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Some operators on Hilbert sequence spaces of generalized means

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Abstract: The idea of sequence spaces of generalized means has recently been introduced by Mursaleen and Noman [2]. In this paper, we study bounded weighted composition operators on some Hilbert sequence spaces of generalized means.

Keywords: Multiplication operators, composition operators, weighted composition operators, Hilbert sequences spaces, sequence spaces of generalized means.

1 Introduction and preliminaries

By w, we denote the space of all complex sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$. Also, we write ϕ for the set of all finite sequences that terminate in zeros. Further, we shall use the conventions that $e = (1,1,1,\dots)$ and $e^{(k)}$ is the sequence whose only non-zero term is 1 in the k^{th} place for each $k \in \mathbb{N}$, where $\mathbb{N} = \{0,1,2,\dots\}$.

Any vector subspace of w is called a *sequence space*. We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Further, by ℓ_p $(1 \le p < \infty)$, we denote the sequence space of all p-absolutely convergent series, that is $\ell_p = \{x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$. The spaces ℓ_∞, c and c_0 are Banach spaces with the usual sup-norm given by $\|x\|_\infty = \sup_k |x_k|$. Also, the space ℓ_p $(1 \le p < \infty)$ is a Banach space with the usual ℓ_p-norm defined by $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$.

For $p = 2$, ℓ_2 is the Hilbert space under the inner product defined as

$$\langle x, y\rangle = \sum_{n=0}^{\infty} x_n \bar{y}_n \ \ \forall\, x, y \in \ell_2.$$

Throughout this paper, let $\mathscr{U}$ and $\mathscr{U}_o$ be the following sets of sequences

$$\mathscr{U} = \{u = (u_k) \in w : u_k \neq 0 \text{ for all } k\} \text{ and } \mathscr{U}_0 = \{u = (u_k) \in w : u_0 \neq 0\}.$$

Let $r, t \in \mathscr{U}$ and $s \in \mathscr{U}_o$. For any sequence $x = (x_n) \in w$, we define the sequence $\bar{x} = (\bar{x}_n)$ of generalized means of x by

$$\bar{x}_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k; \quad (n \in \mathbb{N}), \tag{1}$$

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that is $\bar{x}_n = (s*tx)_n/r_n$ for all $n \in \mathbb{N}$. Further, we define the infinite matrix $\bar{A}(r,s,t)$ of generalized means by

$$(\bar{A}(r,s,t))_{nk} = \begin{cases} s_{n-k}t_k/r_n\,; & (0 \leq k \leq n), \\ 0\,; & (k > n), \end{cases} \tag{2}$$

for all $n,k \in \mathbb{N}$. Then, it follows by (1) that $\bar{x}$ is the $\bar{A}(r,s,t)$-transform of x, that is $\bar{x} = (\bar{A}(r,s,t))x$ for all $x \in w$. It is obvious by (2) that $\bar{A}(r,s,t)$ is a triangle. Moreover, it can easily be seen that $\bar{A}(r,s,t)$ is regular if and only if $s_{n-i} = o(r_n)$ for each $i \in \mathbb{N}$, $\sum_{k=0}^{n} |s_{n-k}t_k| = O(|r_n|)$ and $(s*t)_n/r_n \to 1$ $(n \to \infty)$.

Now, since $\bar{A}(r,s,t)$ is a triangle, it has a unique inverse which is also a triangle. More precisely, we put $D_0^{(s)} = 1/s_0$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}} \begin{vmatrix} s_1 & s_0 & 0 & 0 & \cdots & 0 \\ s_2 & s_1 & s_0 & 0 & \cdots & 0 \\ s_3 & s_2 & s_1 & s_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 \end{vmatrix}; \quad (n = 1,2,3,\ldots).$$

Then, the inverse of $\bar{A}(r,s,t)$ is the triangle $\bar{B} = (\bar{b}_{nk})_{n,k=0}^{\infty}$ defined by

$$\bar{b}_{nk} = \begin{cases} (-1)^{n-k} D_{n-k}^{(s)} r_k/t_n\,; & (0 \leq k \leq n), \\ 0\,; & (k > n) \end{cases} \tag{3}$$

for all $n,k \in \mathbb{N}$. Therefore, we have by (1) that

$$x_n = \frac{1}{t_n} \sum_{k=0}^{n} (-1)^{n-k} D_{n-k}^{(s)} r_k \bar{x}_k; \quad (n \in \mathbb{N}). \tag{4}$$

Recently, Mursaleen and Noman (c.f. [2]) defined the set $X(r,s,t)$ as the matrix domain of the triangle $\bar{A}(r,s,t)$ in X, that is

$$X(r,s,t) = \left\{ x = (x_k) \in w : \bar{x} = \left(\frac{1}{r_n} \sum_{k=0}^{n} s_{n-k}t_k x_k \right)_{n=0}^{\infty} \in X \right\}. \tag{5}$$

The space $\bar{\ell}_p$ $(1 \leq p < \infty)$ is a Banach space with the norm

$$\|x\|_{\bar{\ell}_p} = \|\bar{x}\|_{\ell_p} = \left(\sum_{n=0}^{\infty} \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k}t_k x_k \right|^p \right)^{1/p}.$$

For $p = 2$, $\bar{\ell}_2$ is a Hilbert space under the inner product

$$\langle x,y \rangle = \langle \Lambda x, \Lambda y \rangle$$

where $(\Lambda x)(n) = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k}t_k x_k$, $n \in \mathbb{N}$.

The set $(b_n^{(k)})_{n=0}^{\infty}$ where

$$b_n^{(k)} = \begin{cases} 0;\ (n<k), \\ \qquad\qquad\qquad (n \in \mathbb{N}) \\ (-1)^{n-k} D_{n-k}^{(s)} r_k/t_n;\ (n \geq k). \end{cases} \tag{6}$$

Quite recently in [3], authors studied bounded composition and multiplication operators and weighted composition operators (c.f. [4]) on some Hilbert sequence spaces ℓ_p^{λ} (c.f. [1]) which is an special case of $\bar{\ell}_p$. In this paper, we study such operators on $\bar{\ell}_p$.

2 Composition and multiplication operators

In this section we obtain a condition for bounded composition operators.

Theorem 1. *Let $T: N \to N$ be a mapping. Then $C_T : \ell_p \to \bar{\ell}_p, 1 \leq p < \infty$ is bounded if there exist $M > 0$ such that*

$$\sharp(T^{-1}(\{n\})) \leq M \ \ \forall \ \ n \in N.$$

Proof. For $x \in \ell_p$, consider

$$\begin{aligned}
\| C_T x \|_{\bar{\ell}_p}^p &= \sum_{n=0}^{\infty} |\frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_{T(k)}|^p \\
&\leq \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} (\frac{s_{n-k}t_k}{r_n})^{1/p} |x_{T(k)}| (\frac{s_{n-k}t_k}{r_n})^{1/q} \right]^p \\
&\leq \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} (\frac{s_{n-k}t_k}{r_n}) |x_{T(k)}|^p (\sum_{k=0}^{n} (\frac{s_{n-k}t_k}{r_n}))^{p/q} \right] \\
&\text{(by Holder's inequality)} \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\frac{s_{n-k}t_k}{r_n}) |x_{T(k)}|^p \\
&\leq \sum_{k=0}^{\infty} (s_{n-k}t_k) |x_{T(k)}|^p \sum_{n=k}^{\infty} \frac{1}{r_n} \\
&\leq L \sum_{k=0}^{\infty} |x_{T(k)}|^p, \qquad \text{where } L = \sup_k (s_{n-k}t_k) \sum_{n=k}^{\infty} \frac{1}{r_n} < \infty \\
&= L \sum_{k=0}^{\infty} \sum_{m \in T^{-1}(k)} |x_{T(m)}|^p \\
&= L \sum_{k=0}^{\infty} \sum_{m \in T^{-1}(k)} |x_k|^p \\
&\leq LM \sum_{k=0}^{\infty} |x_k|^p \\
&= LM \| x \|_p^p .
\end{aligned}$$

Therefore we conclude that C_T is a bounded operator.

Corollary 1. *If $T: N \to N$ is a constant function, then C_T is not a bounded operator on $\bar{\ell}_p$.*

Proof. Suppose $T:\mathbb{N}\to\mathbb{N}$ is a constant function. Then $T(n)=n_0\ \forall\ n\in\mathbb{N}$. Take $x\in\bar{\ell}_p$ such that $x_{n_0}\neq 0$. Then from the equality

$$\begin{aligned}\| C_T x \|_{\bar{\ell}_p}^p &= \sum_{n=0}^{\infty} |\frac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_k x_{T(k)}|^p \\ &= \sum_{n=0}^{\infty}\frac{1}{r_n^p}|\sum_{k=0}^{n} s_{n-k}t_k x_{n_0}|^p \\ &= \sum_{n=0}^{\infty}\frac{1}{r_n^p}|r_n x_{n_0}|^p \\ &= \sum_{n=0}^{\infty}|x_{n_0}|^p \\ &= \infty\end{aligned}$$

Thus C_T is not bounded operator.

Example 1. Let $T:N\to N$ be defined by $T(n)=n+1$ and $s_n=1$ for each n. Let $r_n=\sum_{k=0}^{n}t_k=n^2$ for all n. Then, for any $x\in\ell_2$,

$$\begin{aligned}\| C_T x \|_{\bar{\ell}_p}^2 &= \sum_{n=0}^{\infty}|\sum_{k=0}^{n}\frac{s_{n-k}t_k}{r_n}x_{T(k)}|^2 \\ &\leq \sum_{k=0}^{\infty}(k^2-(k-1)^2)|x_{k+1}|^2\sum_{n=k}^{\infty}\frac{1}{n^4} \\ &= \sum_{k=0}^{\infty}|x_{k+1}|^2(2k-1)\sum_{n=k}^{\infty}\frac{1}{n^4} \\ &= M\sum_{k=0}^{\infty}|x_{k+1}|^2, \quad \text{where } M=\sup_k(2k-1)\sum_{n=k}^{\infty}\frac{1}{n^4}<\infty \\ &= M \| x \|^2 .\end{aligned}$$

Hence $C_T:\ell_2\to\bar{\ell}_p$ is a bounded operator, that is, the unilateral shift operator is a bounded operator on $\bar{\ell}_p$.

In the next result we consider the multiplication operators.

Theorem 2. *Let $\theta:N\to C$ be a bounded function . Then $M_\theta:\ell_p\to\bar{\ell}_p$ is a bounded operator.*

Proof. Suppose θ is a bounded function. Then $\exists\, M>0$ such that

$$|\theta(n)|\leq M\ \ \forall\ \ n\in N.$$

For $x \in \ell_p$, consider

$$\begin{aligned}
\| M_\theta x \|_{\bar{\ell}_p}^p &= \sum_{n=0}^{\infty} | \sum_{k=0}^{n} \frac{s_{n-k} t_k}{r_n} \theta(k) x_k |^p \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{s_{n-k} t_k}{r_n} |\theta(k)|^p |x_k|^p \\
&= \sum_{k=0}^{n} t_k |\theta(k)|^p |x_k|^p \sum_{n=k}^{\infty} \frac{s_{n-k}}{r_n} \\
&\leq M^p L \sum_{k=0}^{\infty} |x_k|^p, \quad \text{where } L = \sup_k t_k \sum_{n=k}^{\infty} \frac{s_{n-k}}{r_n} < \infty \\
&= M^p L \| x \|^p .
\end{aligned}$$

Hence $\| M_\theta x \| \leq t \| x \|$ where $t = ML^{\frac{1}{p}}$ This proves that M_θ is a bounded operator.

3 Weighted composition operators

Now, we study the weighted composition operators.

Theorem 3. *Let $w : N \to C$ and $T : N \to N$ be two mappings. If there exists $M > 0$ such that*

$$\sum_{m \in T^{-1}(k)} |w(m)|^p \leq M \quad \forall \ k \in N,$$

then $M_{w,T} : \ell_p \to \bar{\ell}_p$ is a bounded operator.

Proof. For $x \in \ell_p$, consider

$$\begin{aligned}
\| M_{w,T} x \|_{\bar{\ell}_p}^p &= \sum_{n=0}^{\infty} | \sum_{k=0}^{n} \frac{s_{n-k} t_k}{r_n} w(k) x_{T(k)} |^p \\
&\leq \sum_{n=0}^{\infty} [\sum_{k=0}^{n} \frac{s_{n-k} t_k}{r_n} |(w(k)|.|x_{T(k)}|]^p \\
&= \sum_{k=0}^{\infty} t_k |w(k)|^p |x_{T(k)}|^p \sum_{n=k}^{\infty} \frac{s_{n-k}}{r_n} \\
&= \sum_{m=0}^{\infty} t_m |w(m)|^p |x_{T(m)}|^p \sum_{n=m}^{\infty} \frac{s_{n-k}}{r_n} \\
&\leq L \sum_{m=0}^{\infty} |w(m)|^p |x_{T(m)}|^p \text{ where } L = \sup_m t_m \sum_{n=m}^{\infty} \frac{s_{n-k}}{r_n} < \infty \\
&= L \sum_{m=0}^{\infty} (\sum_{k \in T^{-1}(m)} |w(k)|^p) |x_m|^p \\
&\leq ML \sum_{m=0}^{\infty} |f(m)|^p \\
&= LM \| x \| \, |_p^p.
\end{aligned}$$

Hence $\| M_{w,T} x \| \leq t \| x \|_p \quad \forall \ x \in \ell_p$, where $t = (LM)^{\frac{1}{p}}$. This proves that $M_{w,T}$ is a bounded operator.

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