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Some characterizations of constant breadth curves in Euclidean 4-space E^4

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Abstract: In this study, the differential equation characterizations of curves of constant breadth are given in Euclidean 4-space E^4. Furthermore, a criterion for a curve to be the curve of constant breadth in E^4 is introduced. As an example, the obtained results are applied to the case that the curvatures k_1, k_2, k_3 and are discussed.

Keywords: Constant breadth curve, Frenet frame.

1 Introduction

Euler introduced the constant breadth curves in 1778 [7]. He considered these special curves in the plane. Later, many geometers have shown increased interest in the properties of plane convex curves. Struik published a brief review of the most important publications on this subject [20]. Also, Ball [1], Barbier [2], Blaschke [3,4] and Mellish [14] investigated the properties of plane curves of constant breadth. A space curve of constant breadth was obtained by Fujiwara by taking a closed curve whose normal plane at a point P has only one more point Q in common with the curve, and for which the distance $d(P,Q)$ is constant [8].

He also defined and studied constant breadth surfaces. Later, Smakal studied the constant breadth space curves [19]. Furthermore, Blaschke considered the notion of curve of constant breadth on the sphere [4]. Moreover, Reuleaux studied the curves of constant breadth and gave the method related to these curves for the kinematics of machinery [16]. Then, constant breadth curves had an importance for engineering sciences and Tanaka used the constant breadth curves in the kinematics design of Com follower systems [21].

Moreover, Köse has presented some concepts for space curves of constant breadth in Euclidean 3-space in [12] and Sezer has obtained the differential equations characterizing space curves of constant breadth and introduced a criterion for these curves [18]. Constant breadth curves in Euclidean 4-space were given by Mağden and Köse [13]. Moreover, constant breath curves have been studied in Minkowski space. Kazaz, Önder and Kocayiğit have studied spacelike curves of constant breadth in Minkowski 4-space [10]. Önder, Kocayiğit and Candan have obtained and studied the differential equations characterizing constant breadth curves in Minkowski 3-space [15]. Furthermore, Kocayiğit and Önder have showed that constant breadth curves are normal curves, helices, and spherical curves in some special cases [11].

In this paper, we study the differential equations characterizing curves of constant breadth in the Euclidean 4-space E^4. Moreover, we give a criterion characterizing these curves in E^4.

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2 Differential equations characterizing curves of constant breadth in E^4

Let (C) be a unit speed regular curve in E^4 with parametrization $\alpha(s) : I \subset \mathbb{R} \to E^4$. Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{E}\}$ the moving Frenet frame along the curve (C) in E^4. Then, the following Frenet formulate are given,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \\ \mathbf{E}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \\ \mathbf{E} \end{bmatrix}$$

where k_1, k_2 and k_3 are the first, second and third curvatures of the curve (C), respectively [9].

Definition 1. *Let (C) be a unit speed regular curve in E^4 with position vector $\alpha(s)$. If (C) has parallel tangents* $\mathbf{T}$ *and* $\mathbf{T}^*$ *in opposite direction at the opposite points and of the curve and if the distance between these points is always constant then is called a curve of constant breadth in E^4. Moreover, a pair of curves (C) and (C^*) for which the tangents at the corresponding points are parallel and in opposite directions and the distance between these points is always constant is called a curve pair of constant breadth in E^4.*

Let now (C) and (C^*) be a pair of unit speed curves in E^4 with position vector $\alpha(s)$ and $\alpha^*(s^*)$, where s and s^* are arc length parameters of the curves, respectively. Let (C) and (C^*) have parallel tangents in opposite directions at opposite points. Then the curve (C^*) may be represented by the equation

$$\alpha^*(s) = \alpha(s) + m_1(s)\mathbf{T}(s) + m_2(s)\mathbf{N}(s) + m_3(s)\mathbf{B}(s) + m_4(s)\mathbf{E}(s) \tag{1}$$

where $m_i(s)$, $1 \le i \le 4$ are the differentiable functions of s which is the arc length of (C) . Differentiating this equation with respect to s and using the Frenet formulate we obtain

$$\frac{\alpha^*(s)}{ds} = \mathbf{T}^* \frac{ds^*}{ds} = \left(1 + \frac{dm_1}{ds} - m_2 k_1\right)\mathbf{T} + \left(m_1 k_1 + \frac{dm_2}{ds} - m_3 k_2\right)\mathbf{N} + \left(m_2 k_2 + \frac{dm_3}{ds} - m_4 k_3\right)\mathbf{B} + \left(m_3 k_3 + \frac{dm_4}{ds}\right)\mathbf{E}.$$

Since $\mathbf{T} = -\mathbf{T}^*$ at the corresponding points of (C) and (C^*), we have

$$\begin{cases} \left(1 + \frac{dm_1}{ds} - m_2 k_1\right) = -\frac{ds^*}{ds}, \\ \left(m_1 k_1 + \frac{dm_2}{ds} - m_3 k_2\right) = 0, \\ \left(m_2 k_2 + \frac{dm_3}{ds} - m_4 k_3\right) = 0, \\ \left(m_3 k_3 + \frac{dm_4}{ds}\right) = 0. \end{cases} \tag{2}$$

It is well known that the curvature of (C) is $lim(\Delta\varphi/\Delta s) = (d\varphi/ds) = k_1(s)$, where $\varphi = \int_0^s k_1(s)ds$ is the angle between the tangent of the curve (C) and a given fixed direction at the point $\alpha(s)$. Then from (2) we have the following system

$$\begin{aligned} m_1' &= m_2 - f(\varphi),\ m_2' = m_3 \rho k_2, \\ m_3' &= m_4 \rho k_3 - m_2 \rho k_2,\ m_4' = -m_3 \rho k_3. \end{aligned} \tag{3}$$

Here and after we will use $(')$ to show the differentiation with respect to φ . In (3), $f(\varphi) = \rho + \rho^*$ and, $\rho = \frac{1}{k_1}$ and $\rho^* = \frac{1}{k_1^*}$ denote the radius of curvatures at the points α and α^*, respectively. From (3) eliminating m_2, m_3 and m_4 their derivatives we have the following differential equation

$$\frac{d}{d\varphi}\left[\frac{d}{d\varphi}\left[\frac{1}{\rho k_2}\left(\frac{d^2 m_1}{d\varphi^2} + m_1\right)\right] + \frac{k_2}{k_3}\frac{dm_1}{d\varphi}\right] + \frac{k_2}{k_3}\left(\frac{d^2 m_1}{d\varphi^2} + m_1\right) + \frac{d}{d\varphi}\left[\frac{1}{\rho k_2}\frac{d}{d\varphi}\left(\frac{1}{\rho k_2}\frac{df}{d\varphi}\right) + \frac{k_2}{k_3}f\right] + \frac{k_2}{k_3}\frac{df}{d\varphi} = 0. \tag{4}$$

Then we can give the following theorem.

Theorem 1. *The general differential equation characterizing space curves of constant breadth in* E^4 *is given by* (4).

Let now consider the system (3) again. The distance d between the opposite points α and α^* is the breadth of the curves and is constant, that is,

$$d^2 = \|\mathbf{d}\|^2 = \|\alpha^* - \alpha\|^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 = const. \tag{5}$$

Then the system (3) may be written as follows:

$$\begin{aligned} m_2 =& f(\varphi),\ m_2' = m_3\rho k_2,\ m_3' = m_4\rho k_3 - m_2\rho k_2,\\ m_4' =& -m_3\rho k_3,\ m_1 = 0. \end{aligned} \tag{6}$$

or

$$\begin{aligned} m_1' =& m_2,\ m_2' = -m_1 + m_3\rho k_2,\\ m_3' =& m_4\rho k_3 - m_2\rho k_2,\ m_4' = -m_3\rho k_3. \end{aligned} \tag{7}$$

which are the systems describing the curve (C).

Let us consider the system (7) with special chosen $m_1 = const.$. Here, eliminating first m_1, m_2, m_3 and their derivatives, and then m_1, m_2, m_4 and their derivatives, respectively, we obtain the following linear differential equations of second order

$$\begin{cases} (\rho k_3)m_4'' - (\rho k_3)' m_4' + (\rho k_3)^3 m_4 = 0, & \rho k_2 \neq 0, \\ (\rho k_3)m_3'' - (\rho k_3)' m_3' + (\rho k_3)^3 m_3 = 0, & \rho k_3 \neq 0. \end{cases} \tag{8}$$

By changing the variable φ of the form $\xi = \int_0^\varphi \rho(t)k_3(t)dt$, these equations can be transformed into the following differential equations with constant coefficients,

$$\frac{d^2 m_4}{d\xi^2} + m_4 = 0 \text{ and } \frac{d^2 m_3}{d\xi^2} + m_3 = 0, \tag{9}$$

respectively [5]. Then, the general solutions of the differential equations (9) are

$$\begin{cases} m_3 = A\cos\left(\int_0^\varphi \rho k_3 dt\right) + B\sin\left(\int_0^\varphi \rho k_3 dt\right), \\ m_4 = C\cos\left(\int_0^\varphi \rho k_3 dt\right) + D\sin\left(\int_0^\varphi \rho k_3 dt\right). \end{cases} \tag{10}$$

respectively, where A, B, C and D are real constants. Substituting (10) into (7), we obtain $A = -D$, $B = C$, and so, the set of the solutions of the system (7), in the form

$$\begin{Bmatrix} m_1 = c = const.,\ m_2 = 0, \\ m_3 = A\cos\int_0^\varphi \rho k_3 dt + B\sin\int_0^\varphi \rho k_3 dt, \\ m_4 = B\cos\int_0^\varphi \rho k_3 dt - A\sin\int_0^\varphi \rho k_3 dt. \end{Bmatrix} \tag{11}$$

Thus the equation (1) is described and since $d^2 = \|\alpha^* - \alpha\|^2 = const.$, from (11) the breadth of the curve is $d^2 = c^2 + A^2 + B^2$.

Now, let us return to the system (6) with $m_1 = 0$. By changing the variable φ of the form $u = \int_0^\varphi \mu(t)dt$, $\mu = \rho k_3$ and eliminating m_1, m_2, m_4 and their derivatives we have the linear differential equation

$$\frac{d^2m_3}{du^2}+m_3=-\frac{d}{du}(\frac{k_2}{k_3}m_2). \tag{12}$$

which has the following solution

$$m_3=A_1\cos\int_0^{\varphi}\rho k_3dt+B_1\sin\int_0^{\varphi}\rho k_3dt-\int_0^{\varphi}\cos[u(\varphi)-u(t)]\rho k_2 f(t)dt. \tag{13}$$

Then, the general solution of the system (6) is

$$\begin{cases} m_1=0, \\ m_2=f(\varphi), \\ m_3=A_1\cos\int_0^{\varphi}\rho k_3dt+B_1\sin\int_0^{\varphi}\rho k_3dt-\int_0^{\varphi}\cos[u(\varphi)-u(t)]\rho k_2 f(t)dt, \\ m_4=B_1\cos\int_0^{\varphi}\rho k_3dt-A_1\sin\int_0^{\varphi}\rho k_3dt+\int_0^{\varphi}\sin[u(\varphi)-u(t)]\rho k_2 f(t)dt. \end{cases} \tag{14}$$

which determines the constant breadth curve in (1) where A_1, B_1 are real constants.

Furthermore, in this case, i.e., $m_1=0$, from (4) we have the following differential equation

$$\frac{d}{d\varphi}\left[\frac{1}{\rho k_3}\frac{d}{d\varphi}\left(\frac{1}{\rho k_2}\frac{df}{d\varphi}\right)+\frac{k_2}{k_3}f\right]+\frac{k_2}{k_3}\frac{df}{d\varphi}=0. \tag{15}$$

By changing the variable φ of the form $w=\int_0^{\varphi}\rho k_2 d\varphi$, (15) becomes

$$\frac{d}{dw}\left[\frac{k_2}{k_3}\left(\frac{d^2f}{dw^2}+f\right)\right]+\frac{k_3}{k_2}\frac{df}{dw}=0. \tag{16}$$

which also determines the constant breadth curve in (1).

So far we have dealt with a pair of space curves having parallel tangent in opposite directions at corresponding points. Now let us consider a simple closed unit speed space curve (C) in E^4 for which the normal plane of every point P on the curve meets the curve of a single opposite point Q other than P. Then, we may give the following theorem concerning the space curves of constant breadth in E^4.

Theorem 2. *Let (C) be a closed space curve in E^4 having parallel tangents in opposite directions at the opposite points of the curve. If the chord joining the opposite points of (C) is a double-normal, then (C) has constant breadth, and conversely, if (C) is a curve of constant breadth in E^4 then every normal of (C) is a double-normal.*

Proof. Let the vector $\mathbf{d}=\alpha^*-\alpha=m_1\mathbf{T}+m_2\mathbf{N}+m_3\mathbf{B}+m_4\mathbf{E}$ be a double-normal of (C) where m_1, m_2, m_3 and m_4 are the functions of s , the arc length parameter of the curve. Then we get $\langle\mathbf{d},\mathbf{T}^*\rangle=-\langle\mathbf{d},\mathbf{T}\rangle=m_1=0$. Thus from (2) we have

$$m_2\frac{dm_2}{ds}+m_3\frac{dm_3}{ds}+m_4\frac{dm_4}{ds}=0. \tag{17}$$

It follows that $m_2^2+m_3^2+m_4^2=constant$, i.e., the breadth of (C) is constant.

Conversely, if $\|\mathbf{d}\|^2=m_1^2+m_2^2+m_3^2+m_4^2=constant$ then as shown, $m_1=0$. This means that $\mathbf{d}$ is perpendicular to $\mathbf{T}$ and $\mathbf{T}^*$. So, $\mathbf{d}$ is the double-normal of (C).

A simple closed curve having parallel tangents in opposite directions at opposite points may be represented by the system (14). In this case a pair of opposite points of the curve is $(\alpha^*(\varphi), \alpha(\varphi))$ for φ , where $0 \leq \varphi \leq 2\pi$. Since (C) is a simple closed curve we get $\alpha^*(0) = \alpha^*(2\pi)$. Hence from (14) we have

$$\int_0^{2\pi} \rho k_3 dt = 2n\pi, \quad (n \in \mathbb{Z}). \tag{18}$$

Using the equality $ds = \rho d\varphi$, this formula may be given as $\int_C k_3 ds = 2n\pi$, $(n \in \mathbb{Z})$. This says that the integral third curvature of (C) is zero. So, we can give the following corollary.

Corollary 1. *The total third curvature of a simple closed curve (C) of constant breadth is $2n\pi$, $n \in \mathbb{Z}$.*

Furthermore, if we take $\frac{k_2}{k_3} = a = constant$, then from (16) we have

$$\frac{d^3 f}{dw^3} + K \frac{df}{dw} = 0. \tag{19}$$

where $K = 1 + \frac{1}{a^2}$. If we assume $K \neq \pm 1$, the general solution of (19) is

$$f = A_2 \sin \int_0^{\varphi} K \rho k_2 dt + B_2 \cos \int_0^{\varphi} K \rho k_2 dt + C_1. \tag{20}$$

where A_2, B_2 and C_1 are real constants. Since (C) is a simple closed curve, i.e., $\alpha^*(0) = \alpha^*(2\pi)$, from (20) it follows,

$$\int_0^{\varphi} K \rho k_2 dt = 2n\pi, \quad (n \in \mathbb{Z}). \tag{21}$$

Using the equality $ds = \rho d\varphi$, this formula may be given as $\int_C k_2 ds = 2\frac{n}{K}\pi, \quad (K, n \in \mathbb{Z})$. This says that the integral second curvature of (C) is $2\frac{n}{K}\pi, \quad (K, n \in \mathbb{Z})$. So, we can give the following corollary.

Corollary 2. *The total second curvature of a simple closed curve (C) of constant breadth with $a = k_2/k_3 = constant$ is $2\frac{n}{K}\pi$, where $n \in \mathbb{Z}$ and $K = 1 + \frac{1}{a^2}$.*

3 A criterion for curves of constant breadth in E^4

Let us assume that (C) is a curve of constant breadth in E^4 and $\alpha(s)$ denotes the position vector of a generic point of the curve. If (C) is a closed curve, the position vector $\alpha(s)$ must be a periodic function of period $\omega = 2\pi$, where ω is the total length of (C). Then the curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$ are also periodic of the same period. However, periodicity of the curvatures and closeness of the curve are not sufficient to guarantee that a space curve is a constant breadth curve in E^4. That is, if a curve is closed curve (periodic), it may be the curve of constant breadth or not. Therefore, to guarantee that a curve is a constant breadth curve, we may use the system (7) characterizing a curve of constant breadth and follow the similar way given in [6].

For this purpose, first let us consider the following Frenet formulas at a generic point on the curve (C),

$$\frac{d\mathbf{T}}{ds} = k_1 \mathbf{N}, \frac{d\mathbf{N}}{ds} = -k_1 \mathbf{T} + k_2 \mathbf{B}, \frac{d\mathbf{B}}{ds} = -k_2 \mathbf{N} + k_3 \mathbf{E}, \frac{d\mathbf{E}}{ds} = -k_3 \mathbf{B}. \tag{22}$$

Writing the formulas (22) in terms of φ and allowing for $\frac{d\varphi}{ds} = k_1 = \frac{1}{\rho}$ we have

$$\frac{d\mathbf{T}}{d\varphi} = \mathbf{N}, \frac{d\mathbf{N}}{d\varphi} = -\mathbf{T} + \rho k_2 \mathbf{B}, \frac{d\mathbf{B}}{d\varphi} = -\rho k_2 \mathbf{N} + \rho k_3 \mathbf{E}, \frac{d\mathbf{E}}{d\varphi} = -\rho k_3 \mathbf{B}. \tag{23}$$

Furthermore we can write the Frenet vectors **T**, **N**, **B**, **E** in the coordinate forms as follows

$$\mathbf{T}=\sum_{i=1}^{4} t_i\mathbf{e_i}, \mathbf{N}=\sum_{i=1}^{4} n_i\mathbf{e_i}, \mathbf{B}=\sum_{i=1}^{4} b_i\mathbf{e_i}, \mathbf{E}=\sum_{i=1}^{4} \varepsilon_i\mathbf{e_i}. \tag{24}$$

Since $\{\mathbf{T},\mathbf{N},\mathbf{B},\mathbf{E}\}$ is the orthonormal base in E_1^4, putting (24) and their derivatives into (23), we have the systems of linear differential equations

$$\left\{\begin{array}{llll}\frac{dt_1}{d\varphi}=n_1, & \frac{dt_2}{d\varphi}=n_2, & \frac{dt_3}{d\varphi}=n_3, & \frac{dt_4}{d\varphi}=n_4 \\ \frac{dn_1}{d\varphi}=-t_1+\rho k_2 b_1, & \frac{dn_2}{d\varphi}=-t_2+\rho k_2 b_2, & \frac{dn_3}{d\varphi}=-t_3+\rho k_2 b_3, & \frac{dn_4}{d\varphi}=-t_4+\rho k_2 b_4 \\ \frac{db_1}{d\varphi}=\rho k_3\varepsilon_1-\rho k_2 n_1, & \frac{db_2}{d\varphi}=\rho k_3\varepsilon_2-\rho k_2 n_2, & \frac{db_3}{d\varphi}=\rho k_3\varepsilon_3-\rho k_2 n_3, & \frac{db_4}{d\varphi}=\rho k_3\varepsilon_4-\rho k_2 n_4 \\ \frac{d\varepsilon_1}{d\varphi}=-\rho k_3 b_1, & \frac{d\varepsilon_2}{d\varphi}=-\rho k_3 b_2, & \frac{d\varepsilon_3}{d\varphi}=-\rho k_3 b_3, & \frac{d\varepsilon_4}{d\varphi}=-\rho k_3 b_4.\end{array}\right\} \tag{25}$$

From (25), we find that $\{t_1,n_1,b_1,\varepsilon_1\}$, $\{t_2,n_2,b_2,\varepsilon_2\}$, $\{t_3,n_3,b_3,\varepsilon_3\}$ and $\{t_4,n_4,b_4,\varepsilon_4\}$ are four independent solutions of the following system of differential equations:

$$\frac{d\psi_1}{d\varphi}=\psi_2, \frac{d\psi_2}{d\varphi}=-\psi_1+\rho k_2\psi_3, \frac{d\psi_3}{d\varphi}=\rho k_3\psi_4-\rho k_2\psi_2, \frac{d\psi_4}{d\varphi}=-\rho k_3\psi_3. \tag{26}$$

If the curve (C) is the curve of constant breadth, then the systems (7) and (26) must be the same system. So, we observe that $\psi_1=m_1$, $\psi_2=m_2$, $\psi_3=m_3$, $\psi_4=m_4$. For brevity, we can write (7) or (26) in the form

$$\frac{d\psi}{d\varphi}=A(\varphi)\psi. \tag{27}$$

where

$$\psi=\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}, A(\varphi)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \rho k_2 & 0 \\ 0 & -\rho k_2 & 0 & \rho k_3 \\ 0 & 0 & -\rho k_3 & 0 \end{bmatrix}.$$

Obviously, (27) is a special case of the general linear differential equations abbreviated to the form

$$\left\{\begin{array}{l}\frac{d\psi}{dt}=A(t)\psi, \\ \varphi=\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{bmatrix}, A(t)=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, (4\leq n).\end{array}\right. \tag{28}$$

where $a_{ij}(t)$ are assumed to be continuous and periodic of period ω (See [6,17]). Let the initial conditions be $\psi_i(0)=x_i$, $(i=1,2,...,n)$. Let us take $x=\left[x_1,\ x_2,\ ...,\ x_n\right]^T$ and

$$\psi(t,x)=\left[m_1(t,x)\ m_2(t,x)\ ...\ m_n(t,x)\right]^T.$$

Then the equation (28) may be written in the form $\frac{d\psi}{dt}=A(t)\psi$, $\psi(0)=x$ as is well known from [6], the solution $\psi(t,x)$ of this equation is periodic of period ω , if

$$\int_0^{\omega} A(\xi)\psi(\xi,x)d\xi=0$$

and

$$\begin{cases} \psi(t,x) = \{E + M(t)\}x, \ (E = \text{ unit matrix}), \\ M(t) = IA(t) + I^{(2)}A(t) + ... + I^{(n)}A(t) + ..., \\ (IA)(t) = I^{(I)}A(t) = \int_0^t A(\xi)d\xi, \\ (I^{(n)}A)(t) = \int_0^t A(\xi)(I^{(n-1)}A)(\xi)d\xi, \ n > 1. \end{cases} \tag{29}$$

Furthermore, the following theorem is given in [6]:

Theorem 3. *The equations* $\frac{d\psi}{dt} = A(t)\psi$ *possess a non-vanishing periodic solution of period* ω *, if and only if* $det(M(\omega)) = 0$ *. In particular, in order that the equations* $\frac{d\psi}{dt} = A(t)\psi$ *possess linearly independent periodic solutions of period* ω *, the necessary and sufficient condition is that* $M(\omega)$ *be a zero matrix.*

Now, let us apply this theorem to the system (27). If $M(\omega) = 0$, there exist the unit vector functions $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, $\mathbf{E}$ of period ω , such that each set of functions $\{t_i, n_i, b_i, \varepsilon_i\}$, $(i = 1,2,3,4)$ form a solution of the equation (27) corresponding to the initial conditions (A_i, B_i, C_i, D_i). The curve (C) can be described as follows

$$\alpha(s) = \int_0^s \mathbf{T}(s)ds \quad \text{or} \quad \alpha(\varphi) = \int_0^\varphi \rho(\varphi)\mathbf{T}(\varphi)d(\varphi).$$

Here, to find $\mathbf{T}$, we can make use of the equation

$$\begin{bmatrix} t_i \\ n_i \\ b_i \\ \varepsilon_i \end{bmatrix} = \{E + M(\varphi)\} \begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{bmatrix}, \ (i = 1,2,3,4). \tag{30}$$

which is established by (29). If we take the initial conditions as $t_i(0) = A_i$, $n_i(0) = B_i$, $b_i(0) = C_i$, $\varepsilon_i(0) = D_i$, $(i = 1,2,3,4)$ such that (A_1, A_2, A_3, A_4), (B_1, B_2, B_3, B_4), (C_1, C_2, C_3, C_4), (D_1, D_2, D_3, D_4) form an orthonormal frame, then from (30) we obtain

$$t_i = (1 + m_{11})A_i + m_{12}B_i + m_{13}C_i + m_{14}D_i; \ \ (i = 1,2,3,4). \tag{31}$$

When the curve (C) is a curve of constant breadth, which is also periodic of period ω , it is clear that

$$\int_0^\omega \rho t_i d\varphi = 0. \tag{32}$$

Hence, form (31) and (32), we have

$$A_i \int_0^\omega \rho(1 + m_{11})d\varphi + B_i \int_0^\omega \rho m_{12} d\varphi + C_i \int_0^\omega \rho m_{13} d\varphi + D_i \int_0^\omega \rho m_{14} d\varphi = 0; \ (i = 1,2,3,4)$$

Since the coefficient determinant $\Delta \neq 0$ in this system, we obtain the equalities

$$\int_0^\omega \rho(1 + m_{11})d\varphi = 0 = \int_0^\omega \rho m_{12} d\varphi = \int_0^\omega \rho m_{13} d\varphi = \int_0^\omega \rho m_{14} d\varphi. \tag{33}$$

which are the conditions for a curve to be constant breadth curve in E^4. Here, we can take the period $\omega = 2\pi$ because of $0 \leq \varphi \leq 2\pi$. Thus we establish the following corollary.

Corollary 3. Let (C) be a regular curve in E^4 such that $\rho(\varphi) > 0$, $k_2(\varphi)$ and $k_3(\varphi)$ are continuous periodic functions of period ω. Then (C) is a curve of constant breadth, and also periodic of period ω, if and only if

$$M(\omega) = 0, \ \int_0^\omega \rho(1 + m_{11})d\varphi = 0 = \int_0^\omega \rho m_{12} d\varphi = \int_0^\omega \rho m_{13} d\varphi = \int_0^\omega \rho m_{14} d\varphi. \tag{34}$$

holds, where

$$\begin{cases} M(t) = IA(t) + I^{(2)}A(t) + ... + I^{(n)}A(t) + ..., \\ A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \rho k_2 & 0 \\ 0 & -\rho k_2 & 0 & \rho k_3 \\ 0 & 0 & -\rho k_3 & 0 \end{bmatrix}. \end{cases} \tag{35}$$

and $m_{ij}(t)$ are the entries of the matrix $M(t)$.

By means of (29) and (35), the matrix $M(t)$ can be constructed and each m_{ij} involves infinitely many integrations. Hence, we can write the conditions (34) in the following forms:

$$\begin{cases} \int_0^\omega \rho(\varphi)d\varphi - \int_0^\omega \int_0^r \int_0^s \rho(\varphi)dsdtd\varphi + \int_0^\omega \int_0^\phi \int_0^p \int_0^r \int_0^s \rho(\varphi)[1+\lambda(p)\lambda(s)]dtdsdrdpd\varphi - ... = 0 \\ \int_0^\omega \int_0^s \rho(\varphi)dtd\varphi - \int_0^\omega \int_0^p \int_0^r \int_0^s \rho(\varphi)[1+\lambda(t)\lambda(s)]dtdsdrd\varphi + ... = 0 \\ \int_0^\omega \int_0^r \int_0^s \rho(\varphi)\lambda(t)dtdsd\varphi - \int_0^\omega \int_0^\phi \int_0^p \int_0^r \int_0^s \rho(\varphi)[\lambda(t)+\lambda(p)\{\lambda(t)\lambda(s)+\mu(t)\mu(s)\}]dtdsdrdpd\varphi + ... = 0 \\ \int_0^\omega \int_0^p \int_0^r \int_0^s \rho(\varphi)\lambda(s)\mu(t)dtdsdrd\varphi - \int_0^\omega \int_0^q \int_0^\phi \int_0^p \int_0^r \int_0^s \rho(\varphi)\lambda(p)\mu(t)[1+\lambda(t)\lambda(s)+\mu(t)\mu(s)]dtdsdpd\phi d\varphi + ... = 0. \end{cases} \tag{36}$$

where $\lambda(\xi) = p(\xi)k_2(\xi)$, $\mu(\xi) = p(\xi)k_3(\xi)$.

Example 1. Let us consider the special case $\rho = const.$, $k_2 = const.$ and $k_3 = const.$ In this case, from (33), we have

$$\begin{cases} \omega - \frac{\omega^3}{3!} + (1+\rho^2k_2^2)\frac{\omega^5}{5!} - \varepsilon(1+\rho^2k_2^2)^2\frac{\omega^7}{7!} ... = 0 \\ \frac{\omega^2}{2!} - (1+\rho^2k_2^2)\frac{\omega^4}{4!} + (1+\rho^2k_2^2)^2\frac{\omega^6}{6!} - ... = 0 \\ k_2[\frac{\omega^3}{3!} - (1+\rho^2k_2^2+\rho^2k_3^2)\frac{\omega^5}{5!} + (1+\rho^2k_2^2+\rho^2k_3^2)^2\frac{\omega^7}{7!} - ...] = 0 \\ k_2k_3[\frac{\omega^4}{4!} - (1+\rho^2k_2^2+\rho^2k_3^2)\frac{\omega^6}{6!} + ...] = 0 \end{cases} \tag{37}$$

or

$$\begin{cases} \rho^2k_2^2(1+\rho^2k_2^2)^{\frac{1}{2}}\omega + \sin[(1+\rho^2k_2^2)^{\frac{1}{2}}\omega] = 0, \\ \cos[(1+\rho^2k_2^2)^{\frac{1}{2}}\omega] = 1 \ \text{ or } \ (1+\rho^2k_2^2)^{\frac{1}{2}}\omega = 2k\pi, \ k \in \mathbb{Z} \\ k_2[(1+\rho^2k_2^2+\rho^2k_3^2)^{\frac{1}{2}}\omega - \sin[(1+\rho^2k_2^2+\rho^2k_3^2)^{\frac{1}{2}}\omega]] = 0, \\ k_2k_3[-1+(1+\rho^2k_2^2+\rho^2k_3^2)\frac{\omega^2}{2} + \cos[(1+\rho^2k_2^2+\rho^2k_3^2)^{\frac{1}{2}}\omega]] = 0. \end{cases} \tag{38}$$

where $\omega = 2k\pi$. It is seen that all of the equalities (37) or (38) are satisfied simultaneously, if and only if $\rho k_2 = 0$, $\rho k_3 = 0$ that is, $\rho = const. > 0$ and $k_2, k_3 = 0$. Therefore, only ones with $\rho = const. > 0$ and $k_2, k_3 = 0$ of the curves with and are curves of constant breadth, which are circles in E^4.

Now let us construct the relation characterizing these circles. Since $\rho k_2, \rho k_3 = 0$ system (7) becomes

$$m_1' = m_2, \ m_2' = -m_1, \ m_3' = 0, \ m_4' = 0. \tag{39}$$

From (39), the equations with the unknowns m_1, m_2 and m_3 can be written as follows

$$m_1'' + m_1 = 0, \ m_2'' + m_2 = 0, \ m_3' = 0, \ m_4' = 0. \tag{40}$$

The general solution of (40) is

$$\begin{cases} m_1 = A_3\cos(\varphi) + B_3\sin(\varphi), \\ m_2 = C_2\cos(\varphi) + D_1\sin(\varphi), \\ m_3 = L_1, \\ m_4 = L_2. \end{cases} \tag{41}$$

where A_3, B_3, C_2, D_1, L_1 and L_2 are arbitrary constants. Replacing (41) into (39), we have $A_3 = -D_1$, $B_3 = C_2$ and thus we get

$$\{m_1 = A_3\cos(\varphi) + B_3\sin(\varphi),\ m_2 = B_3\cos(\varphi) - A_3\sin(\varphi), m_3 = L_1,\ m_4 = L_2\}. \tag{42}$$

which is the solution set of the system (40). Consequently, replacing (42) into (1), we obtain the equation

$$\alpha^*(\varphi) = \alpha(\varphi) + (A_3\cos(\varphi) + B_3\sin(\varphi))\mathbf{T} + (B_3\cos(\varphi) - A_3\sin(\varphi))\mathbf{N} + L_1\mathbf{B} + L_2\mathbf{E}.$$

which represents the circles with the diameter $d = \|\alpha^* - \alpha\| = (A_3^2 + B_3^2 + L_1^2 + L_2^2)^{\frac{1}{2}}$. In this case, a pair of opposite points of the curve is $(\alpha^*(\varphi), \alpha(\varphi))$ for φ in $0 \leq \varphi \leq 2\pi$.

4 Conclusion

In the characterizations and determinations of the special curves and curve pair, the differential equations have an important role. A differential equation or a system of differential equations with respect to the curvatures can determinate the special curves or curve pairs. In this paper, the differential equations characterizing the curves of constant breadth in E^4 are studied. Furthermore, a criterion for a space curve to be the curve of constant breadth in E^4 is given.

References

[1] Ball, N. H., On Ovals, American Mathematical Monthly, 37(7) (1930), 348-353.
[2] Barbier, E., Note sur le probleme de l'aiguille et le jeu du joint couvert. J. Math. Pures Appl., II. Ser. 5, 273-286 (1860).
[3] Blaschke, W., Konvexe bereiche gegebener konstanter breite und kleinsten inhalts, Math. Annalen, 76 (1915), 504-513.
[4] Blaschke, W., Einige Bemerkungen über Kurven und Flächen von konstanter Breite. Ber. Verh. sächs. Akad. Leipzig, 67, 290-297 (1915).
[5] Breuer, S., Gottlieb, D., The Reduction of Linear Ordinary Differential Equations to Equations with Constant Coefficients, J. Math. Anal. Appl., 32 (1970), no. 1, 62-76.
[6] Chung, H.C., A Differential-Geometric Criterion for a Space Curve to be Closed, Proc. Amer. Math. Soc. 83 (1981), no. 2, 357-361.
[7] Euler, L., De Curvis Triangularibus, Acta Acad. Petropol., (1778), (1780), 3-30.
[8] Fujivara, M., On space curves of constant breadth, Tohoku Math., J., 5 (1914), 180-184.
[9] Hacisalihoğlu, H.H., Diferensiyel Geometri, İnönü University Science and Art Faculty Press, 1983.
[10] Kazaz, M., Önder, M., Kocayiğit H., Spacelike curves of constant breadth in Minkowski 4-space, Int. Journal of Math. Analysis, 2 (2008), no. 22, 1061-1068.
[11] Kocayiğit, H., Önder, M., Spacelike curves of constant breadth in Minkowski 3-space, Annali di Matematica, 192 (5), 2013, 805-814.
[12] Köse, Ö., On space curves of constant breadth, Doğa Tr. J. Math., 10 (1986), no. 1, 11-14.
[13] Mağden A., Köse, Ö., On the curves of constant breadth in space, Turkish J. Math., 21 (1997), no. 3, 277-284.
[14] Mellish, A. P., Notes on Differential Geometry, Annals of Math., 32 (1) (1931), 181-190.
[15] Önder, M., Kocayiğit, H., Candan, E., Differential Equations Characterizing Timelike and Spacelike Curves of Constant Breadth in Minkowski 3-spaceJ. Korean Math. Soc. 48 (2011), No. 4, pp. 849-866.
[16] Reuleaux, F., The Kinematics of Machinery, Trans. By A.B.W. Kennedy, Dover, Pub. Nex York, (1963).
[17] Ross, S.L., Differential Equations, John Wiley and Sons, Inc., New York, (1974), 440-468.
[18] Sezer, M., Differential equations characterizing space curves of constant breadth and a criterion for these curves, Turkish, J. of Math. 13 (1989), no. 2, 70-78.
[19] Smakal, S., Curves of constant breadth, Czechoslovak Mathematical Journal, 23 (1973), no. 1, 86-94.
[20] Struik, D.J., Differential Geometry in the Large, Bull. Amer. Math. Soc., 37 (1931), 49-62.
[21] Tanaka, H., Kinematics Design of Com Follower Systems, Doctoral Thesis, Columbia Univ., (1976).