

A collocation method for linear integral equations in terms of the generalized Bernstein polynomials

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Abstract: In this study, a collocation method based on the generalized Bernstein polynomials is presented and analyzed for the solution of linear Fredholm-Volterra integral equations (FVIEs). Error bounds and convergence of this method are demonstrated. Some examples are also given to illustrate the accuracy, efficiency and applicability of the method.

Keywords: Bernstein polynomial approach, linear integral equations, collocation method.

1 Introduction

Integral equations have been first produced in connection with the famous tautochrone problem by Abel (Italian mathematician) in 1825. These equations are defined as equations in which the unknown function appears under the integral sign. Besides, the most frequently used integral equations are collected two major classes, namely Volterra and Fredholm integral equations. Integral equations are closely related both in pure and applied mathematics. These equations play an important role in various fields of science and numerous applications such as elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc. Many problems included ordinary or partial differential equations can be converted to problems of the integral equations. Therefore, many researchers are interested in numerical methods to get the solution of integral equations.

Bernstein polynomials and their basis polynomials [5,6] defined on the interval $[0, 1]$, can be easily generalized to the interval $[a, b]$ by using transformation $t = \frac{x-a}{b-a}$ as follows.

Definition 1. Generalized Bernstein basis polynomials can be defined on the interval $[a, b]$; by

$$p_{i,n}(x) = \frac{1}{(b-a)^n} \binom{n}{i} (x-a)^i (b-x)^{n-i}; \quad i = 0, 1, \dots, n.$$

Definition 2. Let y be continuous function on the interval $[a, b]$. Generalized Bernstein polynomials of n th-degree are defined by

$$B_n(y; x) = \sum_{i=0}^n y \left(a + \frac{(b-a)i}{n} \right) p_{i,n}(x)$$

such that $p_{i,n}(x)$ are the generalized Bernstein basis polynomials.

The generalized Bernstein polynomials and their basis forms have many useful properties, such as the positivity, continuity, recursion's relation, symmetry, unity partition of the basis set over the interval $[a, b]$, uniform approximation, differentiability and integrability. For this reason, these polynomials can be useful mathematical tools for numerical analysis and approximation theory. In recent years, these polynomials are encountered in research works of numerical solution of Volterra [2, 7, 12] and Fredholm [8, 11] integral equations.

Theorem 1. *If $y(x)$ is continuous function on the interval $[a, b]$, then*

$$\lim_{n \rightarrow \infty} B_n(y; x) = y(x)$$

converges uniformly.

Proof. Proof of the above theorem can be easily seen from the proof of theorem given on the interval $[0, 1]$ by Phillips [9].

Definition 3. *Linear Fredholm -Volterra integral equation of the third kind is given by:*

$$a(x)y(x) = g(x) + \lambda_1 \int_a^b f(x, t)y(t)dt + \lambda_2 \int_a^x v(x, t)y(t)dt \quad (1)$$

such that $a(x) \neq 0$ and $a(x) \neq 1$. Here $a(x), g(x) \in [a, b]$, $f(x, t) : [a, b] \times [a, b] \rightarrow \mathbb{R}$ and $v(x, t) : [a, b] \times [a, b] \rightarrow \mathbb{R}$ are continuous functions, λ_1 and λ_2 are constants, $y(x)$ is unknown function. Privately the equation (1) is called linear Fredholm-Volterra integral equations of the first and second kind respectively for $a(x) \equiv 0$ and $a(x) \equiv 1$.

The remainder of this paper consists four sections: In Section 2, a collocation method is developed directly to get the numerical solution of the integral equations by means of the generalized Bernstein polynomials. In Section 3, error bounds and convergence analysis are given for the proposed method. Section 4 demonstrates the applicability of the presented method. In this part, some linear examples are solved and compared with different methods. Finally, Section 5 is ended with conclusions contained general inferences of the study.

2 Method of solution

The aim of the proposed method is to derive a collocation method for approximate solution of the linear FVIE (1) by means of the generalized Bernstein polynomials:

$$y(x) \cong B_n(y; x) = \sum_{i=0}^n y \left(a + \frac{(b-a)i}{n} \right) p_{i,n}(x). \quad (2)$$

Now, we give a theorem that represents a main matrix relation for the linear integral equation (1).

Theorem 2. *Let x_s be collocation points and $y(x)$ be continuous function on the interval $[a, b]$. The linear FVIE (1) has the following matrix form:*

$$[\mathbf{A}\mathbf{P} - \lambda_1 \mathbf{F} - \lambda_2 \mathbf{V}] \mathbf{Y} = \mathbf{G}. \quad (3)$$

Here $\mathbf{A} = \text{diag}[a(x_s)]$, $\mathbf{F} = [F_{s,i}]$, $\mathbf{V} = [V_{s,i}]$, $\mathbf{P} = [p_{i,n}(x_s)]$ are $(n+1) \times (n+1)$ matrices, and $\mathbf{Y} = \left[y \left(a + \frac{(b-a)i}{n} \right) \right]$, $\mathbf{G} = [g(x_s)]$ are $(n+1) \times 1$ matrices for $i, s = 0, 1, \dots, n$.

Proof. Since solution of the equation (1) is continuous function on the interval $[a, b]$, it has the Bernstein polynomial approach given with the expression (2) from the Theorem 1. So the solution function can be written as the matrix form:

$$y(x) \simeq \mathbf{P}(x)\mathbf{Y} \tag{4}$$

such that

$$\mathbf{P}(x) = \left[p_{0,n}(x) \ p_{1,n}(x) \ \dots \ p_{n,n}(x) \right], \quad \mathbf{Y} = \left[y(a) \ y\left(a + \frac{b-a}{n}\right) \ \dots \ y(b) \right]^T.$$

Substituting the collocation points and the relation (4) into the equation (1), we obtain a linear algebraic equation system

$$a(x_s)\mathbf{P}(x_s)\mathbf{Y} = g(x_s) + \lambda_1 \int_a^b f(x_s, t)\mathbf{P}(t)dt\mathbf{Y} + \lambda_2 \int_a^{x_s} v(x_s, t)\mathbf{P}(t)dt\mathbf{Y}. \tag{5}$$

Here $y(x_s) = B_n(y; x_s)$ ($s = 0, 1, \dots, n$) because of the collocation method. If the integrals at the sides of λ_1 and λ_2 are called respectively $\mathbf{F}(x_s)$ and $\mathbf{V}(x_s)$, then for $i = 0, 1, \dots, n$, the elements of these matrices can be written as

$$\mathbf{F}(x_s) = \left[F_{s,0} \ F_{s,1} \ \dots \ F_{s,n} \right]; \quad F_{s,i} = \int_a^b f(x_s, t) p_{i,n}(t) dt,$$

$$\mathbf{V}(x_s) = \left[V_{s,0} \ V_{s,1} \ \dots \ V_{s,n} \right]; \quad V_{s,i} = \int_a^{x_s} v(x_s, t) p_{i,n}(t) dt.$$

Therefore the equation (5) becomes

$$\left[a(x_s)\mathbf{P}(x_s) - \lambda_1\mathbf{F}(x_s) - \lambda_2\mathbf{V}(x_s) \right]\mathbf{Y} = g(x_s).$$

For $s = 0, 1, \dots, n$, this system is written by the matrix equation (3), and this is completed the proof.

Corollary 1. *The matrix equation (3) can be reduced the following matrix forms respectively for linear FVIEs of the first and second kind:*

$$\left[-\lambda_1\mathbf{F} - \lambda_2\mathbf{V} \right]\mathbf{Y} = \mathbf{G} \quad \text{and} \quad \left[\mathbf{P} - \lambda_1\mathbf{F} - \lambda_2\mathbf{V} \right]\mathbf{Y} = \mathbf{G}.$$

Now we follow the some steps to solve the linear FVIE (1).

- (i) Let the equation (3) be written in the compact form

$$\mathbf{W}\mathbf{Y} = \mathbf{G} \quad \text{or} \quad \left[\mathbf{W}; \mathbf{G} \right]$$

so that $\mathbf{W} = \mathbf{A}\mathbf{P} - \lambda_1\mathbf{F} - \lambda_2\mathbf{V}$. The equation corresponds a linear algebraic equation system with n unknowns.

- (ii) If $\text{rank}(\mathbf{W}) = \text{rank}[\mathbf{W}; \mathbf{G}] = n + 1$, then solution of this system is uniquely determined. This kind of systems can be solved by the Gauss elimination, generalized inverse, LU and QR factorization methods.

3 Convergence and error analysis

Definition 4. Error is denoted by $e_n(x) = y(x) - y_n(x)$ such that $y(x)$ is an exact solution and $y_n(x) = B_n(y;x)$ is a generalized Bernstein approximate solution. Then maximum error can be defined as

$$\|e_n\|_\infty = \max_{a \leq x \leq b} |e_n(x)|.$$

Let $y(x_s) \neq 0$ and $B_n(y;x_s)$ be scalars. Then maximum, mean, root of the mean square and absolute relative errors at the collocation points can be numerically computed respectively by

$$E_{\max} = \max_{x_s \in [a,b]} |e_n(x_s)|, \quad E_{\text{mean}} = \frac{1}{n+1} \sum_{s=0}^n |e_n(x_s)|,$$

$$E_{\text{root}} = \sqrt{\frac{1}{n+1} \sum_{s=0}^n (e_n(x_s))^2}, \quad E_{\text{rel}} = \frac{|e_n(x_s)|}{|y(x_s)|}.$$

Besides, residual error of the presented method can be defined by

$$R_n(x) = a(x)B_n(y;x) - \lambda_1 \int_a^b f(x,t)B_n(y;t)dt - \lambda_2 \int_a^x v(x,t)B_n(y;t)dt - g(x). \quad (6)$$

Definition 5. [10] Let $y(x)$ be defined on $[a, b]$, the modulus of continuity of $y(x)$ on $[a, b]$, $\omega(\delta)$, is defined for $\delta > 0$ by

$$\omega(\delta) = \sup_{\substack{x_1, x_2 \in [a,b] \\ |x_1 - x_2| \leq \delta}} |y(x_1) - y(x_2)|.$$

Lemma 1. [10] $y(x)$ is uniformly continuous on $[a, b]$ iff $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$.

Theorem 3. (Uniqueness Theorem) Let the function $a(x) \in \mathbb{C}[a, b]$ and the kernels $f, v \in \mathbb{C}([a, b]^2)$. The equation (1) has a unique solution whenever $0 < q < 1$, $q = M_1 + |\lambda_1|M_2(b-a) + |\lambda_2|M_3(b-a)$.

Proof. Let $T : \mathbb{C}[a, b] \rightarrow \mathbb{C}[a, b]$ be a mapping. Then the equation (1) can be rewritten as

$$Ty = a(x)y(x) - g(x) - \lambda_1 \int_a^b f(x,t)y(t)dt - \lambda_2 \int_a^x v(x,t)y(t)dt,$$

$$Ty^* = a(x)y^*(x) - g(x) - \lambda_1 \int_a^b f(x,t)y^*(t)dt - \lambda_2 \int_a^x v(x,t)y^*(t)dt$$

for $y, y^* \in \mathbb{C}[a, b]$ such that $y = Ty$ and $y^* = Ty^*$.

Besides, $a(x), f(x,t)$ and $v(x,t)$ are bounded functions such that $|a(x)| \leq M_1$, $|f(x,t)| \leq M_2$ and $|v(x,t)| \leq M_3$; $M_1, M_2, M_3 > 0$ because of $a(x) \in \mathbb{C}[a, b]$ and $f, v \in \mathbb{C}([a, b]^2)$. Considering the maximum norm for one variable

function, we have

$$\begin{aligned} \|Ty - Ty^*\|_\infty &\leq M_1 \|y - y^*\|_\infty + |\lambda_1| M_2 \|y - y^*\|_\infty \int_a^b dt + |\lambda_2| M_3 \|y - y^*\|_\infty \int_a^x dt \\ &\leq M_1 \|y - y^*\|_\infty + |\lambda_1| M_2 (b - a) \|y - y^*\|_\infty + |\lambda_2| M_3 (b - a) \|y - y^*\|_\infty \\ &\leq (M_1 + |\lambda_1| M_2 (b - a) + |\lambda_2| M_3 (b - a)) \|y - y^*\|_\infty. \end{aligned}$$

Denoting $q = M_1 + |\lambda_1| M_2 (b - a) + |\lambda_2| M_3 (b - a)$ such that $0 < q < 1$, the above inequality is

$$\|Ty - Ty^*\|_\infty \leq q \|y - y^*\|_\infty.$$

From well-known Banach fixed-point theorem, there exist a unique solution of the equation (1) and this completes the proof.

Theorem 4. [1] Let $B_n(y; x)$ be generalized Bernstein polynomial solution on the interval $[a, b]$. If $y(x)$ is continuous function on the interval $[a, b]$, then the following inequality and uniform convergence are hold:

$$|e_n(x)| \leq \omega\left(n^{-1/2}\right) \left(1 + \sqrt{(x-a)(b-x)}\right) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|e_n\|_\infty = 0.$$

Here $\omega(\delta)$ is modulus of continuity of $y(x)$ on the interval $[a, b]$.

Theorem 5. Let $a(x), y(x)$ be continuous functions on the interval $[a, b]$ and $f(x, t), v(x, t)$ be continuous functions on the square $[a, b] \times [a, b]$. Then residual error of the generalized Bernstein polynomial approach holds for the linear FVIE (1) as follows:

$$\|R_n\|_\infty \leq (\alpha + (b - a) |\lambda_1| \beta + (x - a) |\lambda_2| \gamma) \|e_n\|_\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \|R_n\|_\infty = 0.$$

Here $\alpha = \|a\|_\infty, \beta = \|f\|_\infty$ and $\gamma = \|v\|_\infty$ are positive constants.

Proof. Since y is continuous function on the interval $[a, b]$, it can be denoted as Bernstein approximate solution. Substituting $g(x)$ given in the equation (1) to the absolute value of the residual error (6), the following inequality can be written for the residual error:

$$\begin{aligned} |R_n(x)| &= \left| a(x) B_n(y; x) - \lambda_1 \int_a^b f(x, t) B_n(y; t) dt - \lambda_2 \int_a^x v(x, t) B_n(y; t) dt - g(x) \right| \\ &\leq |a(x)| |B_n(y; x) - y(x)| + |\lambda_1| \int_a^b |f(x, t)| |B_n(y; t) - y(t)| dt + |\lambda_2| \int_a^x |v(x, t)| |B_n(y; t) - y(t)| dt. \end{aligned}$$

Moreover, by using definitions of the maximum error and properties of the norm for one and two variables functions, the residual error bound is obtained as:

$$\begin{aligned} \|R_n\|_\infty &\leq \|a\|_\infty \|e_n\|_\infty + (b - a) |\lambda_1| \|f\|_\infty \|e_n\|_\infty + (x - a) |\lambda_2| \|v\|_\infty \|e_n\|_\infty \\ &\leq (\|a\|_\infty + (b - a) |\lambda_1| \|f\|_\infty + (x - a) |\lambda_2| \|v\|_\infty) \|e_n\|_\infty. \end{aligned}$$

Denoting $\alpha = \|a\|_\infty = \max_{x \in [a, b]} |a(x)|, \beta = \|f\|_\infty = \max_{x, t \in [a, b]} |f(x, t)|$ and $\gamma = \|v\|_\infty = \max_{x, t \in [a, b]} |v(x, t)|$, the desired inequality is founded.

In view of the Theorem 4, the following relation

$$\|R_n\|_\infty \leq \left(\alpha + (b-a)|\lambda_1|\beta + \max_{x \in [a,b]} (x-a)|\lambda_2|\gamma \right) \|e_n\|_\infty \leq c \|e_n\|_\infty$$

and uniform convergence of the residual error are also proved. Here $c = \alpha + (b-a)|\lambda_1|\beta + (b-a)|\lambda_2|\gamma$ is positive constant.

Corollary 2. *The residual error bounds of linear FVIEs of the first and second kind are respectively*

$$\begin{aligned} \|R_n\|_\infty &\leq ((b-a)|\lambda_1|\beta + (x-a)|\lambda_2|\gamma) \|e_n\|_\infty, \\ \|R_n\|_\infty &\leq (1 + (b-a)|\lambda_1|\beta + (x-a)|\lambda_2|\gamma) \|e_n\|_\infty. \end{aligned}$$

Here α, β and γ are positive constants denoted as the Theorem 5.

Theorem 6. *Let y be a continuous function and x_s ($s = 0, 1, \dots, n$) be collocation points on the interval $[a, b]$. Then, the residual error bound at the collocation points for linear FVIEs of the first, second and third kind are*

$$|R_n(x_s)| < \kappa(x_s)(b-a) \left(1 + \frac{b-a}{2}\right) \omega(n^{-1/2})$$

and $\lim_{n \rightarrow \infty} |R_n(x_s)| = 0$. Here κ is a positive constant depended on the collocation points.

Proof. Since y is continuous function on the interval $[a, b]$, it can be denoted as Bernstein approximate solution. So absolute value of residual error can be expressed at the collocation points for linear FVIE (1) of the third kind as

$$|R_n(x_s)| \leq |a(x_s)| |B_n(y; x_s) - y(x_s)| + |\lambda_1| \int_a^b |f(x_s, t)| |B_n(y; t) - y(t)| dt + |\lambda_2| \int_a^{x_s} |v(x_s, t)| |B_n(y; t) - y(t)| dt.$$

Since $B_n(y; x_s) = y(x_s)$ ($s = 0, 1, \dots, n$) from the collocation method and from the Theorem 4, the following inequality can be written for the absolute residual error:

$$|R_n(x_s)| \leq \omega(n^{-1/2}) \left\{ |\lambda_1| \int_a^b |f(x_s, t)| \left[1 + \sqrt{(t-a)(b-t)}\right] dt + |\lambda_2| \int_a^{x_s} |v(x_s, t)| \left[1 + \sqrt{(t-a)(b-t)}\right] dt \right\}.$$

Denoting $\varepsilon_s = \varepsilon(x_s) = |\lambda_1| \max_{t \in [a,b]} |f(x_s, t)|$, $\rho_s = \rho(x_s) = |\lambda_2| \max_{t \in [a,b]} |v(x_s, t)|$, and taking $\max_{t \in [a,b]} \left[1 + \sqrt{(t-a)(b-t)}\right] = 1 + \frac{b-a}{2}$ the residual error becomes

$$\begin{aligned} |R_n(x_s)| &\leq ((b-a)\varepsilon_s + (x_s-a)\rho_s) \left(1 + \frac{b-a}{2}\right) \omega(n^{-1/2}) \\ &\leq (\varepsilon_s + \rho_s)(b-a) \left(1 + \frac{b-a}{2}\right) \omega(n^{-1/2}) \\ &\leq \kappa(x_s)(b-a) \left(1 + \frac{b-a}{2}\right) \omega(n^{-1/2}). \end{aligned}$$

Table 1: Mean errors of Example 1.

n	$x_s = \frac{s}{n}$	$x_s = \cos(\frac{\pi s}{n})$	n	$x_s = \frac{s}{n}$	$x_s = \cos(\frac{\pi s}{n})$
2	0	$3.7e-017$	11	$2.7e-015$	$3.3e-016$
3	$4.9e-017$	$1.1e-016$	12	$1.4e-015$	$3.6e-016$
4	$2.5e-017$	$1.8e-016$	13	$4.4e-015$	$2.5e-016$
5	$3.6e-016$	$5.4e-017$	14	$2.6e-015$	$1.7e-016$
6	$5.4e-016$	$9.8e-017$	16	$1.5e-015$	$3.5e-016$
7	$4.2e-016$	$9.8e-017$	17	$2.1e-014$	$2.7e-016$
8	$2.8e-016$	$2.4e-016$	18	$2.5e-014$	$2.2e-016$
9	$1.2e-016$	$2.0e-016$	19	$8.8e-014$	$3.2e-016$
10	$2.4e-016$	$2.0e-016$	20	$1.5e-013$	$4.0e-016$

Here $\kappa(x_s)$ is bigger constant than ϵ_s and ρ_s for $a \leq x_s \leq b$. Since $\omega(n^{-1/2}) \rightarrow 0$ in view of the Lemma 1, $|R_n(x_s)| \rightarrow 0$ as $n \rightarrow \infty$. The proof is completed. Besides, this theorem is hold for linear FVIEs of the first and second kind because of $B_n(y; x_s) = y(x_s)$.

4 Numerical results

Four numerical examples are given by using the Bernstein collocation method. The collocation points are selected as uniform $x_s = a + \frac{(b-a)s}{n}$ and nonuniform $x_s = \frac{(a+b) - (b-a)\cos(\frac{\pi s}{n})}{2}$; $s = 0, 1, \dots, n$. Numerical results computed in MATLAB 7.1 with 32 digits are compared with the other methods and they are presented via the Tables and Figures.

Example 1. Consider the

$$y(x) = 1 + \int_{-1}^1 (xt + x^2t^2) y(t) dt; \quad -1 \leq x \leq 1$$

linear Fredholm integral equation of the second kind that the exact solution is $y(x) = 1 + \frac{10}{9}x^2$.

The mean errors of Bernstein collocation method obtained at the uniform and nonuniform collocation points with increasing n are given in Table 1. The Table shows that the numerical results obtained at the nonuniform collocation points are better than the results obtained at the uniform collocation points. Mean errors obtained for $n = 4$ by using the numerical method [8] based on the Bernstein basis polynomials is nearby 10^{-13} , whereas mean errors of the presented method is nearby 10^{-17} for $n = 2$. Therefore, we can say that our method is effective even for smaller value n .

Example 2. Consider the

$$y(x) = \cos(x) - e^x \sin(x) + \int_0^x e^x y(t) dt; \quad 0 \leq x \leq 1$$

linear Volterra integral equation of the second kind that analytic solution is $y(x) = \cos x$.

In Table 2, the root of the mean square errors of collocation method based on the generalized Bernstein polynomials at the collocation points $x_s = (1 - \cos(\frac{\pi s}{n}))/2$ for $n \geq 2$ are compared with the Bernstein approach [7]. It is seen from the Table, the numerical results of proposed method converge more rapidly than the other method for $n \geq 4$.

Table 2: Comparison the E_{root} errors for Example 2.

n	Presented method	Bernstein approach [7]
2	$4.7e-003$	$2.0e-003$
3	$3.1e-004$	$2.3e-004$
4	$4.7e-006$	$6.2e-006$
5	$1.8e-007$	$5.4e-007$
6	$2.1e-009$	$1.1e-008$
7	$1.0e-010$	$1.1e-009$
8	$1.3e-012$	$3.4e-010$
9	$5.2e-014$	$3.3e-010$
10	$6.6e-016$	$3.4e-010$

Table 3: Mean errors of Example 3.

n	$x_s = \frac{s}{n}$	$x_s = \frac{1-\cos(\frac{\pi s}{n})}{2}$	n	$x_s = \frac{s}{n}$	$x_s = \frac{1-\cos(\frac{\pi s}{n})}{2}$
2	$6.0e-004$	$6.0e-004$	12	$1.1e-016$	$1.6e-016$
3	$5.8e-005$	$2.5e-005$	17	$3.4e-016$	$1.9e-016$
4	$1.2e-006$	$4.7e-007$	18	$8.8e-015$	$1.9e-016$
5	$1.1e-007$	$1.2e-008$	19	$3.0e-015$	$3.3e-016$
6	$2.0e-009$	$1.3e-010$	20	$1.6e-014$	$2.5e-016$
7	$1.5e-010$	$3.9e-012$	21	$1.2e-013$	$1.8e-016$
8	$2.2e-012$	$3.8e-014$	22	$1.9e-013$	$3.0e-016$
9	$1.4e-013$	$1.4e-015$	23	$9.0e-015$	$2.2e-016$
10	$1.6e-015$	$1.6e-016$	24	$6.4e-013$	$2.0e-016$
11	$1.7e-016$	$2.3e-016$	25	$5.9e-014$	$2.4e-016$

Table 4: Comparison of the E_{rel} errors for Example 3.

x	$n = 5$		$n = 7$	
	Presented method	Galerkin method [11]	Presented method	Galerkin method [11]
0.1	$1.5e-006$	$1.9e-005$	$1.4e-009$	$1.3e-005$
0.2	$3.4e-007$	$7.2e-006$	$1.2e-009$	$3.8e-006$
0.3	$8.6e-007$	$1.0e-005$	$3.4e-010$	$6.1e-006$
0.4	$3.4e-007$	$1.1e-005$	$3.1e-010$	$3.5e-006$
0.5	$2.1e-008$	$4.8e-007$	$6.4e-010$	$7.6e-006$
0.6	$3.4e-007$	$9.2e-006$	$3.4e-010$	$1.4e-006$
0.7	$7.1e-007$	$7.8e-006$	$3.7e-010$	$5.9e-006$
0.8	$3.4e-007$	$3.6e-006$	$8.8e-010$	$3.3e-006$
0.9	$6.0e-007$	$9.3e-006$	$4.5e-010$	$5.5e-006$
1	$3.4e-007$	$2.1e-005$	$4.5e-010$	$1.5e-005$

Example 3. Consider the linear Fredholm integral equation of the second kind:

$$y(x) = e^x + 2 \int_0^1 e^{x+t} y(t) dt; \quad 0 \leq x \leq 1.$$

Exact solution of the above equation is $y(x) = \frac{e^x}{2-e^2}$.

Mean errors obtained by using the presented method are given in Table 3. When the collocation points are not equally spaced, Table 3 shows that results obtained on these points are much better than the other results. The absolute relative

Table 5: Mean errors of Example 4.

n	$x_s = \frac{s}{n}$	$x_s = \frac{1-\cos(\frac{\pi s}{n})}{2}$	n	$x_s = \frac{s}{n}$	$x_s = \frac{1-\cos(\frac{\pi s}{n})}{2}$
2	$9.8e-018$	$9.3e-018$	10	$3.5e-017$	$8.1e-017$
3	$2.3e-017$	$4.3e-017$	14	$7.6e-017$	$7.6e-017$
4	0	$7.4e-018$	15	$7.7e-017$	$6.7e-017$
5	$3.7e-017$	$1.1e-016$	16	$4.7e-016$	$1.1e-016$
6	$8.2e-017$	$1.9e-017$	18	$5.5e-016$	$5.4e-017$
7	$1.3e-016$	$1.8e-016$	20	$5.0e-015$	$9.3e-017$
8	$2.6e-017$	$6.2e-017$	25	$4.8e-014$	$7.6e-017$
9	$5.9e-017$	$7.0e-017$	30	$2.0e-012$	$7.3e-017$

Table 6: Comparison of the E_{max} errors for Example 4.

n	Presented method	Collocation method [3]	Fixed point method [3]	Taylor expansion method [4]
14	$3.3e-016$	–	–	$2.0e-015$
16	$6.7e-016$	$7.8e-005$	$3.8e-004$	–
32	$2.5e-012$	$4.7e-005$	$9.5e-005$	–

errors are compared with the Galerkin method [11] based on the Bernstein basis polynomials in Table 4. It shows that the results obtained on the collocation points $x_s = s/n$ are better than the results given by the other method.

Example 4. Consider the following linear Fredholm-Volterra integral equation of the third kind:

$$3y(x) = 3x^2 - \sin x(x^2 \sin x + 2x \cos x - 2 \sin x - \sin 1 + 2 \cos 1) + \int_0^1 \sin x \cos t y(t) dt + \int_0^x \sin x \cos t y(t) dt$$

Analytic solution of the above equation is $y(x) = x^2$.

The mean errors of the presented method with increasing n are given in Table 5. From the Table, we can see that the numerical results obtained at the not equally spaced points are better than the numerical results obtained at the equally spaced points.

In Figure 1, the absolute error results obtained at the collocation points $x_s = s/n$ by applying derivated Bernstein collocation method are presented for different values n . The figures show that the numerical results converge more rapidly for increasing values n .

The maximum errors obtained with Bernstein collocation method at the number $n + 1$ collocation points $x_s = (1 - \cos(\frac{\pi s}{n})) / 2$ are compared with the maximum errors presented for number n by Taylor expansion method [4], collocation method and fixed point method [3] in Table 6. It shows that the presented method is more effective than the other methods. Besides, the results of proposed method obtained without iteration are better than the results of collocation and fixed point methods [3] obtained with 8th iteration.

5 Conclusions

In this work, a collocation method based on the generalized Bernstein polynomials has been introduced for the numerical solution of linear FVIEs directly. This method is valid for the integral equations whose solution is continuous on the interval $[a, b]$. The error bounds and convergence of the Bernstein collocation method have been presented. Some numerical examples have been given to show the applicability and accuracy of the proposed method. The collocation

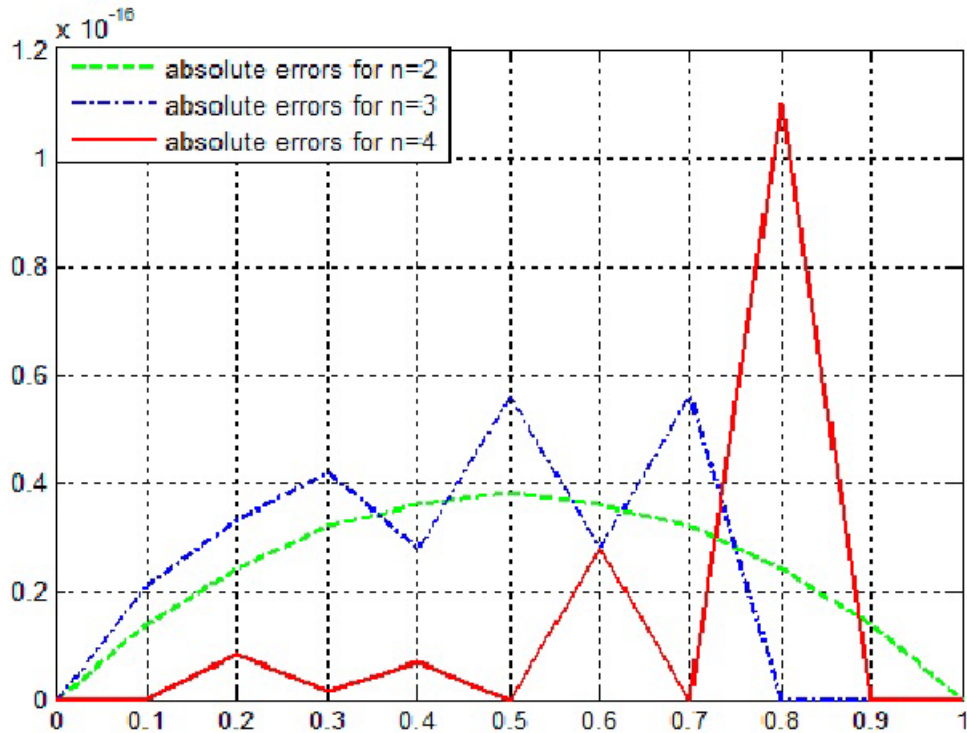


Fig. 1: The results of $|e_n(x)|$ absolute errors for $n = 2, 3, 4$.

method has much better numerical results obtained at nonuniform points than the uniform points for increasing values n . It is also more effective method than the other methods given in the examples (1-4). Consequently, all these positive reasons are encouraged that the proposed method can be applied to the other linear and nonlinear equations.

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