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CONTINUED FRACTION EXPANSIONS OF SOME FUNCTIONS OF POSITIVE DEFINITE MATRICES

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Abstract: In this paper we recall some results of matrix functions with eal cofficients. The aim of this paper is to provide some properties and results of continued fractions with matrix arguments. Then we give continued fractions expansions of some inverse of hyperbolic and circular functions $\arcsin(\mathrm{A})$, $\mathrm{arcsinh}(\mathrm{A})$, $\arccos \mathrm{A}$ and $\mathrm{arcch}(\mathrm{A})$ where A is a positive definite matrix.

Keywords: Continued fraction expansion, positive definite matrix, function of matrices

1. Introduction and mativation

Over the last two hundred years, the theory of continued fractions has been a topic of extensive study. The basic idea of this theory over real numbers is to give an approximation of various real numbers by the rational ones. One of the main reasons why continued fractions are so useful in computation is that they often provide representation for transcendental functions that are much more generally valid than the classical representation by, say, the power series. Further; in the convergent case, the continued fractions expansions have the advantage that they converge more rapidly than other numerical algorithms.

Recently, the extension of continued fractions theory from real numbers to the matrix case has seen several developments and interesting applications (see [5],[7], [11]). The real case is relatively well studied in the literature. However, in contrast to the theoretical importance, one can find in mathematical literature only a few results on the continued fractions with matrices arguments. The main difficulty arises from the fact that the algebra of square matrices is not commutative.

For simplicity and clearness, we restrict ourselves to positive definite matrices, but our results can be, without special difficulties, projected to the case of positive definite operators from an infinite dimensional Hilbert space into itself.

2. Preleminary and notations

Matrix functions play a widespreased role in science and engineering, with applications areas ranging from nuclear magnetic resonance [2]. So for any scalar polynomial $p(z) = \sum_{i=0}^{k} \alpha_i z^i$ gives rise to a matrix polynomial with scalar coefficients by simply substituting A^i ve z^i:

$$P(A) = \sum_{i=0}^{k} \alpha_i A^i$$

More generally, for function f with a series representation on an open disk containing the eigenvalues of A, we are able to define the matrix function $f(A)$ via the Taylor series for f [4].

Alternatively, given a function $f(z)$ that is analytic inside on a closed contour Γ which encircles the eigenvalues of $A, f(A)$ can be defined, by analogy with Cauchy's integral theorem, by

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma} f(z)(zI - A)^{-1}dz$$

The definition is known as the matrix version of Cauchy's integral theorem. We now mention an important result of matrix functions.

Lemma 2.1 *(i) If two matrices $A \in M_m$ and $B \in M_m$ are similar, with*

$$A = XBX^{-1}$$

Then the matrices $f(A)$ and $f(B)$ are also similar, with

$$f(A) = Xf(B)X^{-1}$$

(ii) If $B \in M_m$ is a block diagonal matrix

$$f(B) = diag(f(B_1), f(B_2), \dots, f(B_r))$$

Proof. For $A = XBX^{-1}$ we have $A^k = ZB^kZ^{-1}$. Hence for every polynomial p it follow that

$$p(A) = Xp(B)X^{-1}$$

Therefore if either one of $p(A)$ or $p(B)$ equals zero then so does the other, implying that A and B share the same minimal polynomial. From definition there exists an interpolating polynomial $r(Z)$ such that

$$f(A) = r(A), \;\; f(B) = r(B)$$

and since for every polynomial we have $p(A) = Zp(B)Z^{-1}$, the result follows.

(ii) We deduce it from (i).

Let $A \in M_m, A$ is said to be positive semidefinite (resp. positive definite) if A is symmetric and

$$\forall x \in \mathbb{R}^m, < Ax, x > \geq 0 \; (resp. \forall x \in \mathbb{R}^m, x \neq 0 < Ax, x > > 0)$$

where <.,.> denotes the standard scalar product of $\mathbb{R}^m$.

We observe that positive semidefiniteness induces a partial ordering on the space of symmetric matrices: if A and B are two symmetric matrices, we write $A \leq B \; if \; B - A$ is positive semidefinite.

Henceforth, whenever we say that $A \in M_m$ is positive semidefinite (or positive definite), it will be assumed that A is symmetric.

For any matrices $A, B \in M_m$ with B invertible, we write $\frac{A}{B} = B^{-1}A$. It is easy to verify that for any invertible matrix X we have

$$\frac{A}{B} = \frac{XA}{XB} \neq \frac{AX}{BX}$$

Definition 2.2 Let $(A_n)\; n \geq 0$ and $(B_n)\; n \geq 0$ be two sequences of matrices in M_m. We define the sequences $(P_n)\; n \geq -1$ and $(Q_n)\; n \geq -1$ by

$$\begin{cases} P_{-1} = I, \; P_0 = A_0 \\ Q_{-1} = 0, \; Q_0 = I \end{cases} \quad and \quad \begin{cases} P_n = A_nP_{n-1} + B_nP_{n-2} \\ Q_n = A_nQ_{n-1} + B_nQ_{n-2} \end{cases} \quad n \geq 1 \tag{2.1}$$

The matrix P_n/Q_n is called the n^{th} convergent of $K(B_n/A_n)$, the fraction $\frac{B_n}{A_n}$ is called its n^{th} partial quotient. The proof of the next proposition is elementary and we left it to the reader.

Propotion 2.3. *For any two matrices C and D with C invertible, we have*

$$C\left[A_0;\frac{B_k}{A_k}\right]_{k=1}^{n} \qquad D = \left[CA_0D;\frac{B_1 D}{A_1 C^{-1}},\frac{B_2 C^{-1}}{A_2},\frac{B_k}{A_k}\right]_{k=3}^{n}$$

Definition 2.4. Let $(A_n),(B_n),(C_n)$ *and* (D_n) be four sequences of matrices. We say that the continued fractions $K(B_n/A_n)$ and $K(D_n/C_n)$ are equivalent if we have $F_n = G_n$ for all $n \geq 1$, where F_n and G_n are the n^{th} convergents of $K(B_n/A_n)$ and $K(D_n/C_n)$ respectively.

In order to simplify the statements on some partial quotients of continued fractions with matrices arguments, we need the following proposition which is an example of equivalent continued fractions.

Proposition 2.5. *(see [10]) Let* $\left[A_0;\frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ *be a given continued fraction.*

Then

$$\frac{P_n}{Q_n} = \left[A_0;\frac{B_k}{A_k}\right]_{k=1}^{n} = \left[A_0;\frac{X_k B_k X_{k-2}^{-1}}{X_k A_k X_{k-2}^{-1}}\right]_{k=1}^{n},$$

where $X_{-1} = X_0 = I$ *and* $X_1, X_2, \dots, X_n$ *are arbitrary invertible matrices.*

We also recall the following proposition in real cace.

Proposition 2.6. *Let* (r_n) be a non-zero sequence of real numbers. The following continued fractions

$$\left[a_0;\frac{b_1}{a_1},\frac{b_2}{a_2},\frac{b_3}{a_2},\dots\right] \quad and \quad \left[a_0;\frac{r_1 b_1}{r_1 a_1},\frac{r_2 r_1 b_2}{r_2 a_2},\frac{r_3 r_2 b_3}{r_3 a_2},\dots\right]$$

are equivalent.

We end this section by introducing some topological notions of continued fractions with matrix arguments. We provide M_m with the topology induced by the following classical norm:

$$\forall A \in M_{m,}\ \|A\| = sup_{x\neq 0}\frac{\|Ax\|}{\|x\|} = sup_{\|x\|=1}\|Ax\|$$

Definition 2.7. Let $\{A_n\}$ be a sequence of matrices in M_m. We say that $\{A_n\}$ converges in M_m if there exists a matrix $A \in M_m$ such that $\|A_n - A\|$ tends to 0 when n tens to $+\infty$. In this case we write ,$\lim_{n\to\infty} A_n = A$.

The continued fraction $\left[A_0;\frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ is said to be convergent in M_m if the sequence $\{P_n/Q_n\}_n = \{Q_n^{-1}P_n\}_n$ converges in M_m in the sense that there exists a matrix $F \in M_m$ such that $\lim_{n\to+\infty}\|F_n - F\| = 0$

3. Man Result

Let $A \in M_m$ be a positive definite matrix. Our aim in this section is to give a continued fraction expansion of arcsin, arcsh(A), arccos(A) and arcch(A). For simplicity, we start with the real case and we begin by recalling Laguerre's continued fraction in the following lemma.

Lemma 3.11 *(see [3]) Let x be a real number such that* $0 < x < 1$. *Then there holds*

$$\frac{\arcsin(x)}{(1-x^2)^{1/2}} = \left[0;\frac{x}{1},\frac{-(2k-2)(2k)x^2}{-(4k-1)}, \qquad \frac{-(2k-2)(2k)x^2}{-(4k+1)}\right]_{k=1}^{+\infty} \tag{3.1}$$

Now we establish a main theorem which, is a matrix version of the previous lemma.

Theorem 3.2. *Let* $A \in M_m$ *be a positive definite matrix such that* $\|A\| < 1$. *Then a continued fraction expansion of arcsin(A) is*

$$\arcsin(A) = \left[0; \frac{A(I-A^2)^{1/2}}{I}, \frac{(2k-1)(2k)A^2}{-(4k-1)I}, \frac{(2k-1)(2k)A^2}{-(4k+1)I}\right]_{k=1}^{+\infty} \tag{3.2}$$

Since $arcsh(A) = -iarcsin(A)$, by vertu of proposition 2.6 and the previous theorem, we have the next result.

Corollary 3.3. *A continued fraction expansion of arcsh(A) is given by*

$$\text{arcsh}(A) = \left[0; \frac{A(I-A^2)^{1/2}}{I}, \frac{(2k-1)(2k)A^2}{-(4k-1)I}, \quad \frac{(2k-1)(2k)A^2}{-(4k+1)I}\right]_{k=1}^{+\infty} \tag{3.3}$$

Proof. Let $A \in M_m$ be a positive definite matrix Then there exists an invertible matrix X such that $A = XDX^{-1}$, where $D = diag(\lambda_1, \lambda_2, \dots, \lambda_m)$ $and\ \lambda_i > 0$.

As the function $z \to \arcsin(z)$ is analytic in the open halfplane $\{z \in \mathbb{C}/Re(z) > 0\}$, then

$$\arcsin(A) = X(\arcsin(D))X^{-1} = X\, diag(\arcsin(\lambda_1), \arcsin(\lambda_2), \dots, \arcsin(\lambda_m))X^{-1}.$$

Let us define the sequences $\{P_n\}$ and $\{Q_n\}$ by:

$$\begin{cases} P_{-1} = I, P_0 = 0, P_1 = D \\ Q_{-1} = O, Q_0 = I, Q_1 = I \end{cases}$$

and for $k \geq 1$,

$$\begin{cases} P_{2k} = -(4k-1)P_{2k-1} + (2k-1)(2k)D^2P_{2k-2}, \\ Q_{2k} = -(4k-1)Q_{2k-1} + (2k-1)(2k)D^2Q_{2k-2}, \end{cases}$$

$$\begin{cases} P_{2k+1} = -(4k-1)P_{2k} + (2k-1)(2k)D^2P_{2k-1}, \\ Q_{2k+1} = -(4k-1)Q_{2k} + (2k-1)(2k)D^2Q_{2k-1}, \end{cases}$$

We see that P_n and Q_n are diagonal matrices, by setting $p_n = diag(p_n^1, p_n^2, \dots, p_n^m)$ and $q_n = diag(q_n^1, q_n^2, \dots, q_n^m)$, we obtain for each $i, 1 \leq i \leq m$

$$\begin{cases} p_{-1}^i = 1, p_0^i = 0, p_1^i = \lambda_i \\ q_{-1}^i = 0, q_0^i = 1, q_1^i = 1 \end{cases}$$

and for $k \geq 1$,

$$\begin{cases} p_{2k}^i = -(4k-1)p_{2k-1}^i + (2k-1)(2k)\lambda_i^2 p_{2k-2}^i \\ q_{2k}^i = -(4k-1)q_{2k-1}^i + (2k-1)(2k)\lambda_i^2 q_{2k-2}^i \end{cases}$$

$$\begin{cases} p_{2k+1}^i = -(4k-1)p_{2k}^i + (2k-1)(2k)\lambda_i^2 p_{2k-1}^i \\ q_{2k+1}^i = -(4k-1)q_{2k}^i + (2k-1)(2k)\lambda_i^2 q_{2k-1}^i \end{cases}$$

By lemma 3.1, the convergent p_n^i/q_n^i converges to $(1-\lambda_i^2)^{-1/2} arcsin\lambda_i$. It follows that P_n/Q_n converges to $(I - D2)-1/2$arcsin$D$, so that

$$\frac{\arcsin(D)}{(I-D^2)^{1/2}} = \left[0; \frac{D}{I}, \frac{2D^2}{-3I}, \frac{2D^2}{-5I}, \frac{-(2k-2)(2k)D^2}{-(4k-1)I}, \frac{-(2k-2)(2k)D^2}{-(4k+1)I}\right]_{k=2}^{+\infty}$$

Then, we multiply the continued fraction $\frac{\arcsin(D)}{(I-D^2)^{1/2}}$ by $(I-D^2)^{1/2}$ in the left to obtain

$$\text{arcsin}\,(D) = \left[0; \frac{D}{(I-D)^{-1/2}}, \frac{2D^2(I-D^2)^{1/2}}{-3I}, \frac{2D^2}{-5I}, \frac{(2k-1)(2k)D^2}{-(4k-1)I}, \frac{-(2k-1)(2k)D^2}{-(4k+1)I}\right]_{k=2}^{+\infty}$$

by proposition 2.3, we get

$$\text{X}(\text{arcsin}(D))X^{-1} = \left[0; \frac{DX^{-1}}{(I-D)^{-1/2}X^{-1}}, \frac{2D^2(I-D^2)^{1/2}X^{-1}}{-3I}, \frac{2D^2}{-5I}, \frac{(2k-1)(2k)D^2}{-(4k-1)I}, \frac{-(2k-1)(2k)D^2}{-(4k+1)I}\right]_{k=2}^{+\infty}$$

Let us define the sequence $(X_n)_{n\geq -1}$ by

$$\begin{cases} X_{-1} = X_0 = I \\ X_n = X(I-D^2)^{1/2},\ for\ n \geq 1. \end{cases}$$

Then

$$\begin{cases} \dfrac{X_1(DX^{-1})X_{-1}^{-1}}{X_1((I-D^2)^{\frac{1}{2}}X^{-1})X_0^{-1}} = \dfrac{A(I-A^2)^{1/2}}{I} \\ \dfrac{X_2(2D^2(I-D)^{-\frac{1}{2}}X^{-1})X_0^{-1}}{X_2(-3I)X_1^{-1}} = \dfrac{2A^2}{-3I} \end{cases}$$

For $k \geq 3$, we have

$$\frac{X_k(2k-1)(2k)D^2X_{k-2}^{-1}}{-X_k(4k-1)X_{k-1}^{-1}} = \frac{(2k-1)(2k)A^2}{-(4k-1)I}.$$

By applying the result of proposition 2.5 to the sequence $(X_n)_{n\geq -1}$, we finish the proof of theorem 3.2.

Before giving continued fraction expansions of arccos(A) and arcch(A), we begin with the real case in the following lemma

Lemma 3.4. *(see [7]) Let x be a real number such that 0 < x < 1, Then there holds*

$$\frac{\text{arccos}\,(x)}{(I-x^2)^{1/2}} = \left[0; \frac{x}{I}, \frac{(2k-1)(2k)x^2}{-(4k-1)}, \frac{(2k-1)(2k)x^2}{-(4k+1)}\right]_{k=1}^{+\infty} \tag{3.4}$$

Now we establish a main theorem which, is a matrix version of the lemma 4.

Theorem 3.5. *Let* $A \in M_m$ *be a positive definite matrix such that* $\|A\| < 1$. *Then a continued fraction expansion of arcsin(A) is*

$$\text{arccos}\,(A) = \left[0; \frac{A(I-A^2)^{1/2}}{I}, \frac{(2k-1)(2k)A^2}{-(4k-1)I}, \frac{(2k-1)(2k)A^2}{-(4k+1)I}\right]_{k=1}^{+\infty} \tag{3.5}$$

since $arcch(A) = iarccos(A)$, by vertu of proposition 2.6 and theorem 3.5, we have:

Corollary 3.6. *A continued fraction expansion of arcch(A) is given by*

$$\text{arcch}\,(A) = \left[0; \frac{A(A^2-I)^{1/2}}{I}, \frac{(2k-1)(2k)A^2}{(4k-1)I}, \frac{(2k-1)(2k)A^2}{(4k+1)I}\right]_{k=1}^{+\infty} \tag{3.6}$$

With a similar method as in theorem 3.2, we prove the result of this theorem.

References

[1] T.Ando, Topics on operators inequalities, Ryukyu Univ. Lecture Note Series, 1 (1978).
[2] S. M. Cox & P. C. Matthews: Exponential time diferencing for sti_ systems. J. Comp. Phys., 176 (2); 430-455, 2002.
[3] A. Cuyt & V. Brevik Petersen, Handbook of continued fractions for special functions, Springer (2007).
[4] F. R. Gantmacher: The Theory of Matrices, Vol. I. Chelsa, New York, 1959. Elsevier Science Publishers, (1992).
[5] V. Gen H. Golub & Charles F. Van Loan: Matrix Computations, Johns Hopking University Press, Baltimore, MD, USA, third edition, (1996).
[6] W.B.Jons & W.J. Thron: Continued Fractions, Analytic Theory and Applications, Addison-Westly Publishing Company, 5. (1980).
[7] Peter Lancaster & Miron Tismenetsky. The Theory of Matrices. Academic Press, New York, second editions, (1985).
[8] L.Lorentzen, H.Wadeland, Continued fractions with application, Elseiver Science Publishers, (1992).
[9] M.Raissouli & A.Kacha, Convergence for matrix continued fractions, Linear Algebra and its Applications, bf 320 (2000), pp.115-129.
[10] M.Raissouli & F.Leazizi, Continued fractions expansion of the geometric matrix means and applications, Linear Algebra and its Appl, 359 (2003), pp. 37-57.
[11] R.S.Varga, On higher order stable implicit methods for solving parabolic partial diferential equations, J. Math. Phys, 40 (1961), pp. 220-231.