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An Estimating the *p*-adic Sizes of Common Zeros of Partial Derivative Polynomials

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Abstract: Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ be a vector in the space Z^n with Z ring of integers and q be a positive integer, f a polynomial in $\underline{x}$ with coefficient in Z. The exponential sum associated with f is defined as $S(f;q) = \sum \underline{x} mod q e^{\frac{2\pi i f(x)}{q}}$, where the sum is taken over a complete set of residues modulo q. The value of $S(f;q)$ depend on the estimate of cardinality $|V|$, the number of elements contained in the set $V = \left\{\underline{x} mod q | \underline{f_x} = \underline{0} \ mod q\right\}$ where f_x is the partial derivatives of $\underline{f}$ with respect to $\underline{x}$. To determine the cardinality of V, the p-adic sizes of common zeros of the partial derivative polynomials need to be obtained. In this paper we estimate the *p*-adic sizes of common zeros of partial derivative polynomials of $f(x,y)$ in $Z_p[x,y]$ of degree nine by using Newton polyhedron technique. The degree nine polynomial is of the form $f(x,y) = ax^9 + bx^8y + cx^7y^2 + sx + ty + k$.

Keywords: Exponential sums; cardinality; *p*-adic sizes; Newton polyhedron.

1. Introduction

In our discussion, we use notations the ring of *p*-adic integers Z_p, the completion of algebraic closure of Q_p the field of rational *p*-adic numbers Ω_p and $ord_p x$ is the highest power of p dividing x. It follows that for rational number x and y, $ord_p x = \infty$ if and only if $x = 0$, $ord_p(xy = ord_p x + ord_p y \ and \ ord_p(x+y) \geq min\{ord_p x, ord_p y\}$ with equality if $ord_p x \neq ord_p y$.

Loxton and Vaughan (1985) are the researches who investigate the exponential sums $S(f;q) = \sum_{\underline{x} mod q} exp(2\pi i q)$ where f is a nonlinear polynomial in $Z[\underline{x}]$. They showed that the number of common zeros of the partial derivative polynomials of f with respect to $\underline{x}$ modulo q gives the estimation of $S(f;q)$.

From the works of Loxton and Smith (1982), they found that the *p*-adic sizes of common zeros to partial derivative polynomials associated with f in the neighbourhood of points in the product space $\Omega_p^n, n > 0$, can estimate the cardinality of V. Their result is the estimation of $ord_p(\underline{x} - \xi_i)$ that will lead to a derivation of estimate of $N\left(\underline{f}, p^\alpha\right)$.

The estimations for lower degree two-variable polynomials by using Newton polyhedron technique are found by many researchers such as Mohd. Atan (1986), Chan and Mohd. Atan (1997) who estimates the cardinality $N(f;p^\alpha)$ of the set of solutions to congruence equations modulo a prime power and also Heng and Mohd. Atan (1999). However, results for the polynomials of higher degrees are less complete.

Our approach entails the work developed by Mohd. Atan and Loxton (1986) who presented the *p*-adic Newton polyhedral method of finding the *p*-adic order of polynomials in $\Omega_p[x,y]$ which is an analogue of Newton polygon defined by Koblitz (1977). Sapar and Mohd. Atan (2002) improved the result and then Yap, Sapar and Mohd. Atan (2011) showed that the *p*-adic sizes of common zeros of partial derivative polynomials associated with a cubic form can be found explicitly on the overlapping segment of the indicator diagrams associated with the polynomials by using Newton polyhedron technique.

Our work involves application of the Newton polyhedron technique at the point of intersection in the combination of indicator diagrams to determine explicitly the p-adic sizes of the component (ξ, η) a common root of partial derivative polynomials of $f(x, y)$ in $Z_p[x, y]$ of degree nine.

2. *p*-ADIC Orders of Zeros of A Polynomial

Sapar and Mohd Atan (2002) proved that for every point of intersection of the indicator diagrams, there exist common zeros of both polynomials in $Z_p[x, y]$ whose p-adic orders correspond to point (μ, λ) as mention in Theorem 2.1 below:

Theorem 2.1. Let p be a prime. Suppose f and g are polynomials in $Z_p[x, y]$. Let (μ, λ) be a point of intersection of the indicator diagrams associated with f and g at the vertices or simple points of intersections. Then there are ξ and η in Ω_p satisfying $f\ (\xi, \eta) = g\ (\xi, \eta) = 0$ and $ord_p \xi = \mu,\ ord_p \eta = \lambda$.

Our investigation concentrates on the p-adic sizes of common zeros of partial derivative associated with a polynomial $f(x, y) = ax^9 + bx^8y + cx^7y^2 + sx + ty + k$. First we prove the following lemma.

Lemma 2.1. Let $p > 7$ be a prime, a, b and c in Z_p and λ_1, λ_2 zeros of $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac$. Let

$$\alpha_1 = \frac{4b + \lambda_1 c}{9a + \lambda_1 b}\ and\ \alpha_2 = \frac{4b + \lambda_2 c}{9a + \lambda_2 b}.$$

If $ord_p b^2 > ord_p ac$, then $ord_p \alpha_i = ord_p(\alpha_1 - \alpha_2) = \frac{1}{2} ord_p \frac{c}{a}$, for $i = 1,2$ and $ord_p(\alpha_1 + \alpha_2) = ord_p \frac{b}{a}$.

Proof. The zeros of $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac$ are given by

$$\lambda_1 c = \frac{-b \pm \sqrt{252ac - 63b^2}}{2}, for\ i = 1,2.$$

Since $ord_p b^2 > ord_p ac$ and $p > 7$, we have $ord_p \lambda_i c = \frac{1}{2} ord_p ac, i = 1,2$.

Hence, $ord_p \lambda_i c = \frac{1}{2} ord_p ac < ord_p b$. Therefore,

$$ord_p(4b + \lambda c) = ord_p \lambda c = \frac{1}{2} ord_p ac. \tag{2.1}$$

It can be shown that $ord_p \lambda > ord_p a$.
It follows that, $ord_p(9a + \lambda b) = ord_p a$. (2.2)

From (2.1) and (2.2), since $p > 7$, we have

$$ord_p \alpha_i = ord_p \frac{4b + \lambda_i c}{9a + \lambda_i b} = \frac{1}{2} ord_p ac - ord_p a, \qquad i = 1,2.$$

That is $ord_p \alpha_i = \frac{1}{2} ord_p \frac{c}{a}, for\ i = 1,2.$ (2.3)

Clearly,

$$ord_p(\alpha_1 - \alpha_2) = ord_p \frac{(\lambda_1 - \lambda_2)(9ac - 4b^2)}{(9a + \lambda_1 b)(9a + \lambda_2 b)}$$

where $\lambda_1 - \lambda_2 = \frac{\sqrt{252ac-63b^2}}{c}$.

Thus,

$$ord_p(\alpha_1 - \alpha_2) = ord_p\sqrt{252ac - 63b^2} - ord_p c + ord_p(9ac - 4b^2) - ord_p(9a + \lambda_1 b) - ord_p(9a + \lambda_2 b).$$

Since $p > 7, ord_p b^2 > \ ord_p ac$ and by (2.1), (2.2) and (2.3) we have

$$ord_p(\alpha_1 - \alpha_2) = \frac{1}{2} ord_p \frac{c}{a}, for\ i = 1,2.$$

It can be shown that

$$ord_p(\alpha_1 + \alpha_2) = ord_p \frac{72ab + 2bc\lambda_1\lambda_2 + (9ac - 4b^2)(\lambda_1 + \lambda_2)}{(9a + \lambda_1 b)(9a + \lambda_2 b)} \tag{2.4}$$

where $\lambda_1\lambda_2 = \frac{64b^2-252ac}{4c^2}$ and $\lambda_1 + \lambda_2 = -\frac{b}{a}$.

Since $p > 7, ord_p b^2 > \ ord_p ac$ and $ord_p(9a + \lambda_i b) = ord_p a, i = 1,2$ and simplifying (2.4) we have

$$ord_p(\alpha_1 + \alpha_2) = ord_p \frac{b}{a}$$

as asserted.

Throughout the following discussion,

$$\alpha_1 = \frac{4b + \lambda_1 c}{9a + \lambda_1 b}\ and\ \alpha_2 = \frac{4b + \lambda_2 c}{9a + \lambda_2 b} \tag{2.5}$$

with λ_1, λ_2 zeros of $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac. \alpha_1 \neq \alpha_2$ since $\lambda_1 \neq \lambda_2$.

Lemma 2.2. Suppose (U,V) in Ω_p^2. Let $p > 7$ be a prime, a, b and c coefficient of $\alpha_1$and α_2 as in (2.5) Z_p. If $ord_p b^2 > ord_p ac$, then $ord_p(\alpha_1 V - \alpha_2 U) = ord_p\left[7b(U - V) + \sqrt{252ac - 63b^2}(U + V)\right] - ord_p a.$

Proof.

$$\begin{aligned} ord_p(\alpha_1 V - \alpha_2 U) &= ord_p\left(\frac{4b + \lambda_1 c}{9a + \lambda_1 b}V - \frac{4b + \lambda_2 c}{9a + \lambda_2 b}U\right) \\ &= ord_p[(4b + \lambda_1 c)(9a + \lambda_2 b)V - (4b + \lambda_2 c)(9a + \lambda_1 b)U] - ord_p(9a + \lambda_1 b) - ord_p(9a + \lambda_2 b). \end{aligned} \tag{2.6}$$

Now, let λ_1 and λ_2 be the zeros of $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac$ are of the form

$$\lambda_1 = \frac{-b + \sqrt{252ac - 63b^2}}{2c} \text{ and } \lambda_2 = \frac{-b - \sqrt{252ac - 63b^2}}{2c}.$$

From (2.6), we have

$$(4b + \lambda_1 c)(9a + \lambda_2 b)V - (4b + \lambda_2 c)(9a + \lambda_1 b)U = \left(\frac{9ac-4b^2}{2c}\right)\left[7b(U - V) + \sqrt{252ac - 63b^2}(U + V)\right].$$

Therefore, from (2.6) we have

$$ord_p(\alpha_1 V - \alpha_2 U) = ord_p\left(\frac{9ac - 4b^2}{2c}\right)\left[7b(U - V) + \sqrt{252ac - 63b^2}(U + V)\right] - ord_p(9a + \lambda_1 b) - ord_p(9a + \lambda_2 b)$$

Since $ord_p b^2 > ord_p ac$, and by proof of Lemma 2.1, $ord_p(9a + \lambda_i b) = ord_p a$ for $i = 1,2$, we obtain

$$ord_p(\alpha_1 V - \alpha_2 U) = ord_p\left[7b(U-V) + \sqrt{252ac - 63b^2}(U+V)\right] - ord_p a$$

as asserted.

From the above result, it is clear that to ascertain the p-adic sizes of $ord_p(\alpha_1 V - \alpha_2 U)$ we need to examine the p-adic size of $\left[7b(U-V) + \sqrt{252ac - 63b^2}(U+V)\right]$. To do this, the sizes of each quantity in the expression should be considered. This is done in the proof of the following assertion.

Lemma 2.3. Suppose(x, y) in Ω_p^2 and $U = x^4 + \alpha_1 x^3 y, V = x^4 + \alpha_2 x^3 y$ where α_1 and α_2 as in (2.5). Let $p > 7$ be a prime, a, b and c coefficient of $\alpha_1$and α_2 $Z_p$and $ord_p b^2 > ord_p ac$. Then $ord_p x \geq \frac{1}{4} W$ and $ord_p y \geq \frac{1}{4}\Big[W -$ *12ordpcb6a7* or *ordpy≥14W−12ordpcb6a7−3ε* in an exceptional case with *W=minordpV,ordpU* and some *ε≥0* which can be specified explicitly.

Proof. From $U = x^4 + \alpha_1 x^3 y$ and $V = x^4 + \alpha_2 x^3 y$, we have

$$x = \left(\frac{\alpha_1 V - \alpha_2 U}{\alpha_1 - \alpha_2}\right)^{\frac{1}{4}} \text{ and } y = \frac{U - V}{(\alpha_1 - \alpha_2)x^3}.$$

Thus,$ord_p x = \frac{1}{4} ord_p(\alpha_1 V - \alpha_2 U) - \frac{1}{4} ord_p(\alpha_1 - \alpha_2)$ (2.7)

and $ord_p y = ord_p(U - V) - ord_p(\alpha_1 - \alpha_2) - ord_p x^3$. (2.8)

By (2.7), Lemmas 2.1 and 2.2, we have

$$ord_p x = \frac{1}{4} ord_p\left[7b(U-V) + \sqrt{252ac - 63b^2}(U+V)\right] - \frac{1}{8} ord_p ac.$$

Now, we have to consider two cases.

<u>Case 1:</u> $\left\{ord_p 7b(U-V) \neq ord_p \sqrt{252ac - 63b^2}(U+V)\right\}$

(i) Suppose $min\left\{ord_p 7b(U-V), ord_p \sqrt{252ac - 63b^2}(U+V)\right\} = ord_p \sqrt{252ac - 63b^2}(U+V)$

It follow that, $ord_p x = \frac{1}{4} ord_p \sqrt{252ac - 63b^2}(U+V) - \frac{1}{8} ord_p ac$.

Since $p > 7$ and $ord_p b^2 > ord_p ac$, we have

$$ord_p x = \frac{1}{4} ord_p(U+V) \tag{2.9}$$

It follow that, $ord_p x \geq \frac{1}{4} W$.

From the definition of U and V,

$$ord_p(U+V) = ord_p(2x^4 + (\alpha_1 + \alpha_2)x^3 y).$$

From (2.9),

$$ord_p x^4 = ord_p(U+V).$$

Thus,

$$ord_p x \leq ord_p(\alpha_1 + \alpha_2)y.$$

Hence from (2.8), we have

$$ord_p y \geq ord_p(U-V) - ord_p(\alpha_1 - \alpha_2) - ord_p(\alpha_1 + \alpha_2)^3 - 3ord_p y$$

and from the proof of Lemma 2.1 and simplify it, we have

$$ord_p y \geq ord_p(U-V) - \frac{1}{2}\left[ord_p \frac{c}{a} ord_p \frac{b^3}{a^3}\right]$$

by Lemma 2.1, we have

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right].$$

Hence, in this case,

$$ord_p x \geq \frac{1}{4} W \text{ and } ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right].$$

(ii) Suppose $min\{ord_p 7b(U-V), ord_p\sqrt{252ac - 63b^2}(U+V)\} = ord_p 7b(U-V)$ (2.10)

We have

$$ord_p x = \frac{1}{4} ord_p 7b(U-V) - \frac{1}{8} ord_p ac = \frac{1}{4} ord_p(U-V) + \frac{1}{8}\left(ord_p b^2 - ord_p ac\right). \quad (2.11)$$

Since $ord_p b^2 > ord_p ac$, we have

$$ord_p x^4 \geq ord_p(U-V).$$

That is, $ord_p x = \frac{1}{4} W$.

By (2.10) and (2.11),

$$ord_p x \leq \frac{1}{4} ord_p\sqrt{252ac - 63b^2}(U+V) - \frac{1}{8} ord_p ac = \frac{1}{8} ord_p ac + \frac{1}{4} ord_p(U+V) - -\frac{1}{8} ord_p ac.$$

That is, $ord_p x \leq \frac{1}{4} ord_p(U+V)$.

Now $(U+V) = 2x^4 + (\alpha_1 + \alpha_2)x^3 y$. Thus,

$$ord_p x \leq ord_p(2x + (\alpha_1 + \alpha_2)y).$$

It follows that,

$$ord_p x \leq ord_p(\alpha_1 + \alpha_2)y.$$

By (2.8), and the same argument as in (i) we have,

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right].$$

<u>Case2:</u> $\{ord_p 7b(U-V) = ord_p\sqrt{252ac - 63b^2}(U+V)\}$.

We have

$$ord_p x = \frac{1}{4}\left[ord_p 7b(U-V) + ord_p\sqrt{252ac-63b^2}(U+V)\right] - \frac{1}{8}ord_p ac$$
$$\geq \frac{1}{4}min\left\{ord_p 7b(U-V), ord_p\sqrt{252ac-63b^2}(U+V)\right\} - \frac{1}{8}ord_p ac.$$

Since $ord_p 7b(U-V) = ord_p\sqrt{252ac-63b^2}(U+V)$ and $p > 7$,

$$ord_p x \geq \frac{1}{4}ord_p\sqrt{252ac-63b^2}(U+V) - \frac{1}{8}ord_p ac.$$

Since $p > 7$ and $ord_p b^2 > ord_p ac$, we have

$$ord_p x \geq \frac{1}{4}ord_p(U+V) + \frac{1}{8}\left(ord_p ac - ord_p ac\right).$$

Therefore, $ord_p x \geq \frac{1}{4}ord_p(U+V)$.

It follows that, $ord_p x \geq \frac{1}{4}W$.

From (2.7) and (2.8), we obtain

$$ord_p y = ord_p(U-V) - ord_p(\alpha_1-\alpha_2) - 3\left[\frac{1}{4}ord_p(\alpha_1 V - \alpha_2 U) - \frac{1}{4}ord_p(\alpha_1-\alpha_2)\right].$$

By Lemmas 2.1 and 2.2, we obtain

$$ord_p y = ord_p(U-V) - \frac{1}{8}ord_p\frac{c}{a^7} - \frac{3}{4}ord_p\left[7b(U-V) + \sqrt{252ac-63b^2}(U+V)\right]. \quad (2.12)$$

Let, $\beta = ord_p 7b(U-V) = ord_p\sqrt{252ac-63b^2}(U+V)$. (2.13)

Then, there exist k and l such that,

$7b(U-V) = p^\beta k$ with $ord_p k = 0$ and $\sqrt{252ac-63b^2} = p^\beta l$ with $ord_p l = 0$.

From (2.13), $ord_p(U-V) = \beta - ord_p b$. Hence from (2.12), we have

$$ord_p y = \frac{1}{4}\beta - ord_p b - \frac{1}{8}ord_p\frac{c}{a^7} - \frac{3}{4}ord_p(k+l).$$

Let $\varepsilon = ord_p(k+l)$, then

$$ord_p y = \frac{1}{4}ord_p(U-V) - \frac{1}{8}ord_p\frac{cb^6}{a^7} - \frac{3}{4}\varepsilon.$$

It follows that,

$$ord_p y \geq \frac{1}{4}W - \frac{1}{8}ord_p\frac{cb^6}{a^7} - \frac{3}{4}\varepsilon.$$

Hence, we have

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2}ord_p\frac{cb^6}{a^7} - 3\varepsilon\right]$$

with $W = \left\{ord_p V, ord_p U\right\}$ and $\varepsilon ord_p(k+l)$.

Therefore, $ord_p x \geq \frac{1}{4}W$ and $ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2}ord_p \frac{cb^6}{a^7}\right]$ or $ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2}ord_p \frac{cb^6}{a^7} - 3\varepsilon\right]$

with $W = \{ord_p V, ord_p U\}$ and $\varepsilon \geq 0$ as asserted.

The following lemma gives explicit estimates of the components x, y in U and V in terms of p-adic sizes of integers in Z_p where U and V as in Lemma 2.3. the proof utilizes the result obtained above.

Lemma 2.4. Suppose(x, y) in Ω_p^2 and $U = x^4 + \alpha_1 x^3 y, V = x^4 + \alpha_2 x^3 y$ where α_1 and α_2 as in (2.5). Let $p > 7$ be a prime, a, b, c, s and t in Z_p $ord_p b^2 > ord_p ac, \delta = max\{ord_p a, ord_p b, ord_p c\}$ and $ord_p s, ord_p t \geq \delta$. If $ord_p U = \frac{1}{2}ord_p \frac{s+\lambda_1 t}{9a+\lambda_1 b}$ and $ord_p V = \frac{1}{2}ord_p \frac{s+\lambda_2 t}{9a+\lambda_2 b}$ then $ord_p x \geq \frac{1}{8}(\alpha - \delta)$ and $ord_p y \geq \frac{1}{8}(\alpha - \delta)$ or $ord_p y \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$ for some $\varepsilon \geq 0$.

Proof. Since $U = x^4 + \alpha_1 x^3 y, V = x^4 + \alpha_2 x^3 y$ and $ord_p b^2 > ord_p ac$, we have from Lemma 2.3

$$ord_p x \geq \frac{1}{4}W \tag{2.14}$$

where $W = min\{ord_p U, ord_p V\}$.

Now,

$$ord_p U = \frac{1}{2}ord_p \frac{s + \lambda_1 t}{9a + \lambda_1 b} \text{ and } ord_p V = \frac{1}{2}ord_p \frac{s + \lambda_2 t}{9a + \lambda_2 b}.$$

It follows from (2.14) that

$$ord_p x \geq \frac{1}{8}ord_p \frac{s + \lambda_i t}{9a + \lambda_i b}, \;\; i = 1 \text{ or } 2.$$

By proof of Lemma 2.1, $ord_p(9a + \lambda_i b) = ord_p a$ for $i = 1, 2$. As such

$$ord_p x \geq \frac{1}{8}\left[ord_p(s + \lambda_i t) - ord_p a\right] \tag{2.15}$$

If $min\{ord_p s, ord_p \lambda_i t\} = ord_p s, i = 1, 2$ then

$$ord_p x \geq \frac{1}{8}\left(ord_p s - ord_p a\right).$$

By the hypothesis, we obtain

$$ord_p x \geq \frac{1}{8}(\alpha - \delta).$$

Now, if $min\{ord_p s, ord_p \lambda_i t\} = ord_p \lambda_i t, i = 1, 2$ then

$$ord_p x \geq \frac{1}{8}\left[ord_p \lambda_i t - ord_p a\right].$$

Since $ord_p a \leq ord_p \lambda_i b, i = 1, 2$ it follows that

$$ord_p x \geq \frac{1}{8}\left[ord_p t - ord_p b\right].$$

By the hypothesis, we obtain

$$ord_p x \geq \frac{1}{8}(\alpha - \delta).$$

By Lemma 2.3, we have

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right] \text{ or } ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7} - 3\varepsilon\right] \tag{2.16}$$

for some $\varepsilon \geq 0$ where $W = min\{ord_p U, ord_p V\}$.

For the first inequality we have from (2.16),

$$ord_p y \geq \frac{1}{4}\left[\frac{1}{2} ord_p \left(\frac{s + \lambda_i t}{9a + \lambda_i b}\right) - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right], i = 1, 2.$$

Since $ord_p(9a + \lambda_i b) = ord_p a$ for $i = 1, 2$,

$$ord_p y \geq \frac{1}{8}\left[ord_p(s + \lambda_i t) + ord_p a^6 - ord_p b^6 c\right] \tag{2.17}$$

Since $ord_p b^2 > ord_p ac$, we have

$$ord_p y \geq \frac{1}{8}\left[ord_p(s + \lambda_i t) + ord_p a\right].$$

By using the same method as equation (2.15), we have

$$ord_p y \geq \frac{1}{8}(\alpha - \delta).$$

Now, we consider the second inequality,

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7} - 3\varepsilon_0\right]$$

with $W = min\{ord_p U, ord_p V\}$ and for some $\varepsilon_0 \geq 0$.

By the same argument for the first inequality not involving ε_0, we let $\varepsilon = 3\varepsilon_0$ and we will arrive at

$$ord_p y \geq \frac{1}{8}(\alpha - \delta - \varepsilon).$$

Therefore, $ord_p x \geq \frac{1}{8}(\alpha - \delta)$ and $ord_p y \geq \frac{1}{8}(\alpha - \delta)$ or $ord_p y \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$

as asserted.

The next theorem will gives the p-adic sizes of common zeros of partial derivative polynomials associated with a polynomial $f(x, y)$ in $Z_p[x, y]$, in terms of the coefficients of its dominant terms.

Theorem 2.2. Let $f(x, y) = ax^9 + bx^8 y + cx^7 y^2 + sx + ty + k$ be a polynomial in $Z_p[x, y]$ with $p > 7$. Let $\alpha > 0, \delta = max\{ord_p a, ord_p b, ord_p c\}$ and $ord_p b^2 > ord_p ac$. İf $ord_p f_x(0,0)$, $ord_p f_y(0,0) \geq \alpha > \delta$ then there exists (ξ, η) such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ and $ord_p \xi \geq \frac{1}{8}(\alpha - \delta)$, $ord_p \eta \geq \frac{1}{8}(\alpha - \delta)$ or in an exceptional case $ord_p \eta \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$ with a certain $\varepsilon \geq 0$.

Proof. Let $g = f_x$ and $h = f_y$ and λ be a constant where, $g = f_x = 9ax^8 + 8bx^7 y + 7cx^6 y^2 + s$ and $h = f_y = bx^8 + 2cx^7 y + t$.

Then,

$(g + \lambda h)(x, y) = (9a + \lambda b)x^8 + (8b + 2\lambda c)x^7y + 7cx^6y^2 + s + \lambda t$. That is

$$\frac{(g + \lambda h)(x, y)}{9a + \lambda b} = x^8 + \left(\frac{8b + 2\lambda c}{9a + \lambda b}\right)x^7y + \left(\frac{7c}{9a + \lambda b}\right)x^6y^2 + \frac{s + \lambda t}{9a + \lambda b}. \tag{2.18}$$

By completing the square in (2.18), we have

$$\frac{(g + \lambda h)(x, y)}{9a + \lambda b} = \left(x^4 + \frac{4b + \lambda c}{9a + \lambda b}x^3y\right)^2 + \frac{s + \lambda t}{9a + \lambda b}. \tag{2.19}$$

with

$$\frac{7c}{9a + \lambda b} - \left(\frac{4b + \lambda c}{9a + \lambda b}\right)^2 = 0 \tag{2.20}$$

That is, $c^2\lambda^2 + bc\lambda + 16b^2 - 63ac = 0$.

From the equation (2.20) above, we have

$$\lambda_1 = \frac{-b + \sqrt{252ac - 63b^2}}{2c} \text{ and } \lambda_2 = \frac{-b - \sqrt{252ac - 63b^2}}{2c}.$$

Let the above λ_1, λ_2 be the zeros of the equation (2.20) whose expressions are given in Lemma 2.1. $\lambda_1 \neq \lambda_2$, since $ord_pb^2 > ord_pac$ implies $b^2 \neq 4ac$.

Now let

$$U = x^4 + \frac{4b + \lambda_1 c}{9a + \lambda_1 b}x^3y \tag{2.21}$$

$$V = x^4 + \frac{4b + \lambda_2 c}{9a + \lambda_2 b}x^3y \tag{2.22}$$

$$F(U, V) = (g + \lambda_1 h)(x, y) \tag{2.23}$$

and

$$F(U, V) = (g + \lambda_2 h)(x, y) \tag{2.24}$$

Substitution of U and V in (2.19), for $i = 1, 2$, we have polynomials in (U, V),

$$F(U, V) = (9a + \lambda_1 b)U^2 + s + \lambda_1 t \tag{2.25}$$

$$F(U, V) = (9a + \lambda_2 b)U^2 + s + \lambda_2 t \tag{2.26}$$

The combination of the indicator diagrams associated with the Newton polyhedron of (2.25) and (2.26) is shown in figure below

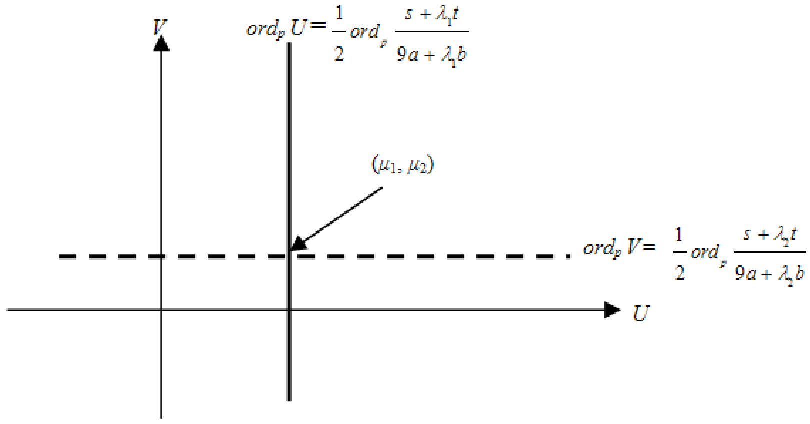

Figure 2.2.1. The indicator diagrams of $F(U,V) = (9a+\lambda_1 b)U^2 + s + \lambda_1 t$ (bold line) and $F(U,V) = (9a+\lambda_2 b)U^2 + s + \lambda_2 t$ (broken line)

From Figure 2.2.1 and by Theorem 2.1, there exists $(\hat{U},\hat{V})$ in Ω_p^2 such that $F(\hat{U},\hat{V}) = 0$, $F(\hat{U},\hat{V}) = 0$ and $ord_p\hat{U} = \mu_1$, $ord_p\hat{V} = \mu_2$ with $\mu_1 = \frac{1}{2}ord_p\frac{s+\lambda_1 t}{9a+\lambda_1 b}$ and $\mu_2 = \frac{1}{2}ord_p\frac{s+\lambda_2 t}{9a+\lambda_2 b}$.

Suppose $U = \hat{U}$ and $V = \hat{V}$ in (2.21) and (2.22). Thus, there exists (x_0, y_0) in Ω_p^2 such that

$$\hat{U} = x_0^4 + \alpha_1 x_0^3 y_0 \qquad (2.27)$$
$$\hat{V} = x_0^4 + \alpha_2 x_0^3 y_0 \qquad (2.28)$$

with $\alpha_1 = \frac{4b+\lambda_1 c}{9a+\lambda_1 b}$ and $\alpha_2 = \frac{4b+\lambda_2 c}{9a+\lambda_2 b}$, λ_1, λ_2 the zeros $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac$. $\alpha_1 \neq \alpha_2$ since $\lambda \neq \lambda$.

By solving (2.27) and (2.28) simultaneously, we have

$$x_0 = \left(\frac{\alpha_1\hat{V} - \alpha_2\hat{U}}{\alpha_1 - \alpha_2}\right)^{\frac{1}{4}} \; and \; y_0 = \frac{\hat{U} - \hat{V}}{(\alpha_1 - \alpha_2)x_0^3}.$$

That is,

$$ord_p x_0 = \frac{1}{4}ord_p(\alpha_1\hat{V} - \alpha_2\hat{U}) - \frac{1}{4}ord_p(\alpha_1 - \alpha_2) \qquad (2.7)$$

and

$$ord_p y_0 = ord_p(\hat{V} - \hat{U}) - ord_p(\alpha_1 - \alpha_2) - ord_p x_0^3.$$

From Lemma 2.4, we have

$$ord_p x_0 \geq \frac{1}{8}(\alpha - \delta), ord_p y_0 \geq \frac{1}{8}(\alpha - \delta) \text{ or } ord_p y_0 \geq \frac{1}{8}(\alpha - \delta - \varepsilon) \text{ for some } \varepsilon \geq 0.$$

Let $x_0 = \xi$ and $y_0 = \eta$. Since $F(\hat{U},\hat{V}) = 0$ and $G(\hat{U},\hat{V}) = 0$, by back substitution in (2.23) and (2.24) we would have $g(\xi,\eta) = f_x(\xi,\eta) = 0$ and $h(\xi,\eta) = f_y(\xi,\eta) = 0$. Thus, $ord_p\xi \geq \frac{1}{8}(\alpha - \delta)$, $ord_p\eta \geq \frac{1}{8}(\alpha - \delta)$ or $ord_p\eta \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$ where (ξ,η) is a common zero of f_x and f_y $\delta = max\{ord_p a, ord_p b, ord_p c\}$, for some $\varepsilon \geq 0$.

3. Conclusion

From this project, we found that if p is a prime, $p > 7, f(x,y) = ax^9 + bx^8 y + cx^7 y^2 + sx + ty + k$ with all coefficients in Z_p such that for $\alpha > 0, \delta = max\{ord_p a, ord_p b, ord_p c\}$ and $ord_p b^2 > ord_p ac$ if $ord_p f_x(0,0)$,

$ord_p f_y(0,0) \geq \alpha > \delta$ then there exists (ξ, η) such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ and $ord_p \xi \geq \frac{1}{8}(\alpha - \delta)$, $ord_p \eta \geq \frac{1}{8}(\alpha - \delta)$ or in an exceptional case $ord_p \eta \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$ with a certain $\varepsilon \geq 0$.

The *p*-adic sizes of common zeros that we obtained in this project can be used to find the cardinality $|V|$ and through that we can solve the exponential sums $S(f;q) = \sum \underline{x} mod q exp(2\pi i/q)$ that depended from estimate of cardinality. Therefore, we also suggest that by using the same technique as in this project, the *p*-adic sizes of common zeros of partial derivative polynomials associated with much higher degree two-variable polynomials also can be found.

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