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Müntz-legendre matrix method to solve the delay fredholm integro-differential equations with constant coefficients

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Abstract: In this study, we present the Müntz-Legendre matrix method to solve the linear delay Fredholm integro-differential equations with constant coeffcients. By using this method, we obtain the approximate solutions in form of the Müntz-Legendre polynomials. The method reduces the problem to a system of the algebraic equations by means of the required matrix relations of the solution form. By solving this system, the approximate solution is obtained. Also, an error estimation scheme based the residual function is presented for the method and the approximate solutions are improved by this error estimation. Finally, the method will be illustrated on the examples.

Keywords: Linear delay Fredholm integro-differential equation, Müntz-Legendre polynomials, residual function.

1 Introduction

In this study, for the linear delay Fredholm integro-differential equations

$$\sum_{k=0}^{m} P_k y^{(k)}(x-\tau_k) = g(x) + \int_0^1 \sum_{s=0}^{m} K_s(x,t) y^{(s)}(x-\gamma_s)dt, \quad 0 \le x,t \le 1 \tag{1}$$

under the boundary conditions

$$\sum_{k=0}^{m-1} \left(a_{jk} y^{(k)}(0) + b_{jk} y^{(k)}(1) \right) = \lambda_j, j = 0,1,...,m-1, \tag{2}$$

the approximate solution based on the Müntz-Legendre polynomials will be obtained in the form

$$y_N(x) = \sum_{n=0}^{N} a_n L_n(x). \tag{3}$$

Here,$y^{(0)}(x) = y(x)$ is the unknown function, and $g(x)$ is the function defined on interval $0 \le x \le 1$ which can be expanded to Maclaurin series and P_k, a_{jk}, b_{jk},λ_j ,γ_s,τ_k are real constants, a_n $(n = 0,1,2,...,N)$ is the unknown Müntz-Legendre

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coefficients; N is any positive integer and $L_n(x)$, $(n = 0,1,2,...)$ denote the Müntz-Legendre polynomials [6] defined by

$$L_n(x) = \sum_{j=n}^{N} (-1)^{N-j} \binom{N+1+j}{N-n} \binom{N-n}{N-j} x^j, \ 0 \le x \le 1. \tag{4}$$

Also an error problem is constructed by the residual error function and the Müntz-Legendre polynomials of this problem are computed and thus the error function is estimated by these solutions. And then, the approximate solutions are improved by summing the Müntz-Legendre polynomial solutions and the estimated error function. We note that Fredholm integro-differential-difference equations are solved numerically by using different methods in studies [1-8].

2 Fundamental matrix relations

Let us consider the equation (1) and find the matrix forms of each term in the equation. For this purpose let us write the matrix form of the differential part on the left hand side of the equation. First we can write the approximate solution (3) in the matrix form as,

$$y(x) = L(x)A \tag{5}$$

where

$$L(x) = [L_0(x)\ L_1(x)\ \dots\ L_N(x)] \text{ and } A = [a_0\ a_1\ \dots\ a_N]^T.$$

Here, the matrix $L(x)$ can be written as

$$L(x) = X(x)F^T \tag{6}$$

so that $X(x) = \left[1\ x\ \dots x^N\right]$ and

$$F = \begin{bmatrix} (-1)^N\binom{N+1}{N} & (-1)^{N-1}\binom{N+2}{N}\binom{N}{N-1} & (-1)^{N-2}\binom{N+3}{N}\binom{N}{N-2} & \cdots & (-1)^1\binom{2N}{N}\binom{N}{1} & (-1)^0\binom{2N+1}{N} \\ 0 & (-1)^{N-1}\binom{N+2}{N-1} & (-1)^{N-2}\binom{N+3}{N-1}\binom{N-1}{N-2} & \cdots & (-1)^1\binom{2N}{N-1}\binom{N-1}{1} & (-1)^0\binom{2N+1}{N-1} \\ 0 & 0 & (-1)^{N-2}\binom{N+3}{N-2} & \cdots & (-1)^1\binom{2N}{N-2}\binom{N-2}{1} & (-1)^0\binom{2N+1}{N-2} \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & (-1)^1\binom{2N}{1} & (-1)^0\binom{2N+1}{1} \\ 0 & 0 & 0 & \cdots & 0 & (-1)^0\binom{2N+1}{0} \end{bmatrix}$$

By putting Eq.(6) into Eq.(5), we have the matrix form

$$y(x) = X(x)F^T A. \tag{7}$$

The kth-order derivative of Eq.(7) is given by

$$y^{(k)}(x) = X(x)B^k F^T A \tag{8}$$

where

$$B=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By placing $x \to x-\gamma_s$ in Eq. (8), we obtain the matrix form

$$y^{(k)}(x-\gamma_s)=X(x)B(-\gamma_s)B^kF^TA \tag{9}$$

where

$$X(x-\gamma_s)=X(x)B(-\gamma_s)$$

and for $\gamma_s \neq 0$:

$$B(-\gamma_s)=\begin{bmatrix} \binom{0}{0}(-\gamma_s)^0 & \binom{1}{0}(-\gamma_s)^1 & \binom{2}{0}(-\gamma_s)^2 & \cdots & \binom{N}{0}(-\gamma_s)^N \\ 0 & \binom{1}{1}(-\gamma_s)^0 & \binom{2}{1}(-\gamma_s)^1 & \cdots & \binom{N}{1}(-\gamma_s)^{N-1} \\ 0 & 0 & \binom{2}{2}(-\gamma_s)^0 & \cdots & \binom{N}{2}(-\gamma_s)^{N-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}(-\gamma_s)^0 \end{bmatrix}$$

and for $\gamma_s = 0$:

$$B(0)=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N+1)\times(N+1)}.$$

Now let us construct the matrix form of the integral part on the right hand side of the equation. The kernel function $K_s(x,t)$ can be approximated by the truncated Taylor series and the truncated Müntz-Legendre series,

$$K_s(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N} {}^Tk_{mn}^s x^m t^n \quad \text{and} \quad K_s(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N} {}^Lk_{mn}^s L_m(x)L_n(t) \tag{10}$$

where

$${}^Tk_{mn}^s=\frac{1}{m!n!}\frac{\partial^{m+n}K_s(0,0)}{\partial x^m \partial t^n}, \; m,n=0,1,2,...,N, \; s=0,1,2,...,m$$

We write the expressions in (10) in the form

$$K_s(x,t)=X(x)K_T^sX^T(t), \;\; K_T^s=[{}^Tk_{mn}^s] \tag{11}$$

and

$$K_s(x,t)=L(x)K_L^sL^T(t), \;\; K_L^s=[{}^Lk_{mn}^s]. \tag{12}$$

From equations (6), (11) and (12),

$$X(x)K_T^sX^T(t) = L(x)K_L^sL^T(t) \Rightarrow X(x)K_T^sX^T(t) = X(x)F^TK_L^sFX^T(t)$$

$$K_T^s = F^TK_L^sF \text{ or } K_L^s = (F^T)^{-1}K_T^sF^{-1} \tag{13}$$

By writing the matrix forms (9) and (13) into the integral part in the equation, we have the matrix relation we have

$$\begin{aligned}\int_0^1 \sum_{s=0}^m K_s(x,t)y^{(s)}(x-\gamma_s)dt &= \int_0^1 \sum_{s=0}^m L(x)K_L^sL^T(t)X(t)B(-\gamma_s)B^sF^TAdt \\ &= \sum_{s=0}^m \int_0^1 L(x)K_L^sL^T(t)X(t)B(-\gamma_s)B^sF^TAdt\end{aligned}$$

$$= \sum_{s=0}^m L(x)K_L^sQ_sA \tag{14}$$

where

$$\begin{aligned}Q_s &= \int_0^1 L^T(t)X(t)B(-\gamma_s)B^sF^Tdt \\ &= \int_0^1 FX^T(t)X(t)B(-\gamma_s)B^sF^Tdt \\ &= FHB(-\gamma_s)B^sF^T;\end{aligned}$$

$$H = \int_0^1 X^T(t)X(t)dt = [h_{rs}];\ h_{rs} = \frac{1}{r+s+1},\ r,s = 0,1,2,...,N.$$

We put the matrix form (6) into the equation (14) we have the matrix relation,

$$\int_0^1 \sum_{s=0}^m K_s(x,t)y^{(s)}(x-\gamma_s)dt = \sum_{s=0}^m X(x)F^TK_L^sQ_sA.$$

On the other hand by using the Maclaurin expansion, the matrix form of the function $g(x)$ can be written as

$$g(x) = X(x)G_T \tag{15}$$

where

$$g(x) = \sum_{k=0}^N \frac{g^{(k)}(0)}{k!}x^k = X(x)G_T,\ G_T = \left[\begin{array}{cccc}\frac{g^{(0)}(0)}{0!} & \frac{g^{(1)}(0)}{1!} & \dots & \frac{g^{(N)}(0)}{N!}\end{array}\right]^T.$$

Also we can obtain the matrix relations for conditions by means of the relation (8)

$$\sum_{k=0}^{m-1}\left[a_{jk}X(0)+b_{jk}X(1)\right]B^kF^TA = [\lambda_j],\ j = 0,1,...,m-1.$$

3 Method of solution

We are now ready to construct the fundamental matrix equation [8,9] for the Eq.(1). For this purpose if we substitute the relations (9) and (14) into the Eq.(1) and simplify the equation, we obtain the matrix equation

$$\left\{\sum_{k=0}^{m} P_k B(-\tau_k) B^k F^T - \sum_{s=0}^{m} F^T K_L^s Q_s\right\} A = G_T. \tag{16}$$

This equation can be written briefly

$$WA = G \quad \text{or} \quad [W;G] \tag{17}$$

where

$$W = [w_{p,q}] = \sum_{k=0}^{m} P_k B(-\tau_k) B^k F^T - \sum_{s=0}^{m} F^T K_L^s Q_s, \qquad p,q = 0,1,2,...,N.,$$

Here the matrix equation (17) of Eq. (1) corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown Müntz-Legendre coefficients $a_0, a_1, \ldots, a_N$. By using the relation (8), the matrix form of the conditions (2) becomes

$$U_j A = [\lambda_j] \quad \text{or} \quad [U_j, \lambda_j], \quad j = 0,1,...,m-1 \tag{18}$$

where

$$U_j = [u_0\ u_1\ ...\ u_N] = \left[\sum_{k=0}^{m-1} \big(a_{jk} X(0) + b_{jk} X(1)\big)\right] B^k F^T = [\lambda_j], \ \ j = 0,1,...,m-1.$$

To obtain the solution of Eq. (1) under the conditions (2), by replacing the last m rows of matrix (17) by the m row matrices (18) we have the new augmented matrix

$$\tilde{W} A = \tilde{G} \text{ or } \left[\tilde{W};\tilde{G}\right].$$

If rank$\tilde{W}$ = rank$[\tilde{W};\tilde{G}] = N+1$, the unknown coefficients matrix **A** becomes

$$A = (\tilde{W})^{-1} \tilde{G}.$$

Thus, the Müntz-Legendre coefficients matrix A is uniquely determined. Finally, by substituting the determined coefficients $a_0, a_1, \ldots, a_N$ into Eq.(3), we get the Müntz-Legendre polynomial solution

$$y_N(x) = \sum_{n=0}^{N} a_n L_n(x). \tag{19}$$

4 Error estimation and improved approximate solutions

In this section, we develop an error estimation for the Müntz-Legendre approximate solution for the problem by means of the residual correction method [9,11] and we improve the approximate solution (19) by using this error estimation. The residual error estimation was presented for the Bessel approximate solutions of the system of the linear multi-pantograph equations [12]. For the problem (1)-(2), we modify the error estimation considered in [9-12].

Let us call $e_N(x) = y(x) - y_N(x)$ as the error function of the Müntz-Legendre approximation $y_N(x)$ to $y(x)$, where $y(x)$ is the exact solution of problem (1)-(2). Hence, $y_N(x)$ satisfies the following problem:

$$\sum_{k=0}^{m} P_k y_N^{(k)}(x-\tau_k) - \int_0^1 \sum_{s=0}^{m} K_s(x,t) y_N^{(s)}(x-\gamma_s)dt = g(x) + R_N(x), \qquad 0 \leq x,t \leq 1 \tag{20}$$

$$\sum_{k=0}^{m-1} \left(a_{jk} y_N^{(k)}(0) + b_{jk} y_N^{(k)}(1) \right) = \lambda_j, j = 0,1,...,m-1. \tag{21}$$

can be obtained by substituting $y_N(x)$ into the Eq. (1) and in here $R_N(x)$ is the residual function associated with $y_N(x)$.

By using the method defined in Section 3, we purpose to find an approximation $e_{N,M}(x)$ to the $e_N(x)$.

Subtracting (20) and (21) from (1) and (2), respectively, the error function $e_N(x)$ satisfy the equation

$$\sum_{k=0}^{m} P_k e_N^{(k)}(x-\tau_k) - \int_0^1 \sum_{s=0}^{m} K_s(x,t) e_N^{(s)}(x-\gamma_s)dt = -R_N(x) \quad 0 \leq x,t \leq 1 \tag{22}$$

with the homogeneous conditions

$$\sum_{k=0}^{m-1} \left(a_{jk} e_N^{(k)}(0) + b_{jk} e_N^{(k)}(1) \right) = 0, j = 0,1,...,m-1. \tag{23}$$

Solving the error problem (22)-(23) by our method, we obtain the approximation $e_{N,M}(x)$ to $e_N(x)$.

Consequently, we have the improved approximate solution

$$y_{N,M}(x) = y_N(x) + e_{N,M}(x).$$

Note that if the exact solution of the problem is not known, then we can estimate the error function by $e_{N,M}(x)$.

5 Numerical examples

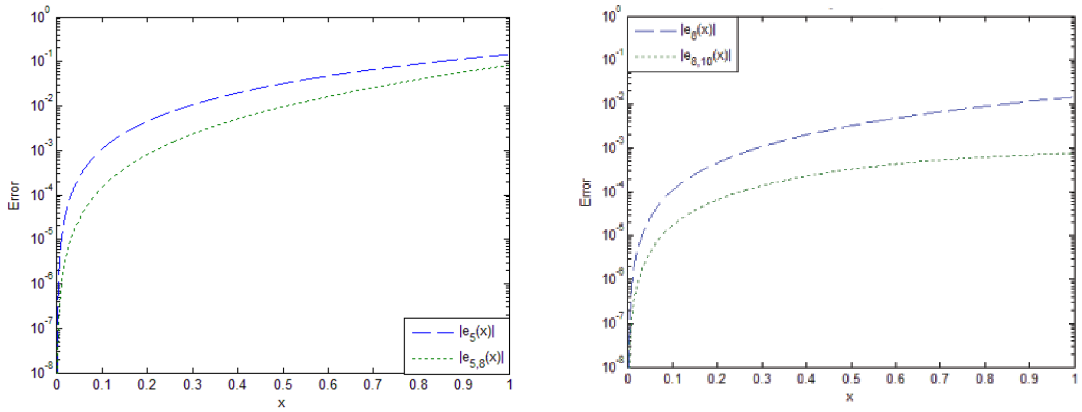
In this section, the efficiency of the method is shown with two examples. In Tables and Figures, we give the values of the exact solution $y(x)$, the approximate solution $y_N(x)$, the absolute error function $|e_N(x)| = |y(x) - y_N(x)|$ and the estimated absolute error function $|e_{N,M}(x)|$ at the selected points of the given interval. All examples have been solved by a computer code written in Matlab.

Example 1. Let us first consider the linear delay Fredholm integro-differential equation,

$$y^{(2)}(x+1) + 2y^{(1)}(x-0.5) - y(x-0.2) = g(x) + \int_0^1 \left[\cos(x)ty(t+1) - \sin(x)ty^{(1)}(x+0.5) \right] dt \tag{24}$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$ and the exact solution $y(x) = \cos(x)$.

Here, $g(x) = -\cos(x+1) - 2\sin(x-0.5) - \cos(x-0.2) + \cos(x)\cos(1) + \sin(x)\sin(0.5) - \cos(x)\cos(2)$



(a) **(b)**

Fig. 1: (**a**) Comparison of the absolute error function $|e_N(x)| = |y(x) - y_N(x)|$ and the estimated absolute error functions $|e_{N,M}(x)|$ for $N = 5$ and $M = 8$ of Eq. (24). (**b**) Comparison of the absolute error function $|e_N(x)| = |y(x) - y_N(x)|$ and the estimated absolute error functions $|e_{N,M}(x)|$ for $N = 8$ and $M = 10$ of Eq. (24).

$-\cos(x)\sin(2) - \sin(x)\sin(1.5) + \sin(x)\cos(1.5)$, $F_0 = -1, F_1 = 2, F_2 = 1, m = 2, K_0(x,t) = \cos(x)t$ and $K_1(x,t) = -\sin(x)t$.

By applying the presented method for $(N,M) = (5,8), (8,10)$, we have the approximate solutions

$$y_5(x) = 1 - 0.3993512391x^2 + (0.5982450134e-1)x^3 + (0.3075266846e-1)x^4 - (0.8062702167e-2x^5),$$

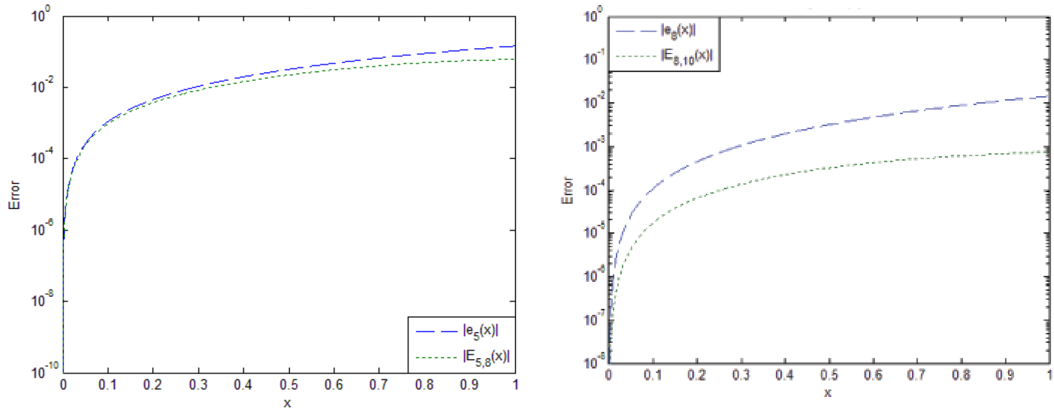
$$y_8(x) = 1 + (0.1831867991e-13)x - (0.5100618414)x^2 - (0.5949528704e-2)x^3 + (0.4260225056e-1)x^4 + (0.7838088704e-3)x^5 - (0.1410035029e-2)x^6 - (0.3747359679e-4)x^7 + (0.2435534701e-4)x^8$$

and the corrected approximate solutions,

$$y_{5,8}(x) = 1 - (0.1692641575e-16)x - 0.4084679337x^2 + (0.6315050695e-2)x^3 + (0.2308413299e-1)x^4 - (0.1214983732e-2)x^5 - (0.1582323762e-1)x^6 + (0.3994724839e-2)x^8 - (0.7404730922e-2)x^7$$

$$y_{8,10}(x) = 1 + (0.1831889218e-13)x - 0.5082977935x^2 - (0.6646869084e-2)x^3 + (0.4207819017e-1)x^4 + (0.8604247297e-3)x^5 - (0.1367667904e-2)x^6 - (0.4057980289e-4)x^7 + (0.2322066163e-4)x^8 + (0.3261778921e-3)x^9 - (0.2277367369e-3)x^{10}.$$

By using the error estimation in Section 4, the error functions for the above approximate solutions are estimated. For some values of N and M, the actual absolute error functions are compared with the estimated absolute error functions in Figure 1-(a)-(b). Figure 2-(a)-(b) show the comparisons of the absolute error functions and the corrected absolute error functions for $(N,M) = (5,8)$ and $(N,M) = (8,10)$.



(a) **(b)**

Fig. 2: (**a**) Comparison of the absolute error function $|e_N(x)| = |y(x) - y_N(x)|$ and the corrected absolute error functions $|E_{N,M}(x)|$ for $N = 5$ and $M = 8$ of Eq. (24). (**b**) Comparison of the absolute error function $|e_N(x)| = |y(x) - y_N(x)|$ and the corrected absolute error functions $|E_{N,M}(x)|$ for $N = 8$ and $M = 10$ of Eq. (24).

Example 2. [6] Now we consider the Firedholm integro-diferantial equation,

$$y'(x) - y(x) = +\frac{1 - e^{x+1}}{x+1} + \int_0^1 e^{tx} y(t) dt \tag{25}$$

with the initial condition $y(0) = 1$. The exact solution of the problem is given by $y(x) = e^x$.

The absolute errors are compared with the Homotopy perturbation method (HPM) [5], the differential transformation method (DTM) [6] and the CAS wavelet method (CASWM) [7] in Table 1.

Table 1 Comparison of the absolute errors of Eq. (36)

	HPM [5]	DTM [6]	CASWM [7]	Present method	
x_i				$e_5(x_i)$	$e_8(x_i)$
0.1	2.314814815e-06	1.00118319e-02	1.34917637e-03	2.7827e-004	3.8002e-007
0.2	9.259259259e-06	2.78651355e-02	1.15960044e-03	6.0702e-004	8.2895e-007
0.4	3.703703704e-05	7.55356316e-02	5.93105645e-02	1.4505e-003	1.9742e-006
0.6	8.333333333e-05	1.09551714e-01	4.39287720e-02	2.6506e-003	3.5560e-006
0.8	1.481481481e-04	6.94512700e-02	1.34514117e-02	4.5072e-003	6.0074e-006
0.9	1.875000000e-04	1.00034260e-02	1.32045209e-02	5.8749e-003	8.0600e-006

6 Conclusions

In this study, we have presented a matrix method based the Müntz-Legendre polynomials for the delay linear Fredholm integro-differential equations with constant coefficients. Also, we have given error estimation for method in terms of the residual function. It is seen from Example 1 that error estimation is very effective. If the exact solution of the problem is unknown, then the absolute errors can be computed with this error estimation, approximately. The comparisons of the suggested method by the other methods show that our method is very effective. A considerable advantage of the method is that the approximate solutions are computed very easily by using a well-known symbolic software such as Matlab, Maple and Mathematica.

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