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The proof of theorem which characterizes a slant helix

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Abstract: Firstly, the axis of a slant helix is found. Secondly, the theorem which characterizes a slant helix is proved in detail. The importance of this theorem is stemed from that it has been led to doing many papers about slant helices.

Keywords: Slant helix, principal normal indicatrix, the axis, geodesic curvature.

1 Introduction

Slant helices are special space curves like Salkowski, Bertrand and Mannheim curves, or general helices. Slant helix concept was first introduced by Izumiya and Takeuchi [1] and then studied by so many authors.

Izumiya and Takeuchi [1] defined slant helices which are generalizations of the notion of general helices. Kula and Yaylı [2] investigated the spherical indicatrices of slant helices and showed that the tangent and binormal indicatrices of them are spherical helices. Also, Kula et al. [3] gave some characterizations for slant helices in Euclidean 3-space. Ali and Turgut [4] extended the notion of slant helix from Euclidean 3-space to Euclidean n-space. They introduced type-2 harmonic curvatures of a regular curve and gave necessary and sufficient conditions for a curve to be a slant helix in Euclidean n-space.

Recently, Ali and Turgut [5] researched the position vector of a timelike slant helix in Minkowski 3-space $\mathbb{E}_1^3$. They determined the parametric representation of the position vector ψ from intrinsic equations in $\mathbb{E}_1^3$ for the timelike slant helix. Besides, Ali and Lopez [6] looked into slant helices in Minkowski 3-space. They gave characterizations for spacelike, timelike and lightlike slant helices and also investigated tangent indicatrix, binormal indicatrix and involutes of a non-null curve.

More recently, Dogan [7], and Dogan and Yayli [8,9] studied isophote curves on surfaces in Euclidean 3-space and Minkowski 3-space. An isophote curve on a surface can be regarded as a nice consequence of Lambert's cosine law in the optics. Lambert's law states that the intensity of illumination on a diffuse surface is proportional to the cosine of the angle generated between the surface normal vector N and the light vector d. According to this law the intensity is irrespective of the actual viewpoint; hence the illumination is the same when viewed from any direction. In other words, isophotes of a surface are curves with the property that their points have the same light intensity from a given source (curves of constant illumination intensity).

When the source light is at infinity, we may consider that the light flow consists of parallel lines. They [7,8,9] showed that the there is a close relation between isophote curves and slant helices, i.e., a curve which is both a geodesic and a

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slant helix is an isophote curve.

This paper is organized as follows. Section 2 presents basic concepts concerning curve and surface theory in $\mathbb{E}^3$. Firstly, the axis of a slant helix is found and secondly, the theorem which characterizes a slant helix is proved in section 3.

2 Preliminaries

In this section, we give some basic notions about curves and surfaces. The differential geometry of curves starts with a smooth map of s; let us call it $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ that parametrizes a spatial curve and it will be denoted again with α. The curve α is parametrized by arc-length if $\left\|\alpha^{'}(s)\right\| = 1$ (unit-speed), where $\alpha^{'}(s)$ is the first derivative of α with respect to s. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a regular curve with an arc-length parameter s and $\kappa(s) = \left\|\alpha^{''}(s)\right\|$, where κ is the curvature of α, and $\alpha^{''}$ is the second derivative of α. For $\kappa > 0$, the Frenet frame $\{T, N, B\}$ is well-defined along the curve α and as follows.

$$\begin{aligned} T(s) &= \alpha^{'}(s), \\ N(s) &= \frac{\alpha^{''}(s)}{\left\|\alpha^{''}(s)\right\|}, \\ B(s) &= T(s) \times N(s), \end{aligned} \tag{1}$$

where T, N and B are the tangent, the principal normal, and the binormal of α, respectively. The derivative of the Frenet frame (Frenet equations) are given by

$$\begin{aligned} T^{'}(s) &= \kappa(s)N(s), \\ N^{'}(s) &= -\kappa(s)T(s) + \tau(s)B(s), \\ B^{'}(s) &= -\tau(s)N(s), \end{aligned} \tag{2}$$

where $\tau(s) = \dfrac{\left\langle \alpha^{'}(s) \times \alpha^{''}(s), \alpha^{'''}(s) \right\rangle}{\kappa^2(s)}$ is the torsion of α; "$\langle , \rangle$" is the standart inner product, and "$\times$" is the cross product on $\mathbb{R}^3$.

Let $\mathbb{M}$ be a regular surface and $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{M}$ be a unit-speed curve. Then the Darboux frame $\{T, Y = U \times T, U\}$ is well-defined along the curve α, where T is the tangent of α and U is the unit normal of $\mathbb{M}$. If we rotate the Darboux frame $\{T, Y = U \times T, U\}$ by ϕ about T, we obtain the Frenet frame $\{T, N, B\}$.

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} T \\ Y \\ U \end{bmatrix}$$

$$\begin{aligned} T &= T, \\ N &= \cos\phi Y + \sin\phi U, \\ B &= -\sin\phi Y + \cos\phi U. \end{aligned} \tag{3}$$

The derivative formulas for the Darboux frame are given by

$$T^{'} = k_g Y + k_n U, \quad (4)$$
$$Y^{'} = -k_g T + \tau_g U,$$
$$U^{'} = -k_n T - \tau_g Y,$$

where " $'$ " denotes the derivative of T, B, and N with respect to s along the curve $U3b1$; k_n, k_g, and τ_g are the normal curvature, the geodesic curvature, and the geodesic torsion of α, respectively. With the above notations, let ϕ denote the angle between the surface normal U and the binormal B. Using equations (2), (3) and (4) we obtain

$$\kappa^2 = k_g^2 + k_n^2, \quad (5)$$
$$k_g = \kappa \cos\phi,$$
$$k_n = \kappa \sin\phi,$$
$$\tau_g = \tau - \phi^{'}.$$

Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a regular curve with an arc-length parameter s and $\kappa > 0$. Since $\|N\| = 1$,the curve $\beta : I \subset \mathbb{R} \longrightarrow \mathbb{S}^2$, $\beta(s) = N(s)$ lies on the unit sphere $\mathbb{S}^2$. It is called the principal normal indicatrix of α.

3 A Slant helix and its axis

In this section, we find the fixed vector (axis) of a slant helix. By means of this axis, we prove the theorem which characterizes a slant helix in detail. A space curve whose its principal normal vectors make a constant angle with a fixed vector is called a slant helix. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a unit-speed slant helix with $\kappa(s) \neq 0$. By the definition of slant helix

$$\langle N, d \rangle = \cos\theta, \quad (6)$$

where N is the principal normal, d is the fixed vector of α, and θ is the constant angle between N and d, respectively. If we differentiate Eq.(6) with respect to s along the curve α and then use the Frenet equations, we obtain

$$\left\langle N^{'}, d \right\rangle = 0 \quad (7)$$

$$\langle -\kappa T + \tau B, d \rangle = 0$$

$$\kappa \langle T, d \rangle = \tau \langle B, d \rangle$$

$$\langle T, d \rangle = \frac{\tau}{\kappa} \langle B, d \rangle .$$

If we take $\langle B, d \rangle = c$, we get

$$d = \frac{\tau}{\kappa} cT + \cos\theta N + cB.$$

Since $\|d\| = 1$, it follows that

$$\frac{\tau^2}{\kappa^2} c^2 + \cos^2\theta + c^2 = 1$$

$$(\frac{\tau^2}{\kappa^2} + 1)c^2 = \sin^2\theta$$

$$c = \mp \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \sin\theta.$$

Therefore, the vector d can be written as

$$d = \mp\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin\theta T + \cos\theta N \mp \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin\theta B. \tag{8}$$

Here, d is actually a constant vector. By differentiating Eq.(7) with respect to s along the curve α, we obtain

$$\begin{aligned}
\left\langle N^{''}, d\right\rangle &= 0 \\
\left\langle \kappa^{'}T + (\kappa^2+\tau^2)N - \tau^{'}B, \mp\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin\theta T + \cos\theta N \mp \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin\theta B\right\rangle &= 0 \\
\mp\frac{\kappa^{'}\tau - \tau^{'}\kappa}{(\kappa^2+\tau^2)^{3/2}}\tan\theta + 1 &= 0 \\
\tan\theta &= \mp\frac{(\kappa^2+\tau^2)^{3/2}}{\tau^{'}\kappa - \kappa^{'}\tau} \\
\cot\theta &= \mp\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)^{'}.
\end{aligned} \tag{9}$$

From Eq.(8) and the Frenet equations, the derivative of d becomes

$$\begin{aligned}
d^{'} &= \mp\left[\sin\theta\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)^{'}T + \sin\theta\frac{\tau}{\sqrt{\kappa^2+\tau^2}}(\kappa N)\right] + \cos\theta\,(-\kappa T + \tau B) \\
&\mp\left[\sin\theta\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)^{'}B + \sin\theta\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}(-\tau N)\right] \\
&= \left[-\kappa \mp \tan\theta\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)^{'}\right]T + \left[\tau \mp \tan\theta\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)^{'}\right]B.
\end{aligned}$$

If we substitute Eq.(9) in the last equality above, we get

$$\begin{aligned}
d^{'} &= \left[-\kappa + \frac{(\kappa^2+\tau^2)^{3/2}}{\tau^{'}\kappa - \kappa^{'}\tau}\left(\frac{\tau^{'}(\kappa^2+\tau^2) - \tau(\kappa\kappa^{'} + \tau\tau^{'})}{(\kappa^2+\tau^2)^{3/2}}\right)\right]T \\
&+ \left[\tau + \frac{(\kappa^2+\tau^2)^{3/2}}{\tau^{'}\kappa - \kappa^{'}\tau}\left(\frac{\kappa^{'}(\kappa^2+\tau^2) - \kappa(\kappa\kappa^{'} + \tau\tau^{'})}{(\kappa^2+\tau^2)^{3/2}}\right)\right]B.
\end{aligned}$$

By a direct calculation, it can be seen that the coefficients of T and B are zero. Then $d^{'} = 0$, in other words, d is a constant vector.

Theorem 1. *A unit-speed curve $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ with $\kappa(s) \neq 0$ is a slant helix if and only if*

$$\sigma(s) = \mp\left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)^{'}\right)(s)$$

is a constant function [1].

Proof. The vectors that make a constant angle with a fixed vector construct a cone. Then the unit vectors in $\mathbb{E}^3$, which make a constant angle with a fixed vector construct a circular cone whose the base curve lies on the unit sphere $\mathbb{S}^2$. Therefore, α is the unit-speed slant helix if and only if its principal normal indicatrix is a circle on the unit sphere $\mathbb{S}^2$. In other words, if we compute the normal indicatrix $\beta : I \subset \mathbb{R} \longrightarrow \mathbb{S}^2$, $\beta(s) = N(s)$ along the curve α, the geodesic curvature of β becomes

$\sigma(s)$ as obtained below.

$$\begin{aligned} N' &= -\kappa T + \tau B, \\ N'' &= -\kappa' T - (\kappa^2+\tau^2)N + \tau' B, \\ N' \times N'' &= \tau(\kappa^2+\tau^2)T + \kappa^2(\frac{\tau}{\kappa})' N + \kappa(\kappa^2+\tau^2)B, \\ \kappa_\beta &= \frac{\left\| N' \times N'' \right\|}{\left\| N' \right\|^3}, \\ &= \sqrt{\frac{(\kappa^2+\tau^2)^3 + (\kappa^2(\frac{\tau}{\kappa})')^2}{(\kappa^2+\tau^2)^3}}, \end{aligned}$$

where κ_β is the curvature of β. Let k_g and k_n be the geodesic curvature and normal curvature of β on $\mathbb{S}^2$, respectively. Since the normal curvature $k_n = 1$ on $\mathbb{S}^2$, if we substitute k_n and κ_β in the following equation, we get the geodesic curvature k_g as follows.

$$\begin{aligned} k_g^2 + k_n^2 &= (\kappa_\beta)^2, \\ k_g(s) = \sigma(s) = \cot\theta &= \mp\left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}(\frac{\tau}{\kappa})'\right)(s), \end{aligned}$$

where θ is the constant angle between the principal normal N and the axis d. Because of the fact that θ is a constant, $\sigma(s)$ is a constant. Then, a unit-speed curve $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ is a slant helix if and only if the spherical image (indicatrix) of its principal normal $\beta : I \subset \mathbb{R} \longrightarrow \mathbb{S}^2$ is a circle.

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