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New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2016216999

Some Hermite-Hadamard-Fejer type inequalities for harmonically convex functions via fractional integral

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Received: 13 November 2015, Revised: 19 November 2015, Accepted: 29 December 2015
Published online: 30 March 2016

Abstract: In this paper, we gave the new general identity for differentiable functions. As a result of this identity some new and general inequalities for differentiable harmonically-convex functions are obtained.

Keywords: Harmoically-convex, Hermite-Hadamard-Fejer type inequality, fractional integral.

1 Introduction

The classical or the usual convexity is defined as follows,

Definition 1. *A function $f : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality*

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

A number of papers have been written on inequalities using the classical convexty and one of the most captivating inequalities in mathematical analysis is stated as follows,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \subseteq \mathbb{R} \longrightarrow$ be a convex mapping and $a, b \in I$ with $a \leq b$. Both the inequalities hold in reversed direction if f is concave. The inequalities stated in (1) are known as Hermite-Hadamard inequalities.

For more results on (1) which provide new proof, significantly extensions, generalizations, refinements, counterparts, new Hermite-Hadamard-type inequalities and numerous applications, we refer the interested reader to [2,3,5,6,8,9,12, 13,15,16] and the references there in.

The usual notion of convex function have been generalized in diverse manners. One of them is the so called harmonically s-convex functions and is stated in the definition below.

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Definition 2. [5,7] *Let $I \subset (0,\infty)$ be a real interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically s-convex(concave), if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (\geq) t^s f(y) + (1-t)^s f(x)$$

holds for all $x,y \in I$ and $t \in [0,1]$, and for some fixed $s \in (0,1]$.

It can be easily seen that for $s = 1$ in Defination 2 reduces to following Defination 3,

Definition 3. [6] *A function $f : I \subseteq \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ is said to be harmonically-convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq t f(y) + (1-t) f(x)$$

holds for all $x,y \in I$ and $t \in [0,1]$. If the inequality is reversed, then f is said to be harmonically concave.

Proposition 1. [6] *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval and $f : I \to \mathbb{R}$ is function, then:*

(i) *if $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is harmonically convex.*
(ii) *if $I \subset (0,\infty)$ and f is harmonically convex and nonincreasing function then f is convex.*
(iii) *if $I \subset (-\infty,0)$ and f is harmonically convex and nondecreasing function then f is convex.*
(iv) *if $I \subset (-\infty,0)$ and f is convex and nonincreasing function then f is harmonically convex.*

For the properties of harmonically-convex functions and harmonically-s-convex function, we refer the reader to [1,5,6,7, 8,10,11] and the reference there in.

Most recently, a number of findings have been seen on Hermite-Hadamard type integral inequalities for harmonically-convex and for harmonically-s-convex functions.

In [14], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1).

Theorem 1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{2}$$

holds, where $g : [a,b] \longrightarrow \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve, and extend the inequalities (1) and (2) see [15].

In [6], İşcan gave defination of harmonically convex functions and established following Hermite- Hadamard type inequality for harmonically convex functions as follows.

Theorem 2. [15] *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

In [11], Iscan and Wu represented Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral form as follows.

Theorem 3. [11] *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a,b \in I$ with $a < b$. If f is harmonically-convex on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{1/a^-}(f\circ h)(1/b) + J^{\alpha}_{1/b^+}(f\circ h)(1/a)\right\} \le \frac{f(a)+f(b)}{2}, \tag{4}$$

with $\alpha > 0$ and $h(x) = 1/x$.

Definition 4. *A function $g : [a,b] \subseteq \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ is said to be harmonically symmetric with respect to $2ab/a+b$ if*

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right) \tag{5}$$

holds for all $x \in [a,b]$.

Theorem 4. *In [1] Chan and Wu represented Hermite-Hadamard-Fejer inequality for harmonically convex functions as follows:*

Theorem 5. *Suppose that $f : I \subseteq \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ be harmonically-convex function and $a,b \in I$, with $a < b$. If $f \in L[a,b]$ and $g : [a,b] \subseteq \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a+b$,then*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx \le \int_a^b \frac{f(x)g(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx \tag{6}$$

In [10] İşcan and Kunt represented Hermite-Hadamard-Fejer type inequality for harmonically convex functions in fractional integral forms and established following identity as follows:

Theorem 6. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be harmonically convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \longrightarrow \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a+b$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{1/a^-}(g\circ h)(1/b) + J^{\alpha}_{1/b^+}(g\circ h)(1/a)\right] &\le \left[J^{\alpha}_{1/a^-}(fg\circ h)(1/b) + J^{\alpha}_{1/b^+}(fg\circ h)(1/a)\right] \\ &\le \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/a^-}(g\circ h)(1/b) + J^{\alpha}_{1/b^+}(g\circ h)(1/a)\right] \end{aligned} \tag{7}$$

with $\alpha > 0$ and $h(x) = 1/x$, $x \in \left[\frac{1}{b},\frac{1}{a}\right]$.

Definition 5. *Let $f \in L[a,b]$. The right-hand side and left-hand side Hadamard fractional integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x > a$$

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ x < b$$

respectively where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}$ and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$

Lemma 1. *For $0 < \theta \le 1$ and $0 < a \le b$ we have*

$$\left|a^{\theta} - b^{\theta}\right| \le (b-a)^{\theta}.$$

In [4] D. Y. Hwang found out a new identity and by using this identity, established a new inequalities. Then in [12] İ. İşcan and S. Turhan used this identity for GA-convex functions and obtain generalized new inequalities. In this paper, we established a new inequality similar to inequality in [12] and then we obtained some new and general integral inequalities for differentiable harmonically-convex functions using this lemma. The following sections, let the notion, $L(t)=\frac{aH}{tH+(1-t)a}$, $U(t)=\frac{bH}{tH+(1-t)b}$ and $H=H(a,b)=\frac{2ab}{a+b}$.

2 Main result

Throughout this section, let $\|g\|_{\infty}=\sup_{t\in[a,b]}|g(x)|$, for the continuous function $g:[a,b]\longrightarrow[0,\infty)$ be differentiable mapping I^{o}, where $a,b\in I$ with $a\leq b$, and $h:[a,b]\longrightarrow[0,\infty)$ be differentiable mapping.

Lemma 2. *If $f'\in L[a,b]$then the following inequality holds:*

$$[h(b)-2h(a)]\frac{f(a)}{2}+h(b)\frac{f(b)}{2}-\int_{a}^{b}f(x)h'(x)dx \tag{8}$$
$$=\frac{b-a}{4ab}\left\{\int_{0}^{1}[2h(L(t))-h(b)]f'(L(t))(L(t))^{2}dt+\int_{0}^{1}[2h(U(t))-h(b)]f'(U(t))(U(t))^{2}dt\right\}.$$

Proof. By the integration by parts, we have

$$I_{1}=\int_{0}^{1}[2h(L(t))-h(b)]d(f(L(t)))=[2h(L(t))-h(b)]f(L(t))|_{0}^{1}-\left(\frac{1}{a}-\frac{1}{b}\right)\int_{0}^{1}f(L(t))h'(L(t))(L(t))^{2}dt$$

and

$$I_{2}=\int_{0}^{1}[2h(U(t))-h(b)]d(f(U(t)))=[2h(U(t))-h(b)]f(U(t))|_{0}^{1}-\left(\frac{1}{a}-\frac{1}{b}\right)\int_{0}^{1}f(U(t))h'(U(t))(U(t))^{2}dt.$$

Therefore

$$\frac{I_{1}+I_{2}}{2}=[h(b)-2h(a)]\frac{f(a)}{2}+h(b)\frac{f(b)}{2}-\frac{b-a}{2ab}\left\{\int_{0}^{1}f(L(t))h'(L(t))(L(t))^{2}dt+\int_{0}^{1}f(U(t))h'(U(t))(U(t))^{2}dt\right\}. \tag{9}$$

This complete the proof.

Lemma 3. *For $a,H,b>0$, we have*

$$\zeta_{1}(a,b)=\int_{0}^{1}|2h(L(t))-h(b)|(1-t)(L(t))^{2}dt \tag{10}$$

$$\zeta_{2}(a,b)=\int_{0}^{1}t(L(t))^{2}|2h(L(t))-h(b)|dt+\int_{0}^{1}t((U(t))^{2}|2h(U(t))-h(b)|dt \tag{11}$$

$$\zeta_{3}(a,b)=\int_{0}^{1}|2h(U(t))-h(b)|(1-t)(U(t))^{2}dt. \tag{12}$$

Theorem 7. *Let $f : I \subseteq \mathbb{R} = (0,\infty) \longrightarrow \mathbb{R}$ be differentiable mapping I^o, where $a,b \in I$ with $a<b$. If the mapping $|f'|$ is harmonically-convex on $[a,b]$, then the following inequality holds:*

$$\left|[h(b)-2h(a)]\frac{f(a)}{2}+h(b)\frac{f(b)}{2}-\int_a^b f(x)h'(x)dx\right| \le \frac{b-a}{4ab}\left[\zeta_1(a,b)\left|f'(a)\right|+\zeta_2(a,b)\left|f'(H)\right|+\zeta_3(a,b)\left|f'(b)\right|\right] \quad (13)$$

where $\zeta_1(a,b), \zeta_2(a,b), \zeta_3(a,b)$ are defined in Lemma 3.

Proof. Continuing equality (8) in Lemma 2

$$\left|[h(b)-2h(a)]\frac{f(a)}{2}+h(b)\frac{f(b)}{2}-\int_a^b f(x)h'(x)dx\right| \quad (14)$$
$$\le \frac{b-a}{4ab}\left\{\int_0^1 |2h(L(t))-h(b)|\left|f'(L(t))(L(t))^2\right|dt+\int_0^1 |2h(U(t))-h(b)|\left|f'(U(t))(U(t))^2\right|dt\right\}.$$

Using $|f'|$ is harmoncally-convex in (14).

$$\left|[h(b)-2h(a)]\frac{f(a)}{2}+h(b)\frac{f(b)}{2}-\int_a^b f(x)h'(x)dx\right| \le \frac{b-a}{4ab}\left\{\int_0^1 |2h(L(t))-h(b)|\left\{t\left|f'(H)\right|+(1-t)\left|f'(a)\right|\right\}(L(t))^2dt\right.$$
$$\left.+\int_0^1 |2h(U(t))-h(b)|\left\{t\left|f'(H)\right|+(1-t)\left|f'(b)\right|\right\}(U(t))^2dt\right\}, \quad (15)$$

by (15) and Lemma 2, this proof is complete.

Corollary 1. *Let $h(t)=\int_{1/t}^{1/a}\left[\left(x-\frac{1}{b}\right)^{\alpha-1}+\left(\frac{1}{a}-x\right)^{\alpha-1}\right]g\circ\varphi(x)dx$ for all $1/t\in\left[\frac{1}{b},\frac{1}{a}\right]$, $\alpha>0$ and $g:[a,b]\longrightarrow[0,\infty)$ be continuous positive mapping and symmetric to $\frac{2ab}{a+b}$ in Teorem 7, we obtain:*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b^+}g\circ\varphi(1/a)+J^{\alpha}_{1/a^-}g\circ\varphi(1/b)\right]-\left[J^{\alpha}_{1/b^+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(fg\circ\varphi)(1/b)\right]\right| \quad (16)$$
$$\le \frac{(b-a)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1}(ab)^{\alpha+1}\Gamma(\alpha+1)}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(H)\right|+C_3(\alpha)\left|f'(b)\right|\right]$$

where

$$C_1(\alpha)=\int_0^1(1-t)\left[(1+t)^\alpha-(1-t)^\alpha\right](L(t))^2dt$$

$$C_2(\alpha)=\int_0^1 t\left[(1+t)^\alpha-(1-t)^\alpha\right]\left[(L(t))^2+(U(t))^2\right]dt$$

$$C_3(\alpha)=\int_0^1(1-t)\left[(1+t)^\alpha-(1-t)^\alpha\right](L(t))^2dt.$$

Specially in (16) and using Lemma 1, for $0<\alpha\leq 1$ *we have:*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b+}g\circ\varphi(1/a)+J^{\alpha}_{1/a-}g\circ\varphi(1/b)\right]-\left[J^{\alpha}_{1/b+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a-}(fg\circ\varphi)(1/b)\right]\right| \tag{17}$$
$$\leq\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{2(ab)^{\alpha+1}\Gamma(\alpha+1)}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(H)\right|+C_3(\alpha)\left|f'(b)\right|\right]$$

where

$$C_1(\alpha)=\int_0^1(1-t)t^{\alpha}(L(t))^2dt,\ C_2(\alpha)=\int_0^1 t^{\alpha+1}\left[(L(t))^2+(U(t))^2\right]dt,\ C_3(\alpha)=\int_0^1(1-t)t^{\alpha}(U(t))^2dt.$$

Proof. By left side of inequality (15) in Teorem 7, when we write $h(t)=\int_{1/t}^{1/a}\left[\left(x-\frac{1}{b}\right)^{\alpha-1}+\left(\frac{1}{a}-x\right)^{\alpha-1}\right]g\circ\varphi(x)dx$ for all $x\in[1/b,1/a]$ and $\varphi(x)=1/x$, we have

$$\left|\Gamma(\alpha)\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b+}g\circ\varphi(1/a)+J^{\alpha}_{1/a-}g\circ\varphi(1/b)\right]-\Gamma(\alpha)\left[J^{\alpha}_{1/b+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a-}(fg\circ\varphi)(1/b)\right]\right|.$$

On the other hand, right side of inequality (15), with

$$\Psi(x,a,b)=\left(x-\frac{1}{b}\right)^{\alpha-1}+\left(\frac{1}{a}-x\right)^{\alpha-1} \tag{18}$$
$$\leq\frac{b-a}{4ab}\left\{\int_0^1\left|2\int_{1/L(t)}^{1/a}[\Psi(x,a,b)]\,g\circ\varphi(x)dx-\int_{1/b}^{1/a}[\Psi(x,a,b)]\,g\circ\varphi(x)dx\right|\left\{t\left|f'(H)\right|+(1-t)\left|f'(a)\right|\right\}(L(t))^2dt\right.$$
$$\left.+\int_0^1\left|2\int_{1/U(t)}^{1/a}[\Psi(x,a,b)]\,g\circ\varphi(x)dx-\int_{1/b}^{1/a}[\Psi(x,a,b)]\,g\circ\varphi(x)dx\right|\left\{t\left|f'(H)\right|+(1-t)\left|f'(b)\right|\right\}(U(t))^2dt\right\}.$$

Since $g(x)$ is symmetric to $x=\frac{2ab}{a+b}$, we have

$$\left|2\int_{1/L(t)}^{1/a}[\Psi(x,a,b)]\,g\circ\varphi(x)dx-\int_{1/b}^{1/a}[\Psi(x,a,b)]\,(g\circ\varphi)(x)dx\right|=\left|\int_{1/U(t)}^{1/L(t)}[\Psi(x,a,b)]\,(g\circ\varphi)(x)dx\right| \tag{19}$$

and

$$\left|2\int_{1/U(t)}^{1/a}[\Psi(x,a,b)]\,g\circ\varphi(x)dx-\int_{1/b}^{1/a}[\Psi(x,a,b)]\,(g\circ\varphi)(x)dx\right|=\left|\int_{1/U(t)}^{1/L(t)}[\Psi(x,a,b)]\,(g\circ\varphi)(x)dx\right| \tag{20}$$

for all $t\in[0,1]$. By (18)- (20), we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b+}g\circ\varphi(1/a)+J^{\alpha}_{1/a-}g\circ\varphi(1/b)\right]-\left[J^{\alpha}_{1/b+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a-}(fg\circ\varphi)(1/b)\right]\right| \tag{21}$$
$$\leq\frac{b-a}{4ab\Gamma(\alpha)}\left\{\int_0^1\left|\left[\int_{1/U(t)}^{1/L(t)}\Psi(x,a,b)\right]g\circ\varphi(x)dx\right|\left\{t\left|f'(H)\right|+(1-t)\left|f'(a)\right|\right\}(L(t))^2dt\right.$$

$$+\int_0^1 \left| \int_{1/U(t)}^{1/L(t)} [\Psi(x,a,b)]\, g\circ\varphi(x)dx \right| \left\{t\left|f'(H)\right|+(1-t)\left|f'(b)\right|\right\}(U(t))^2dt \Bigg\}$$

$$\leq \frac{(b-a)\|g\|_\infty}{4ab\Gamma(\alpha)} \left\{ \int_0^1 \left[\int_{1/U(t)}^{1/L(t)} |\Psi(x,a,b)|\,dx \right] \left\{t\left|f'(H)\right|+(1-t)\left|f'(a)\right|\right\}(L(t))^2dt \right.$$

$$\left. + \int_0^1 \left[\int_{1/U(t)}^{1/L(t)} |\Psi(x,a,b)|\,dx \right] \left\{t\left|f'(H)\right|+(1-t)\left|f'(b)\right|\right\}(U(t))^2dt \right\}.$$

In the last inequality,

$$\int_{1/U(t)}^{1/L(t)} |\Psi(x,a,b)|\,dx = \int_{1/U(t)}^{1/L(t)} \left(x-\frac{1}{b}\right)^{\alpha-1} dx + \int_{1/U(t)}^{1/L(t)} \left(\frac{1}{a}-x\right)^{\alpha-1} dx = \frac{2^{1-\alpha}}{\alpha}\left(\frac{b-a}{ab}\right)^{\alpha} \left\{(1+t)^{\alpha}-(1-t)^{\alpha}\right\}. \quad (22)$$

By Lemma 1, we have

$$\int_{1/U(t)}^{1/L(t)} |\Psi(x,a,b)|\,dx = \int_{1/U(t)}^{1/L(t)} \left(x-\frac{1}{b}\right)^{\alpha-1} dx + \int_{1/U(t)}^{1/L(t)} \left(\frac{1}{a}-x\right)^{\alpha-1} dx \leq \frac{2}{\alpha}\left(\frac{b-a}{ab}\right)^{\alpha} t^{\alpha}$$

A combination of (21) and (22), we have (16). This complete is proof.

Corollary 2. *In Corollary 1,*

(i) *If $\alpha=1$ is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (17):*

$$\left| \left[\frac{f(a)+f(b)}{2}\right] \int_a^b \frac{g(x)}{x^2}dx - \int_a^b f(x)\frac{g(x)}{x^2}dx \right| \leq \frac{(b-a)^2}{4(ab)^2}\|g\|_\infty \left[C_1(1)\left|f'(a)\right|+C_2(1)\left|f'(H)\right|+C_3(1)\left|f'(b)\right|\right] \quad (23)$$

where for $a,b,H>0$, we have

$$C_1(1)=\int_0^1 (1-t)t(L(t))^2dt$$

$$C_2(1)=\int_0^1 t^2\left[(L(t))^2+(U(t))^2\right]dt$$

$$C_3(1)=\int_0^1 (1-t)t(U(t))^2dt$$

(ii) *If $g(x)=1$ is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (16):*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)-\frac{(ab)^{\alpha}\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{1/b+}(f\circ\varphi)(1/a)+J^{\alpha}_{1/a-}(f\circ\varphi)(1/b)\right]\right| \le \frac{(b-a)}{2^{\alpha+2}ab}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(H)\right|+C_3(\alpha)\left|f'(b)\right|\right]. \tag{24}$$

(iii) *If $g(x)=1$ and $\alpha=1$ is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (17):*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{(b-a)}\int_a^b\frac{f(x)}{x^2}dx\right|\le\frac{(b-a)}{4(ab)}\left[C_1(1)\left|f'(a)\right|+C_2(1)\left|f'(H)\right|+C_3(1)\left|f'(b)\right|\right]. \tag{25}$$

Theorem 8. *Let $f:I\subseteq\mathbb{R}\backslash\{0\}\longrightarrow\mathbb{R}$ be differentiable mapping I^o, where $a,b\in I$ with $a<b$. If the mapping $|f'|^q$ is harmonically-convex on $[a,b]$, then the following inequality holds:*

$$\left|[h(b)-2h(a)]\frac{f(a)}{2}+h(b)\frac{f(b)}{2}-\int_a^b f(x)h'(x)dx\right|\le\frac{b-a}{4ab}\left\{\eta_1^{1-\frac{1}{q}}\times\eta_2^{\frac{1}{q}}+\eta_3^{1-\frac{1}{q}}\times\eta_4^{\frac{1}{q}}\right\} \tag{26}$$

where

$$\eta_1=\left(\int_0^1\left|2h(L(t))-h(b)\right|dt\right),$$
$$\eta_2=\left(\int_0^1(\left|2h(L(t))-h(b)\right|dt)\times\left(t\,(L(t))^{2q}\left|f'(a)\right|^q+(1-t)\,(L(t))^{2q}\left|f'(H)\right|^q\right)\right),$$
$$\eta_3=\left(\int_0^1\left|2h(U(t))-h(b)\right|dt\right),$$
$$\eta_4=\left(\int_0^1(\left|2h(U(t))-h(b)\right|dt)\times\left(t\,(U(t))^{2q}\left|f'(a)\right|^q+(1-t)\,(U(t))^{2q}\left|f'(H)\right|^q\right)\right).$$

Proof. Continuing from (14) in Theorem 7, we use Hölder Inequality and we use that $|f'|^q$ is harmonically-convex. Thus this proof is complete.

Corollary 3. *Let $h(t)=\int_{1/t}^{1/a}\left[\left(x-\frac{1}{b}\right)^{\alpha-1}+\left(\frac{1}{a}-x\right)^{\alpha-1}\right](g\circ\varphi)(x)dx$ for all $t\in[a,b]$ and $g:[a,b]\longrightarrow[0,\infty)$ be continuous positive mapping and symmetric to $\frac{2ab}{a+b}$ in Teorem 8, we obtain:*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b+}(g\circ\varphi)(1/a)+J^{\alpha}_{1/a-}(g\circ\varphi)(1/b)\right]-\left[J^{\alpha}_{1/b+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a-}(fg\circ\varphi)(1/b)\right]\right| \le\frac{(b-a)^{\alpha+1}\left\|g\right\|_{\infty}}{2^{\alpha+1}(ab)^{\alpha+1}\Gamma(\alpha+1)}\left(\frac{2^2(2^{\alpha}-1)}{\alpha+1}\right)^{1-\frac{1}{q}}\left[C_1(\alpha,q)\left|f'(a)\right|^q+C_2(\alpha,q)\left|f'(H)\right|^q+C_3(\alpha,q)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \tag{27}$$

where for $q>1$

$$C_1(\alpha,q)=\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]t\,(L(t))^{2q}\,dt$$
$$C_2(\alpha,q)=\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right](1-t)\left((L(t))^{2q}+(U(t))^{2q}\right)dt$$
$$C_3(\alpha,q)=\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]t\,(U(t))^{2q}\,dt.$$

Proof. Continuing from (22) of Corollary 1 and (26) in Theorem 8,

$$\left|\left(\frac{f(a)+f(b)}{2}\right)[\zeta_1]-[\zeta_2]\right|\le\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}\{\ell_1\times\ell_2+\ell_1\times\ell_3\}$$
$$\le\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{2^{\alpha+1}(ab)^{\alpha+1}\Gamma(\alpha+1)}(\zeta_0)^{1-\frac{1}{q}}[\ell_2+\ell_3] \quad (28)$$

where

$$\zeta_0=\frac{2^{\alpha+1}-2}{\alpha+1},$$
$$\zeta_1=J^{\alpha}_{a^+}g(b)+J^{\alpha}_{b^-}g(a),$$
$$\zeta_2=J^{\alpha}_{a^+}(fg)(b)+J^{\alpha}_{b^-}(fg)(a),$$
$$\ell_1=\left(\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]dt\right)^{1-\frac{1}{q}},$$
$$\ell_2=\left(\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]\left(t\,(L(t))^{2q}\left|f'(a)\right|^q+(1-t)\,(L(t))^{2q}\left|f'(H)\right|^q\right)dt\right)^{\frac{1}{q}},$$
$$\ell_3=\left(\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]\left(t\,(U(t))^{2q}\left|f'(b)\right|^q+(1-t)\,(U(t))^{2q}\left|f'(H)\right|^q\right)dt\right)^{\frac{1}{q}},$$

By the power-mean inequality $\left(a^r+b^r<2^{1-r}(a+b)^r\, for \quad a>0,b>0, \quad r<1\right)$ and $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{2^{\alpha+1}(ab)^{\alpha+1}\Gamma(\alpha+1)}(\zeta_0)^{1-\frac{1}{q}}[\ell_4+\ell_5]\le\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{2^{\alpha+1}(ab)^{\alpha+1}\Gamma(\alpha+1)}\left(\frac{2^2(2^{\alpha}-1)}{\alpha+1}\right)^{\frac{1}{p}}\left[\int_0^1(\xi_1+\xi_2+\xi_3)\,dt\right]^{\frac{1}{q}}, \quad (29)$$

where

$$\xi_1=\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]t\,(L(t))^{2q}\left|f'(a)\right|^q,$$
$$\xi_2=\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right](1-t)\left((L(t))^{2q}+(U(t))^{2q}\right)\left|f'(H)\right|^q,$$
$$\xi_3=\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]t\,(U(t))^{2q}\left|f'(b)\right|^q.$$

Corollary 4. *When $\alpha = 1$ and $g(x) = 1$ is taken in Corollary 3, we obtain:*

$$\left| \left(\frac{f(a)+f(b)}{2} \right) - \frac{ab}{(b-a)} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{(b-a)}{2^{2+\frac{1}{q}}(ab)} \left[C_1(1,q) \left| f'(a) \right|^q + C_2(1,q) \left| f'(H) \right|^q + C_3(1,q) \left| f'(b) \right|^q \right]^{\frac{1}{q}}. \quad (30)$$

This proof is complete.

References

[1] F. Chen and S. Wu, Hermite-Hadamard type inequalities for harmonically *s*-convex functions, Sci. World (2014), 7, Article ID 279158.

[2] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, Ann. Funct. Anal. 1(1) (2010), pp. 51-58.

[3] S. S. Dragomir, Hermite Hadamard' s type inequalities for convex functions of selfadjoint operators in Hilbert spaces, Linear Algebra Appl. 436 (2012), no.5, 1503-1515.

[4] D-Y. Hwang, Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables, Applied Mathematics and Computation, 217 (2011), 9598-9605.

[5] I. Iscan, M. Kunt, Fejér and Hermite-Hadamard-Fejér type inequalities for harmonically *s*-convex functions via Fractional Integrals, The Australian Journal of Mathematical Analysis and Applications, (2015), Vol: 12, 1 ,Article 10, pp 1-6.

[6] I. Iscan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and Statistics 43 (6) (2014), 935-942.

[7] I. Iscan, Ostrowski type inequalities for harmonically *s*-convex functions, Konuralp Jurnal of Mathematics, Volume 3, No 1 (2015), pp. 63-74..

[8] I. Iscan, Some new general integral inequalities for h-convex and h-concave functions, Adv. Pure Appl. Math. 5(1) (2014), pp. 21-29 . doi: 10.1515/apam-2013-0029.

[9] I. Iscan, Generalization of different type integral inequalitiesfor s-convex functions via fractional integrals, Applicable Analysis, 2013. doi: 10.1080/00036811.2013.851785.

[10] I. Iscan, M. Kunt, Hermite-Hadamard-Fejer type inequalities for harmonically convex functions via fractional integrals, RGMIA Research Report Collection, 18(2015), Article 107, pp. 1-16.

[11] I. Iscan, S. Wu Hermite-Hadamard type inequalities for harmonically-convex functions via fractional integrals, Applied Mathematics and Computation, 238 (2014), 237–244.

[12] İ. İşcan, S. Turhan, Generalized Hermite-Hadamard-Fejer type inequalities for GA-convex functions via Fractional integral, arXiv:1511.03308v1 [math.CA], 10 Nov 2015.

[13] M.A. Latif, New Hermite Hadamard type integral inequalities for GA-convex functions with applications. Volume 34, Issue 4 (Nov 2014).

[14] L. Fejér, Uberdie Fourierreihen, II, Math. Naturwise. Anz. Ungar. Akad. , Wiss, 24 (1906), pp. 369-390, (in Hungarian)

[15] M. Z. Sarıkaya, On new Hermite Hadamard Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai Math., 57(3) (2012), pp. 377–386.

[16] M. Z. Sarıkaya, E. Set, H. Yaldız and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, 57(9) (2013), pp. 2403-2407.