15].

In 2006, Marino and Xu [16] introduced the viscosity iterative method for nonexpansive mappings. They considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)\, T x_n, \; \forall n \geq 0, \tag{2}$$

where f is a contraction, T is a nonexpansive mapping and A is a strongly positive bounded linear operator on H; that is, there is a constant $\gamma > 0$ such that $\langle Ax, x \rangle \geq \gamma \|x\|$, $\forall x \in H$. They proved that the sequence $\{x_n\}$ generated by (2) converges strongly to the unique solution of the variational inequality

$$\left\langle \left(\gamma f - A\right) x^*, x - x^* \right\rangle \leq 0, \; \forall x \in C, \tag{3}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x)$$

where h is a potential function for γf, i.e., $h'(x) = \gamma f(x)$ for all $x \in H$.

On the other hand, in 2010, Tian [4] proposed an implicit and an explicit schemes on combining the iterative methods of Yamada [9] and Marino and Xu [16]. He also proved the strong convergence of these two schemes to a fixed point of a nonexpansive mapping T defined on a real Hilbert space under suitable conditions. In the same year, Ceng et al. [17] investigated the following iterative method:

$$x_{n+1} = P_C \left[\alpha_n \rho V x_n + \left(1 - \alpha_n \mu F\right) T x_n\right], \; \forall n \geq 0, \tag{4}$$

where F is a L-Lipschitzian and η-strongly monotone operator with constants $L, \eta > 0$ and V is a γ-Lipschitzian (possibly non-self) mapping with constant $\gamma \geq 0$ such that $0 < \mu < \frac{2\eta}{L^2}$ and $0 \leq \rho\gamma < 1 - \sqrt{1 - \mu\left(2\eta - \mu L^2\right)}$. They proved that under some approximate assumptions on the operators and parameters, the sequence $\{x_n\}$ generated by (4) converges strongly to the unique solution of the variational inequality

$$\left\langle \left(\rho V - \mu F\right) x^*, x - x^* \right\rangle \leq 0, \; \forall x \in Fix(T). \tag{5}$$

Fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$ and let $\{T_n\}$ be a sequence of mappings from C into H. Then, the sequence $\{T_n\}$ is called a sequence of nearly nonexpansive mappings [18,19] with respect to a sequence $\{a_n\}$ if

$$\left\| T_n x - T_n y \right\| \leq \left\| x - y \right\| + a_n, \; \forall x, y \in C, \; \forall n \geq 1. \tag{6}$$

It is obvious that the sequence of nearly nonexpansive mappings is a wider class of sequence of nonexpansive mappings. Recently, in 2012, Sahu et al. [19] introduced the following iterative process for the sequence of nearly nonexpansive mappings $\{T_n\}$ defined by (6)

$$x_{n+1} = P_C\left[\alpha_n \rho V x_n + \left(1 - \alpha_n \mu F\right) T_n x_n\right], \ \forall n \geq 1. \tag{7}$$

They proved that the sequence $\{x_n\}$ generated by (7) converges strongly to the unique solution of the variational inequality (5).

Very recently, in 2013, Wang and Xu [20] investigated an iterative method for a hierarchical fixed point problem by

$$\begin{cases} y_n = \beta_n S x_n + \left(1 - \beta_n\right) x_n, \\ x_{n+1} = P_C\left[\alpha_n \rho V x_n + \left(I - \alpha_n \mu F\right) T y_n\right], \ \forall n \geq 0 \end{cases} \tag{8}$$

where $S : C \to C$ is a nonexpansive mapping. They proved that under some approximate assumptions on the operators and parameters, the sequence $\{x_n\}$ generated by (8) converges strongly to the unique solution of the variational inequality (5). In addition to all these methods, similar methods are considered in several papers, see [24, 25, 26, 27, 28].

In this paper, motivated by the work of Wang and Xu [20] and Sahu et al. [19] and by the recent work going in this direction, we introduce a modified iterative projection method and prove a strong convergence theorem based on this method for computing an element of the set of common fixed points of a sequence $\{T_n\}$ of nearly nonexpansive mappings defined by (6) which is also an unique solution of the variational inequality (5). The presented method improves and generalizes many known results for solving variational inequality problems and hierarchical fixed point problems, see, e.g., [4, 16, 17, 19, 20] and relevant references cited therein.

2 Preliminaries

Let $\{x_n\}$ be a sequence in a Hilbert space H and $x \in H$. Throughout this paper, $x_n \to x$ denotes that $\{x_n\}$ strongly converges to x and $x_n \rightharpoonup x$ denotes that $\{x_n\}$ weakly converges to x.

Let C be a nonempty subset of a real Hilbert space H and $T_1, T_2 : C \to H$ be two mappings. We denote $\mathscr{B}(C)$, the collection of all bounded subsets of C. The deviation between T_1 and T_2 on $B \in \mathscr{B}(C)$, denoted by $\mathfrak{D}_B(T_1, T_2)$, is defined by

$$\mathfrak{D}_B(T_1, T_2) = \sup\{\|T_1 x - T_2 x\| : x \in B\}.$$

The following lemmas will be used in the next section.

Lemma 1. [18] *Let C be a nonempty closed bounded subset of a Banach space X and $\{T_n\}$ be a sequence of nearly nonexpansive self-mappings on C with a sequence $\{a_n\}$ such that $\mathfrak{D}_C(T_n, T_{n+1}) < \infty$. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to some point of C. Moreover, if T is a mapping from C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$, then T is nonexpansive and $\lim_{n\to\infty} \mathfrak{D}_C(T_n, T) = 0$.*

Lemma 2. [17] *Let $V : C \to H$ be a γ-Lipschitzian mapping with a constant $\gamma \geq 0$ and let $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator with constants $L, \eta > 0$. Then for $0 \leq \rho\gamma < \mu\eta$,*

$$\left\langle \left(\mu F - \rho V\right) x - \left(\mu F - \rho V\right) y, x - y \right\rangle \geq \left(\mu\eta - \rho\gamma\right) \left\|x - y\right\|^2, \ \forall x, y \in C.$$

That is, $\mu F - \rho V$ is strongly monotone with coefficient $\mu\eta - \rho\gamma$.

Lemma 3. [9] *Let C be a nonempty subset of a real Hilbert space H. Suppose that $\lambda \in (0,1)$ and $\mu > 0$. Let $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator on C. Define the mapping $G : C \to H$ by*

$$Gx = x - \lambda\mu Fx,\ \forall x \in C.$$

Then, G is a contraction that provided $\mu < \frac{2\eta}{L^2}$. More precisely, for $\mu \in \left(0, \frac{2\eta}{L^2}\right)$,

$$\left\|Gx - Gy\right\| \le (1 - \lambda v)\left\|x - y\right\|,\ \forall x, y \in C,$$

where $v = 1 - \sqrt{1 - \mu\left(2\eta - \mu L^2\right)}$.

Lemma 4. [21] *Let C be a nonempty closed convex subset of a real Hilbert space H, and T be a nonexpansive self-mapping on C. If $Fix(T) \neq \emptyset$, then $I - T$ is demiclosed; that is whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

Lemma 5. [22] *Assume that $\{x_n\}$ is a sequence of nonnegative real numbers satisfying the conditions*

$$x_{n+1} \le (1 - \alpha_n)x_n + \alpha_n\beta_n,\ \forall n \ge 1$$

where $\{\alpha_n\}$ and $\left\{\beta_n\right\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$
(ii) $\limsup_{n\to\infty}\beta_n \le 0$.

Then $\lim_{n\to\infty} x_n = 0$.

3 Main results

Now, we give the main results in this paper.

Theorem 1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to H$ be a nonexpansive mapping and $\{T_n\}$ be a sequence of nearly nonexpansive mappings with the sequence $\{a_n\}$ such that $F := \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. Suppose that $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and $Fix(T) = F$. Let $V : C \to H$ be a γ-Lipschitzian mapping, $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator such that these coefficients satisfy $0 < \mu < \frac{2\eta}{L^2}$, $0 \le \rho\gamma < v$, where $v = 1 - \sqrt{1 - \mu\left(2\eta - \mu L^2\right)}$. For an arbitrarily initial value x_1, consider the sequence $\{x_n\}$ in C generated by*

$$\begin{cases} y_n = P_C\left[\beta_n S x_n + \left(1 - \beta_n\right)x_n\right], \\ x_{n+1} = P_C\left[\alpha_n\rho V x_n + \left(I - \alpha_n\mu F\right)T_n y_n\right],\ n \ge 1, \end{cases} \tag{9}$$

where $\{\alpha_n\}$ and $\left\{\beta_n\right\}$ are sequences in $[0,1]$ satisfying the conditions:

$(C1)$ $\lim\limits_{n\to\infty}\alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty}\alpha_n = \infty$;

$(C2)$ $\lim\limits_{n\to\infty}\dfrac{a_n}{\alpha_n} = 0,\ \lim\limits_{n\to\infty}\dfrac{\beta_n}{\alpha_n} = 0,\ \lim\limits_{n\to\infty}\dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$ *and* $\lim\limits_{n\to\infty}\dfrac{\left|\beta_n - \beta_{n-1}\right|}{\alpha_n} = 0$;

$(C3)$ $\lim\limits_{n\to\infty}\mathfrak{D}_B(T_n, T_{n+1}) = 0$ *and* $\lim\limits_{n\to\infty}\dfrac{\mathfrak{D}_B(T_n, T_{n+1})}{\alpha_n} = 0$ *for each* $B \in \mathscr{B}(C)$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \mathfrak{F}$, where x^* is the unique solution of the variational inequality

$$\left\langle\left(\rho V - \mu F\right)x^*, x - x^*\right\rangle \le 0,\ \forall x \in \mathfrak{F}. \tag{10}$$

In particular, the point x^* is the unique solution of the quadratic minimization problem —

In particular, the point x^* is the minimum norm fixed point of T, that is x^* is the unique solution of the quadratic minimization problem

$$x^* = argmin_{x\in\mathfrak{F}}\|x\|^2.$$

Proof. Since the mapping T is defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, by Lemma 1, T is a nonexpansive mapping, and $Fix(T) \neq \emptyset$. Moreover, since the operator $\mu F - \rho V$ is $(\mu\eta - \rho\gamma)$-strongly monotone by Lemma 2, we get the uniqueness of the solution of the variational inequality (10). Let denote this solution by $x^* \in Fix(T) = \mathfrak{F}$.

Now, we divide our proof into six steps.

Step 1. First we show that the sequences $\{x_n\}$ is bounded. From hypothesis (C2), without loss of generality, we may assume that $\beta_n \le \alpha_n$, for $n \ge 1$. Hence, we get $\lim_{n\to\infty}\beta_n = 0$. Let $p \in \mathfrak{F}$ and $t_n = \alpha_n\rho V x_n + (I - \alpha_n\mu F) T_n y_n$. Then we have

$$\begin{aligned}
\|y_n - p\| &= \|P_C[\beta_n S x_n + (1-\beta_n)x_n] - P_C p\| \\
&\le \|\beta_n S x_n + (1-\beta_n)x_n - p\| \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n\|Sx_n - p\| \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n\|Sx_n - Sp\| + \beta_n\|Sp - p\| \\
&\le \|x_n - p\| + \beta_n\|Sp - p\|, \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - p\| &= \|P_C t_n - P_C p\| \\
&\le \|t_n - p\| \\
&= \|\alpha_n\rho V x_n + (I - \alpha_n\mu F) T_n y_n - p\| \\
&\le \alpha_n\|\rho V x_n - \mu F p\| + \|(I - \alpha_n\mu F) T_n y_n - (I - \alpha_n\mu F) T_n p\| \\
&\le \alpha_n\rho\gamma\|x_n - p\| + \alpha_n\|\rho V p - \mu F p\| \\
&\quad + (1 - \alpha_n\nu)(\|y_n - p\| + a_n). \qquad (12)
\end{aligned}$$

From (11) and (12), we get

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n\rho\gamma\|x_n - p\| + \alpha_n\|\rho V p - \mu F p\| \\
&\quad + (1 - \alpha_n\nu)(\|x_n - p\| + \beta_n\|Sp - p\| + a_n) \\
&\le (1 - \alpha_n(\nu - \rho\gamma))\|x_n - p\| + \alpha_n(\|\rho V p - \mu F p\| + \|Sp - p\| + a_n) \\
&\le (1 - \alpha_n(\nu - \rho\gamma))\|x_n - p\| \\
&\quad + \alpha_n(\nu - \rho\gamma)\left[\frac{1}{(\nu - \rho\gamma)}\left(\|\rho V p - \mu F p\| + \|Sp - p\| + \frac{a_n}{\alpha_n}\right)\right]. \qquad (13)
\end{aligned}$$

Note that $\frac{a_n}{\alpha_n} \to 0$ as $n \to \infty$, so there exists a constant $M > 0$ such that

$$\|\rho V p - \mu F p\| + \|Sp - p\| + \frac{a_n}{\alpha_n} \le M, \forall n \ge 1.$$

Thus, from (13) we have

$$\|x_{n+1} - p\| \le (1 - \alpha_n(\nu - \rho\gamma))\|x_n - p\| + \alpha_n(\nu - \rho\gamma)\frac{M}{(\nu - \rho\gamma)}.$$

By induction, we get

$$\|x_{n+1}-p\| \leq \max\left\{\|x_1-p\|, \frac{M}{(\nu-\rho\gamma)}\right\}.$$

Hence, we obtain that $\{x_n\}$ is bounded. So, the sequences $\{y_n\}$,$\{Tx_n\}$,$\{Sx_n\}$,$\{Vx_n\}$ and $\{FTy_n\}$ are bounded.

Step 2. Now, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. By using the iteration (9), we have

$$\begin{aligned}\|y_n-y_{n-1}\| &= \|P_C[\beta_n Sx_n+(1-\beta_n)x_n]-P_C[\beta_{n-1}Sx_{n-1}-(1-\beta_{n-1})x_{n-1}]\| \\ &\leq \beta_n\|Sx_n-Sx_{n-1}\|+(1-\beta_n)\|x_n-x_{n-1}\| \\ &+|\beta_n-\beta_{n-1}|(\|Sx_{n-1}\|+\|x_{n-1}\|) \\ &\leq \|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|M_1,\end{aligned} \tag{14}$$

where M_1 is a constant such that $\sup_{n\geq 1}\{\|Sx_n\|+\|x_n\|\}\leq M_1$. Also, by using the inequality (14), we get

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq \|P_Ct_n-P_Ct_{n-1}\| \\ &\leq \|\alpha_n\rho Vx_n+(I-\alpha_n\mu F)T_ny_n \\ &-\alpha_{n-1}\rho Vx_{n-1}+(I-\alpha_{n-1}\mu F)T_{n-1}y_{n-1}\| \\ &\leq \|\alpha_n\rho V(x_n-x_{n-1})+(\alpha_n-\alpha_{n-1})\rho Vx_{n-1}+(I-\alpha_n\mu F)T_ny_n \\ &-(I-\alpha_n\mu F)T_ny_{n-1}+T_ny_{n-1}-T_{n-1}y_{n-1} \\ &+\alpha_{n-1}\mu FT_{n-1}y_{n-1}-\alpha_n\mu FT_ny_{n-1}\| \\ &\leq \alpha_n\rho\gamma\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\|Vx_{n-1}\| \\ &+(1-\alpha_n\nu)\|T_ny_n-T_ny_{n-1}\|+\|T_ny_{n-1}-T_{n-1}y_{n-1}\| \\ &+\mu\|\alpha_{n-1}FT_{n-1}y_{n-1}-\alpha_nFT_ny_{n-1}\| \\ &\leq \alpha_n\rho\gamma\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\|Vx_{n-1}\| \\ &+(1-\alpha_n\nu)[\|y_n-y_{n-1}\|+a_n]+\|T_ny_{n-1}-T_{n-1}y_{n-1}\| \\ &+\mu\|\alpha_{n-1}(FT_{n-1}y_{n-1}-FT_ny_{n-1})-(\alpha_n-\alpha_{n-1})FT_ny_{n-1}\| \\ &\leq \alpha_n\rho\gamma\|x_n-x_{n-1}\|+\gamma\|Vx_{n-1}\|+(1-\alpha_n\upsilon)\|x_n-x_{n-1}\| \\ &+(1-\alpha_n\upsilon)|\beta_n-\beta_{n-1}|M_1+(1-\alpha_n\upsilon)a_n+\mathfrak{D}_B(T_n,T_{n-1}) \\ &+\mu\alpha_{n-1}L\mathfrak{D}_B(T_n,T_{n-1})+|\alpha_n-\alpha_{n-1}|\|FT_ny_{n-1}\| \\ &\leq (1-\alpha_n(\upsilon-\rho\gamma))\|x_n-x_{n-1}\| \\ &+|\alpha_n-\alpha_{n-1}|(\gamma\|Vx_{n-1}\|+\|FT_ny_{n-1}\|) \\ &+(1+\mu\alpha_{n-1}L)\mathfrak{D}_B(T_n,T_{n-1})+|\beta_n-\beta_{n-1}|M_1+a_n \\ &\leq (1-\alpha_n(\upsilon-\rho\gamma))\|x_n-x_{n-1}\|+\alpha_n(\upsilon-\rho\gamma)\delta_n,\end{aligned}$$

where

$$\delta_n=\frac{1}{(\nu-\rho\gamma)}\left[\begin{array}{c}(1+\mu\alpha_{n-1}L)\frac{\mathfrak{D}_B(T_n,T_{n-1})}{\alpha_n} \\ +\left(\left|\frac{\alpha_n-\alpha_{n-1}}{\alpha_n}\right|+\left|\frac{\beta_n-\beta_{n-1}}{\alpha_n}\right|\right)M_2+\frac{a_n}{\alpha_n}\end{array}\right],$$

and

$$\sup_{n\geq 1}\{\gamma\|Vx_{n-1}\|+\|FT_ny_{n-1}\|, M_1\}\leq M_2.$$

Since $\limsup_{n\to\infty}\delta_n\leq 0$, it follows from Lemma 5, conditions (C2) and (C3) that

$$\|x_{n+1}-x_n\|\to 0 \text{ as } n\to\infty. \tag{15}$$

Step 3. Next, we show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ as $n\to\infty$. Note that

$$\begin{aligned}
\|x_n - T_n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \\
&\le \|x_n - x_{n+1}\| + \|P_C t_n - P_C T_n x_n\| \\
&\le \|x_n - x_{n+1}\| + \left\|\alpha_n \rho V x_n + (I - \alpha_n \mu F) T_n y_n - T_n x_n\right\| \\
&\le \|x_n - x_{n+1}\| + \left\|\alpha_n (\rho V x_n - \mu F T_n y_n) + T_n y_n - T_n x_n\right\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n \left\|\rho V x_n - \mu F T_n y_n\right\| + \left\|y_n - x_n\right\| + a_n \\
&\le \|x_n - x_{n+1}\| + \alpha_n \left\|\rho V x_n - \mu F T_n y_n\right\| + \beta_n \|S x_n - x_n\| + a_n.
\end{aligned}$$

Since $a_n \to 0$, by using (15) and condition (C1), we obtain

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

Hence, we have

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - T x_n\| \\
&\le \|x_n - T_n x_n\| + \mathfrak{D}_B(T_n, T) \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Step 4. Next, we show that $\limsup_{n\to\infty}\langle(\rho V - \mu F)x^*, x_n - x^*\rangle \le 0$, where x^* is the unique solution of variational inequality (10). Since the sequence $\{x_n\}$ is bounded, it has a weak convergent subsequence $\{x_{n_k}\}$ such that

$$\limsup_{n\to\infty}\langle(\rho V - \mu F)x^*, x_n - x^*\rangle = \limsup_{k\to\infty}\langle(\rho V - \mu F)x^*, x_{n_k} - x^*\rangle.$$

Let $x_{n_k} \rightharpoonup \tilde{x}$, as $k\to\infty$. It follows from Lemma 4 that $\tilde{x} \in Fix(T) = \mathfrak{F}$. Hence

$$\limsup_{n\to\infty}\langle(\rho V - \mu F)x^*, x_n - x^*\rangle = \langle(\rho V - \mu F)x^*, \tilde{x} - x^*\rangle \le 0.$$

Step 5. Now, we show that the sequence $\{x_n\}$ converges strongly to x^* as $n\to\infty$. By using the iteration (9), we have

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &= \langle P_C t_n - x^*, x_{n+1} - x^*\rangle \\
&= \langle P_C t_n - t_n, x_{n+1} - x^*\rangle + \langle t_n - x^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{16}$$

Since the metric projection P_C satisfies the inequality

$$\langle x - P_C x, y - P_C x\rangle \le 0,\ \forall x \in H, y \in C,$$

and from (16), we get

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &\le \langle t_n - x^*, x_{n+1} - x^*\rangle \\
&= \langle \alpha_n \rho V x_n + (I - \alpha_n \mu F) T_n y_n - x^*, x_{n+1} - x^*\rangle \\
&= \langle \alpha_n (\rho V x_n - \mu F x^*) + (I - \alpha_n \mu F) T_n y_n \\
&\quad - (I - \alpha_n \mu F) T_n x^*, x_{n+1} - x^*\rangle \\
&= \alpha_n \rho \langle V x_n - V x^*, x_{n+1} - x^*\rangle + \alpha_n \langle \rho V x^* - \mu F x^*, x_{n+1} - x^*\rangle \\
&\quad + \langle (I - \alpha_n \mu F) T_n y_n - (I - \alpha_n \mu F) T_n x^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

Hence, from (11) and Lemma 3, we obtain

$$\begin{aligned}
\left\|x_{n+1}-x^*\right\|^2 &\leq \alpha_n\rho\gamma\left\|x_n-x^*\right\|\left\|x_{n+1}-x^*\right\|+\alpha_n\left\langle\rho Vx^*-\mu Fx^*,x_{n+1}-x^*\right\rangle\\
&+(1-\alpha_n\nu)\left(\left\|y_n-x^*\right\|+a_n\right)\left\|x_{n+1}-x^*\right\|\\
&\leq \alpha_n\rho\gamma\left\|x_n-x^*\right\|\left\|x_{n+1}-x^*\right\|+\alpha_n\left\langle\rho Vx^*-\mu Fx^*,x_{n+1}-x^*\right\rangle\\
&+(1-\alpha_n\nu)\left(\left\|x_n-x^*\right\|+\beta_n\left\|Sx^*-x^*\right\|+a_n\right)\left\|x_{n+1}-x^*\right\|\\
&=\left(1-\alpha_n\left(\upsilon-\rho\gamma\right)\right)\left\|x_n-x^*\right\|\left\|x_{n+1}-x^*\right\|\\
&+\alpha_n\left\langle\rho Vx^*-\mu Fx^*,x_{n+1}-x^*\right\rangle\\
&+(1-\alpha_n\upsilon)\beta_n\left\|Sx^*-x^*\right\|\left\|x_{n+1}-x^*\right\|\\
&+(1-\alpha_n\upsilon)a_n\left\|x_{n+1}-x^*\right\|\\
&\leq\frac{\left(1-\alpha_n\left(\upsilon-\rho\gamma\right)\right)}{2}\left(\left\|x_n-x^*\right\|^2+\left\|x_{n+1}-x^*\right\|^2\right)\\
&+\alpha_n\left\langle\rho Vx^*-\mu Fx^*,x_{n+1}-x^*\right\rangle+\beta_n\left\|Sx^*-x^*\right\|\left\|x_{n+1}-x^*\right\|\\
&+a_n\left\|x_{n+1}-x^*\right\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\left\|x_{n+1}-x^*\right\|^2 &\leq \frac{\left(1-\alpha_n\left(\nu-\rho\gamma\right)\right)}{\left(1+\alpha_n\left(\nu-\rho\gamma\right)\right)}\left\|x_n-x^*\right\|^2\\
&+\frac{2\alpha_n}{\left(1+\alpha_n\left(\nu-\rho\gamma\right)\right)}\left\langle\rho Vx^*-\mu Fx^*,x_{n+1}-x^*\right\rangle\\
&+\frac{2\beta_n}{\left(1+\alpha_n\left(\nu-\rho\gamma\right)\right)}\left\|Sx^*-x^*\right\|\left\|x_{n+1}-x^*\right\|\\
&+\frac{2a_n}{\left(1+\alpha_n\left(\nu-\rho\gamma\right)\right)}\left\|x_{n+1}-x^*\right\|\\
&\leq\left(1-\alpha_n\left(\nu-\rho\gamma\right)\right)\left\|x_n-x^*\right\|^2+\alpha_n\left(\nu-\rho\gamma\right)\theta_n,
\end{aligned}$$

where

$$\theta_n=\frac{2\alpha_n}{\left(1+\alpha_n\left(\nu-\rho\gamma\right)\right)\left(\nu-\rho\gamma\right)}\left[\begin{array}{c}\left\langle\rho Vx^*-\mu Fx^*,x_{n+1}-x^*\right\rangle+\frac{\beta_n}{\alpha_n}M_3\\ +\frac{a_n}{\alpha_n}\left\|x_{n+1}-x^*\right\|\end{array}\right],$$

and

$$\sup_{n\geq 1}\left\{\left\|Sx^*-x^*\right\|\left\|x_{n+1}-x^*\right\|\right\}\leq M_3.$$

Since $\frac{\beta_n}{\alpha_n}\to 0$ and $\frac{a_n}{\alpha_n}\to 0$, we get

$$\limsup_{n\to\infty}\theta_n\leq 0.$$

So, it follows from Lemma 5 that the sequence $\{x_n\}$ generated by (9) converges strongly to $x^*\in\mathfrak{F}$ which is the unique solution of variational inequality (10).

Step 6. Finally, since the point x^* is the unique solution of variational inequality (10), in particular if we take $V=0$ and $F=I$ in the variational inequality (10), then we get

$$\left\langle -\mu x^*,x-x^*\right\rangle\leq 0,\ \forall x\in\mathfrak{F}.$$

So we have

$$\left\langle x^*,x^*-x\right\rangle=\left\langle x^*,x^*\right\rangle-\left\langle x^*,x\right\rangle\leq 0\Longrightarrow\left\|x^*\right\|^2\leq\left\|x^*\right\|\left\|x\right\|.$$

Hence, x^* is the unique solution to the quadratic minimization problem $x^* = argmin_{x\in\mathfrak{F}} \|x\|^2$. This completes the proof.

From Theorem 1, we can deduce the following interesting corollaries.

Corollary 1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to H$ be a nonexpansive mapping and $\{T_n\}$ be a sequence of nonexpansive mappings such that $\mathfrak{F} \neq \emptyset$. Suppose that $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Let $V : C \to H$ be a γ-Lipschitzian mapping, $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator such that these coefficients satisfy $0 < \mu < \frac{2\eta}{L^2}$, $0 \leq \rho\gamma < \nu$, where $\nu = 1 - \sqrt{1-\mu\left(2\eta - \mu L^2\right)}$. For an arbitrarily initial value $x_1 \in C$, consider the sequence $\{x_n\}$ in C generated by (9) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions (C1)-(C3) of Theorem 1 except the condition $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$. Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \mathfrak{F}$, where x^* is the unique solution of variational inequality (10).*

Let $\lambda_i > 0$ $(i = 1,2,3,\ldots N)$ such that $\sum_{i=1}^N \lambda_i = 1$ and $T_1, T_2, \ldots T_N$ be nonexpansive self mappings on C such that $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Then, $\sum_{i=1}^N \lambda_i T_i$ is nonexpansive self mapping on C (see [23, Proposition 6.1]).

Corollary 2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\lambda_i > 0$ $(i = 1,2,3,\ldots N)$ such that $\sum_{i=1}^N \lambda_i = 1$ and $S, T_1, T_2, \ldots T_N$ be nonexpansive self mappings on C such that $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Let $V : C \to H$ be a γ-Lipschitzian mapping, $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator such that these coefficients satisfy $0 < \mu < \frac{2\eta}{L^2}$, $0 \leq \rho\gamma < \nu$, where $\nu = 1 - \sqrt{1-\mu\left(2\eta - \mu L^2\right)}$. For an arbitrarily initial value $x_1 \in C$, consider the sequence $\{x_n\}$ in C generated by*

$$\begin{cases} y_n = \beta_n S x_n + \left(1-\beta_n\right) x_n, \\ x_{n+1} = P_C\left[\alpha_n \rho V x_n + \left(I - \alpha_n \mu F\right) \sum_{i=1}^N \lambda_i T_i y_n\right], \forall n \geq 1 \end{cases} \tag{17}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions (C1) and (C2) of Theorem 1 except the condition $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$. Then, the sequence $\{x_n\}$ in C generated by (17) converges strongly to $x^* \in \bigcap_{i=1}^N Fix(T_i)$, where x^* is the unique solution of variational inequality

$$\left\langle (\rho V - \mu F) x^*, x - x^* \right\rangle \leq 0, \ \forall x \in \bigcap_{i=1}^N Fix(T_i).$$

Remark. Our results can be reduced to some corresponding results in the following ways:

(1) In our iterative process (9), if we take $S = I$ (I is the identity operator of C), then we derive the iterative process (7) which is studied by Sahu et. al. [19]. Therefore, Theorem 1 generalizes the main result of Sahu et. al. [19, Theorem 3.1]. Also, Corollary 1 and Corollary 2 extends the Corollary 3.4 and Theorem 4.1 of Sahu et. al. [19], respectively. So, our results extends the corresponding results of Ceng et. al. [17] and of many other authors.
(2) If we take S as a nonexpansive self mapping on C and $T_n = T$ for all $n \geq 1$ such that T is a nonexpansive mapping in (9), then we get the iterative process (8) of Wang and Xu. [20]. Hence, Theorem 1 generalizes the main result of Wang and Xu [20, Theorem 3.1]. So, our results extend and improve the corresponding results of [4, **?**].
(3) The problem of finding the solution of variational inequality (10), is equivalent to finding the solutions of hierarchical fixed point problem

$$\langle (I-S) x^*, x^* - x \rangle \leq 0, \forall x \in \mathfrak{F},$$

where $S = I - \left(\rho V - \mu F\right)$.

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