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Mannheim partner D-curves in the Euclidean 3-space E^3

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Abstract: In this paper, we consider the idea of Mannheim partner curves for curves lying on surfaces. By considering the Darboux frames of surface curves, we define Mannheim partner D-curves and give the characterizations for these curves. We also find the relations between geodesic curvatures, normal curvatures and geodesic torsions of these associated curves. Furthermore, we show that definition and characterizations of Mannheim partner D-curves include those of Mannheim partner curves in some special cases.

Keywords: Mannheim partner D-curves; Darboux frame; geodesic curvature; normal curvature; geodesic torsion.

1. Introduction

Associated curves, the curves for which at the corresponding points of curves one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of other curve, are very interesting study and an important problem of the fundamental curve theory and characterizations of space curves. The well-known of such curves is Bertrand curve which is characterized as a kind of corresponding relation between the two curves. The relation is that the principal normal of a curve is the principal normal of another curve i.e, the Bertrand curve is a curve which shares the normal line with another curve. Over years many mathematicians have studied on Bertrand curves in different spaces and consider the properties of these curves[1-5,15]. Moreover, Ravani and Ku consider the notion of Bertrand curves for ruled surfaces and defined Bertrand offsets of ruled surfaces [12].

Recently, a new definition of the associated curves was given by Liu and Wang [9,14]. They called these new curves as Mannheim partner curves: Let x and x_1 be two curves in the three dimensional Euclidean E^3. If there exists a corresponding relationship between the space curves x and x_1 such that, at the corresponding points of the curves, the principal normal lines of x coincides with the binormal lines of x_1, then x is called a Mannheim curve, and x_1 is called a Mannheim partner curve of x. The pair $\{x, x_1\}$ is said to be a Mannheim pair. They showed that a curve $x_1(s_1)$ is a Mannheim partner curve of the curve $x(s)$ if and only if the curvature κ_1 and the torsion τ_1 of $x_1(s_1)$ satisfy the following equation for some non-zero constant λ.

$$\dot{\tau} = \frac{d\tau}{ds_1} = \frac{\kappa_1}{\lambda}(1+\lambda^2\tau_1^2)$$

Mannheim partner curves have been studied in Minkowski 3-space by Kahraman et al [8]. Similar to the Bertrand offsets, Orbay, Kasap and Aydemir have defined and characterized the Mannheim offsets of ruled surfaces [11]. The corresponding characterizations of the Mannheim offsets of timelike and spacelike ruled surfaces in Minkowski 3-space have been given by Önder and Uğurlu [6,7].

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In this paper we consider the notion of Mannheim partner curve for the curves lying on different surfaces. We call these new associated curves as Mannheim partner D-curves and by using the Darboux frame of curves we give definition and characterizations of these curves.

2. Darboux Frame of a Curve Lying on a Surface

Let S be an oriented surface in 3-dimensional Euclidean space E^3 and let consider a curve $x(s)$ lying on S entirely where s is the arclength of $x(s)$. Since the curve $x(s)$ is also in space, there exists a Frenet frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ at each points of the curve where $\mathbf{T}$ is unit tangent vector, $\mathbf{N}$ is principal normal vector and $\mathbf{B}$ is binormal vector, respectively. The Frenet equations of the curve $x(s)$ is given by

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N} \\ \mathbf{N}' &= -\kappa \mathbf{T} + \tau \mathbf{B} \\ \mathbf{B}' &= -\tau \mathbf{N} \end{aligned}$$

where κ and τ are curvature and torsion of the curve $x(s)$, respectively and (') denotes the derivative with respect to s [10,13].

Since the curve $x(s)$ lies on the surface S there exists another frame of the curve $x(s)$ which is called Darboux frame and denoted by $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$. In this frame $\mathbf{T}$ is the unit tangent of the curve, $\mathbf{n}$ is the unit normal of the surface Salong the curve and $\mathbf{g}$ is a unit vector given by $\mathbf{g} = \mathbf{n} \times \mathbf{T}$. Since the unit tangent $\mathbf{T}$ is common in both Frenet frame and Darboux frame, the vectors $\mathbf{N}$, $\mathbf{B}$, $\mathbf{g}$ and $\mathbf{n}$ lie on the same plane. Then the relations between these frames can be given as follows

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where φ is the angle between the vectors $\mathbf{g}$ and $\mathbf{N}$. The derivative formulae of the Darboux frame is

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\vec{g}} \\ \dot{\vec{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix} \tag{1}$$

where k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve [10].

The relations between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given as follows

$$k_g = \kappa\cos\varphi,\ k_n = \kappa\sin\varphi,\ \tau_g = \tau + \frac{d\varphi}{ds}. \tag{2}$$

Furthermore, the geodesic curvature k_g and geodesic torsion τ_g of the curve $x(s)$ can be calculated as follows

$$k_g = \left\langle \frac{d\mathbf{x}}{ds}, \frac{d^2\mathbf{x}}{ds^2} \times \mathbf{n} \right\rangle,\ \tau_g = \left\langle \frac{d\mathbf{x}}{ds}, \mathbf{n} \times \frac{d\mathbf{n}}{ds} \right\rangle \tag{3}$$

In the differential geometry of surfaces, for a curve $x(s)$lying on a surface S the followings are well-known

(i) $x(s)$ is a geodesic curve $\Leftrightarrow \ k_g = 0$,

(ii) $x(s)$ is an asymptotic line $\Leftrightarrow \ k_n = 0$,

(iii) $x(s)$ is a principal line $\Leftrightarrow$ $\tau_g = 0$ [10].

Through every point of the surface passes a geodesic in every direction. A geodesic is uniquely determined by an initial point and the tangent at that point. All straight lines on a surface are geodesics. Along all curved geodesics the principal normal coincides with the surface normal. Along asymptotic lines osculating planes and tangent planes coincide, along geodesics they are normal. Through a point of a nondevelopable surface pass two asymptotic lines which can be real or imaginary [13].

3. Mannheim Partner D-Curves in Euclidean 3-space E^3

In this section, by considering the Darboux frame, we define Mannheim partner D-curves and give the characterizations of these curves.

Definition 1. Let S and S_1 be oriented surfaces in 3-dimensional Euclidean space E^3 and let consider the arc-length parameter curves $x(s)$ and $x_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames of $x(s)$ and $x_1(s_1)$ by $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$ and $\{\mathbf{T}_1,\mathbf{g}_1,\mathbf{n}_1\}$, respectively. If there exists a corresponding relationship between the curves x and x_1 such that, at the corresponding points of the curves, the Darboux frame element $\mathbf{g}$ of x coincides with the Darboux frame element $\mathbf{n}_1$ of x_1, i.e., the vectors $\mathbf{g}$ and $\mathbf{n}_1$ lie on a line, then x is called a Mannheim D-curve, and x_1 is a Mannheim partner D-curve of x. Then, the pair $\{x,x_1\}$ is said to be a Mannheim D-pair. If there exist such curves lying on the oriented surfaces S and S_1, respectively, we call the pair $\{S,S_1\}$ as Mannheim surface pair.

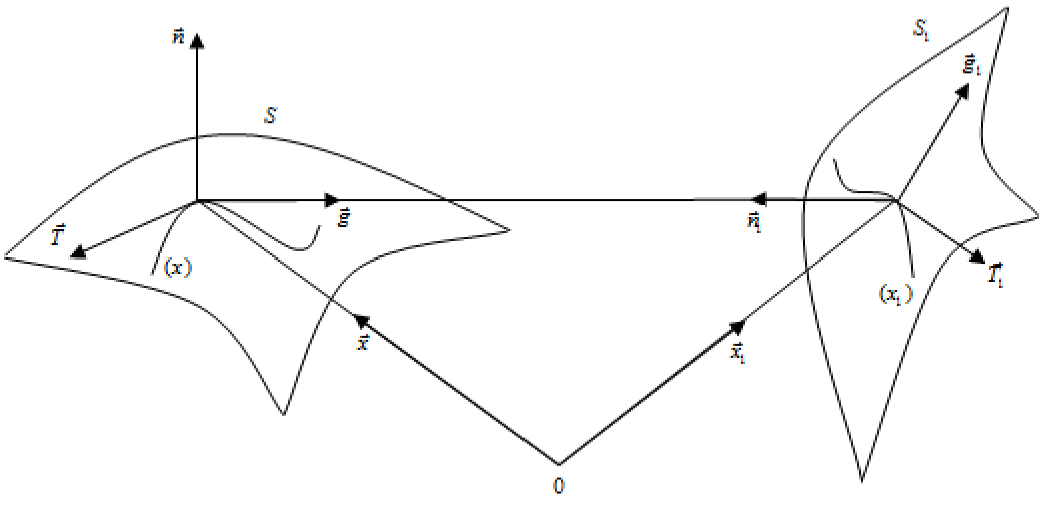

Fig. 1: Mannheim partner D-curves

Theorem 1. Let S be an oriented surface and $x(s)$ be a Mannheim D-curve in E^3 with arc length parameter s fully lying on S. If S_1 is another oriented surface and $x_1(s_1)$ is a curve with arc length parameter s_1 fully lying on S_1, then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$ if and only if the following equality holds for some nonzero constants λ,

$$-\lambda\dot{\tau}_{g_1} = \left(\frac{(1-\lambda k_{n_1})^2+\lambda^2\tau_{g_1}^2}{(1-\lambda k_{n_1})}\right)\left(\frac{\lambda k_{n_1}-1}{\cos\theta}k_n - k_{g_1}\right)+\frac{\lambda^2\tau_{g_1}\dot{k}_{n_1}}{1-\lambda k_{n_1}}$$

where θ is the angle between the tangent vectors $\mathbf{T}$ and $\mathbf{T}_1$ at the corresponding points of x and x_1.

Proof: Suppose that S is an oriented surface and $x(s)$is a Mannheim D-curve fully lying on S. Denote the Darboux frames of $x(s)$ and $x_1(s_1)$ by $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$ and $\{\mathbf{T}_1,\mathbf{g}_1,\mathbf{n}_1\}$, respectively. Then by the definition we can assume that

$$\mathbf{x}(s_1)=\mathbf{x}_1(s_1)+\lambda(s_1)\mathbf{n}_1(s_1) \tag{4}$$

for a smooth function $\lambda(s_1)$. By taking derivative of (4) with respect to s_1 and applying the Darboux formulae (1) we have

$$\mathbf{T}\frac{ds}{ds_1}=(1-\lambda k_{n_1})\mathbf{T}_1+\dot{\lambda}\mathbf{n}_1-\lambda\tau_{g_1}\mathbf{g}_1 \tag{5}$$

Since the direction of $\mathbf{n}_1$ coincides with the direction of $\mathbf{g}$, we get

$$\dot{\lambda}(s_1)=0.$$

This means that λ is a nonzero constant and equality (5) becomes

$$\mathbf{T}\frac{ds}{ds_1}=(1-\lambda k_{n_1})\mathbf{T}_1-\lambda\tau_{g_1}\mathbf{g}_1 \tag{6}$$

On the other hand we have

$$\mathbf{T}=\cos\theta\mathbf{T}_1+\sin\theta\mathbf{g}_1 \tag{7}$$

where θ is the angle between the tangent vectors $\mathbf{T}$ and $\mathbf{T}_1$ at the corresponding points of x and x_1. Differentiating (7) with respect to s_1, it follows

$$(k_g\mathbf{g}+k_n\mathbf{n})\frac{ds}{ds_1}=-(\dot{\theta}+k_{g_1})\sin\theta\mathbf{T}_1+(\dot{\theta}+k_{g_1})\cos\theta\mathbf{g}_1+(k_{n_1}\cos\theta+\tau_{g_1}\sin\theta)\mathbf{n}_1 \tag{8}$$

From this equation and the fact that

$$\mathbf{n}=\sin\theta\mathbf{T}_1-\cos\theta\mathbf{g}_1 \tag{9}$$

we get

$$\begin{aligned}(k_n\sin\theta\mathbf{T}_1-k_n\cos\theta\mathbf{g}_1+k_g\mathbf{g})\tfrac{ds}{ds_1}&=-(\dot{\theta}+k_{g_1})\sin\theta\mathbf{T}_1+(\dot{\theta}+k_{g_1})\cos\theta\mathbf{g}_1\\&+(k_{n_1}\cos\theta+\tau_{g_1}\sin\theta)\mathbf{n}_1\end{aligned} \tag{10}$$

Since the direction of $\mathbf{n}_1$ is coincident with $\mathbf{g}$ we have

$$\dot{\theta}=-\left(k_n\frac{ds}{ds_1}+k_{g_1}\right) \tag{11}$$

Using (6), (7) and the fact that $\mathbf{T}_1$ is orthogonal to $\mathbf{g}_1$, we obtain

$$\frac{ds}{ds_1}=\frac{1-\lambda k_{n_1}}{\cos\theta}=-\frac{\lambda\tau_{g_1}}{\sin\theta} \tag{12}$$

Equality (12) gives us

$$\tan\theta=-\frac{\lambda\tau_{g_1}}{1-\lambda k_{n_1}} \tag{13}$$

By taking the derivative of this equation and applying (11) we get

$$-\lambda \dot{\tau}_{g_1} = \left(\frac{(1-\lambda k_{n_1})^2 + \lambda^2 \tau_{g_1}^2}{(1-\lambda k_{n_1})} \right) \left(\frac{\lambda k_{n_1} - 1}{\cos\theta} k_n - k_{g_1} \right) + \frac{\lambda^2 \tau_{g_1} \dot{k}_{n_1}}{1-\lambda k_{n_1}} \tag{14}$$

that is desired.

Conversely, assume that equation (14) holds for some nonzero constants λ. From (14) we have

$$-k_n \left(\frac{ds}{ds_1} \right)^3 = -\lambda \dot{\tau}_{g_1} (1-\lambda k_{n_1}) - \lambda^2 \tau_{g_1} \dot{k}_{n_1} + \left((1-\lambda k_{n_1})^2 + \lambda^2 \tau_{g_1}^2 \right) k_{g_1} \tag{15}$$

Let define a curve

$$\mathbf{x}(s_1) = \mathbf{x}_1(s_1) + \lambda \mathbf{n}_1(s_1) \tag{16}$$

where λ is a non-zero constant. We will prove that x is a Mannheim D-curve and x_1 is the Mannheim partner D-curve of x. By taking the derivative of (16) with respect to s_1 twice, we get

$$\mathbf{T}\frac{ds}{ds_1} = (1-\lambda k_{n_1})\mathbf{T}_1 - \lambda \tau_{g_1} \mathbf{g}_1 \tag{17}$$

and

$$\begin{aligned} (k_g \mathbf{g} + k_n \mathbf{n}) \left(\tfrac{ds}{ds_1} \right)^2 + \mathbf{T} \tfrac{d^2 s}{ds_1^2} = {} & (-\lambda \dot{k}_{n_1} + \lambda \tau_{g_1} k_{g_1}) \mathbf{T}_1 + ((1-\lambda k_{n_1}) k_{g_1} - \lambda \dot{\tau}_{g_1}) \mathbf{g}_1 \\ & + \left((1-\lambda k_{n_1}) k_{n_1} - \lambda \tau_{g_1}^2 \right) \mathbf{n}_1 \end{aligned} \tag{18}$$

respectively. Taking the cross product of (17) with (18) we have

$$\begin{aligned} [k_g \mathbf{n} - k_n \mathbf{g}] \left(\tfrac{ds}{ds_1} \right)^3 = {} & \left(-\lambda \tau_{g_1} k_{n_1} (1-\lambda k_{n_1}) + \lambda^2 \tau_{g_1}^3 \right) \mathbf{T}_1 - \left((1-\lambda k_{n_1})^2 k_{n_1} - \lambda \tau_{g_1}^2 (1-\lambda k_{n_1}) \right) \mathbf{g}_1 \\ & + \left(k_{g_1} (1-\lambda k_{n_1})^2 - \lambda \dot{\tau}_{g_1} (1-\lambda k_{n_1}) - \lambda^2 \tau_{g_1} \dot{k}_{n_1} + \lambda^2 \tau_{g_1}^2 k_{g_1} \right) \mathbf{n}_1 \end{aligned} \tag{19}$$

By substituting (15) in (19) we get

$$\begin{aligned} [k_g \mathbf{n} - k_n \mathbf{g}] \left(\tfrac{ds}{ds_1} \right)^3 = {} & \left(-\lambda \tau_{g_1} k_{n_1} (1-\lambda k_{n_1}) + \lambda^2 \tau_{g_1}^3 \right) \mathbf{T}_1 \\ & - \left(k_{n_1} (1-\lambda k_{n_1})^2 - \lambda \tau_{g_1}^2 (1-\lambda k_{n_1}) \right) \mathbf{g}_1 - k_n \left(\tfrac{ds}{ds_1} \right)^3 \mathbf{n}_1 \end{aligned} \tag{20}$$

Taking the cross product of (17) with (20) we have

$$\begin{aligned} -[k_g \mathbf{g} + k_n \mathbf{n}] \left(\tfrac{ds}{ds_1} \right)^4 = {} & k_n \left(\tfrac{ds}{ds_1} \right)^3 \lambda \tau_{g_1} \mathbf{T}_1 + k_n \left(\tfrac{ds}{ds_1} \right)^3 (1-\lambda k_{n_1}) \mathbf{g}_1 \\ & + \left((1-\lambda k_{n_1})^2 + \lambda^2 \tau_{g_1}^2 \right) \left(\lambda \tau_{g_1}^2 - k_{n_1} (1-\lambda k_{n_1}) \right) \mathbf{n}_1 \end{aligned} \tag{21}$$

From (20) and (21) we have

$$\begin{aligned} (k_g^2 + k_n^2) \left(\tfrac{ds}{ds_1} \right)^4 \mathbf{n} = {} & \left[k_g \tfrac{ds}{ds_1} \left(-\lambda \tau_{g_1} k_{n_1} (1-\lambda k_{n_1}) + \lambda^2 \tau_{g_1}^3 \right) - \lambda \tau_{g_1} k_n^2 \left(\tfrac{ds}{ds_1} \right)^3 \right] \mathbf{T}_1 \\ & - \left[k_g \tfrac{ds}{ds_1} \left(k_{n_1} (1-\lambda k_{n_1})^2 - \lambda \tau_{g_1}^2 (1-\lambda k_{n_1}) \right) + (1-\lambda k_{n_1}) k_n^2 \left(\tfrac{ds}{ds_1} \right)^3 \right] \mathbf{g}_1 \\ & - \left[k_n k_g \left(\tfrac{ds}{ds_1} \right)^4 + k_n \left((1-\lambda k_{n_1})^2 + \lambda^2 \tau_{g_1}^2 \right) \left(\lambda \tau_{g_1}^2 - k_{n_1} (1-\lambda k_{n_1}) \right) \right] \mathbf{n}_1 \end{aligned} \tag{22}$$

Furthermore, from (17) and (20) we get

$$\begin{cases} \left(\frac{ds}{ds_1}\right)^2 = (1-\lambda k_{n_1})^2 + \lambda^2 \tau_{g_1}^2 , \\ k_g \left(\frac{ds}{ds_1}\right)^2 = k_{n_1}(1-\lambda k_{n_1}) - \lambda \tau_{g_1}^2 , \end{cases} \tag{23}$$

respectively. Substituting (23) in (22) we obtain

$$\begin{aligned} \left(k_g^2 + k_n^2\right)\left(\frac{ds}{ds_1}\right)^4 \mathbf{n} = & \left[k_g \frac{ds}{ds_1}\left(-\lambda \tau_{g_1} k_{n_1}(1-\lambda k_{n_1}) + \lambda^2 \tau_{g_1}^3\right) - \lambda \tau_{g_1} k_n^2 \left(\frac{ds}{ds_1}\right)^3\right] \mathbf{T}_1 \\ & - \left[k_g \frac{ds}{ds_1}\left(k_{n_1}(1-\lambda k_{n_1})^2 - \lambda \tau_{g_1}^2 (1-\lambda k_{n_1})\right) + (1-\lambda k_{n_1}) k_n^2 \left(\frac{ds}{ds_1}\right)^3\right] \mathbf{g}_1 \end{aligned} \tag{24}$$

Equality (17) and (24) shows that the vectors $\mathbf{T}$ and $\mathbf{n}$ lie on the plane $sp\{\mathbf{T}_1, \mathbf{g}_1\}$. So, at the corresponding points of the curves, the Darboux frame element $\mathbf{g}$ of x coincides with the Darboux frame element $\mathbf{n}_1$ of x_1, i.e, the curves x and x_1 are Mannheim partner D-curves.

Let now give the characterizations of Mannheim partner D-curves in some special cases. Assume that $x(s)$ is an asymptotic line. Then, from (14) we have the following special cases:

(i) Consider that $x_1(s_1)$ is a geodesic curve. Then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$if and only if the following equation holds,

$$\dot{\tau}_{g_1} = -\frac{\lambda \tau_{g_1} \dot{k}_{n_1}}{1-\lambda k_{n_1}}.$$

(ii) Assume that $x_1(s_1)$ is also an asymptotic line. Then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$if and only if the geodesic curvature $k_{g_1}$and the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfy the following equation,

$$\lambda \dot{\tau}_{g_1} = (1+\lambda^2 \tau_{g_1}^2) k_{g_1}.$$

In this case, the Frenet frame of the curve $x_1(s_1)$ coincides with its Darboux frame. From (2) we have $k_{g_1} = \kappa_1$ and $\tau_{g_1} = \tau_1$. So, the Mannheim partner D-curves become the Mannheim partner curves, i.e., if both $x(s)$ and $x_1(s_1)$ are asymptotic lines. Then, the definition and the characterizations of the Mannheim partner D-curves involve those of the Mannheim partner curves in Euclidean 3-space.

(iii) If $x_1(s_1)$ is a principal line then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$ if and only if either the geodesic curvature $k_{g_1} = 0$ i.e, $x_1(s_1)$ is also a geodesic curve or $k_{n_1} = 1/\lambda = const$.

Proposition 1. Let the pair $\{x, x_1\}$ be a Mannheim D-pair. Then the relation between geodesic curvature k_g, geodesic torsion τ_g of $x(s)$ and the normal curvature k_{n_1}, the geodesic torsion τ_{g_1} of $x_1(s_1)$ is given as follows

$$k_g - k_{n_1} = \lambda(k_g k_{n_1} - \tau_g \tau_{g_1}).$$

Proof: Let $x(s)$be a Mannheim D-curve and $x_1(s_1)$ be a Mannheim partner D-curve of $x(s)$. Then from (16) we can write

$$\mathbf{x}_1(s_1) = \mathbf{x}(s_1) - \lambda \mathbf{n}_1(s_1) \tag{25}$$

for some constants λ. By differentiating (25) with respect to s_1 we have

$$\mathbf{T}_1 = (1+\lambda k_g)\frac{ds}{ds_1}\mathbf{T} - \lambda\tau_g\frac{ds}{ds_1}\mathbf{n} \tag{26}$$

Since

$$\mathbf{T}_1 = \cos\theta\mathbf{T} + \sin\theta\mathbf{n} \tag{27}$$

from (26) and (27) we obtain

$$\cos\theta = (1+\lambda k_g)\frac{ds}{ds_1}, \quad \sin\theta = -\lambda\tau_g\frac{ds}{ds_1}. \tag{28}$$

Using (12) and (28) it is easily seen that

$$k_g - k_{n_1} = \lambda(k_g k_{n_1} - \tau_g\tau_{g_1}).$$

From Proposition 1, we obtain the following special cases.

Let the pair $\{x,x_1\}$be a Mannheim D-pair. Then,
(i) if x_1 is an asymptotic line, then $k_g = -\lambda\tau_g\tau_{g_1}$.
(ii) if x_1 is a principal line, then $k_g - k_{n_1} = \lambda k_g k_{n_1}$.
(iii) if x is a geodesic curve, then $k_{n_1} = \lambda\tau_g\tau_{g_1}$.
(iv) if x is a principal line then $k_g - k_{n_1} = \lambda k_g k_{n_1}$.

Theorem 2. Let $\{x,x_1\}$ be Mannheim D-pair. Then the following relations hold:

(i) $k_{g_1} = -\left(k_n\frac{ds}{ds_1} + \frac{d\theta}{ds_1}\right)$
(ii) $\tau_g\frac{ds}{ds_1} = -k_{n_1}\sin\theta + \tau_{g_1}\cos\theta$
(iii) $k_g\frac{ds}{ds_1} = k_{n_1}\cos\theta + \tau_{g_1}\sin\theta$
(iv) $\tau_{g_1} = (k_g\sin\theta + \tau_g\cos\theta)\frac{ds}{ds_1}$

Proof: (i) By differentiating the equation $\langle\mathbf{T},\mathbf{T}_1\rangle = \cos\theta$ with respect to s_1 we have

$$\left\langle (k_g\mathbf{g} + k_n\mathbf{n})\frac{ds}{ds_1}, \mathbf{T}_1\right\rangle + \langle\mathbf{T}, k_{g_1}\mathbf{g}_1 + k_{n_1}\mathbf{n}_1\rangle = -\sin\theta\frac{d\theta}{ds_1}.$$

Using the fact that the direction of n_1 coincides with the direction of g and

$$\begin{cases} \mathbf{T}_1 = \cos\theta\mathbf{T} + \sin\theta\mathbf{n}, \\ \mathbf{g}_1 = \sin\theta\mathbf{T} - \cos\theta\mathbf{n}, \end{cases} \tag{29}$$

we easily get that

$$k_{g_1} = -\left(k_n\frac{ds}{ds_1} + \frac{d\theta}{ds_1}\right).$$

(ii) By differentiating the equation $\langle\mathbf{n},\mathbf{n}_1\rangle = 0$ with respect to s_1 we have

$$\left\langle (-k_n\mathbf{T} - \tau_g\mathbf{g})\frac{ds}{ds_1}, \mathbf{n}_1\right\rangle + \langle\mathbf{n}, -k_{n_1}\mathbf{T}_1 - \tau_{g_1}\mathbf{g}_1\rangle = 0.$$

By (29) we obtain

$$\tau_g \frac{ds}{ds_1} = -k_{n_1} \sin\theta + \tau_{g_1} \cos\theta.$$

(iii) By differentiating the equation $\langle \mathbf{T}, \mathbf{n}_1 \rangle = 0$ with respect to s_1 we get

$$\left\langle (k_g \mathbf{g} + k_n \mathbf{n}) \frac{ds}{ds_1}, \mathbf{n}_1 \right\rangle + \left\langle \mathbf{T}, -k_{n_1} \mathbf{T}_1 - \tau_{g_1} \mathbf{g}_1 \right\rangle = 0.$$

From (29) it follows that

$$k_g \frac{ds}{ds_1} = k_{n_1} \cos\theta + \tau_{g_1} \sin\theta.$$

(iv) By differentiating the equation $\langle \mathbf{g}, \mathbf{g}_1 \rangle = 0$ with respect to s_1 we obtain

$$\left\langle (-k_g \mathbf{T} + \tau_g \mathbf{n}) \frac{ds}{ds_1}, \mathbf{g}_1 \right\rangle + \left\langle \mathbf{g}, -k_{g_1} \mathbf{T}_1 + \tau_{g_1} \mathbf{n}_1 \right\rangle = 0.$$

By considering (29) we get

$$\tau_{g_1} = (k_g \sin\theta + \tau_g \cos\theta) \frac{ds}{ds_1}.$$

Let now x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x. From the first equation of (3) it follows

$$\begin{aligned} k_{g_1} &= \left\langle \dot{\vec{x_1}}, \ddot{\vec{x_1}}, \times \vec{n_1} \right\rangle = \left\langle \dot{\vec{x_1}}, \ddot{\vec{x_1}}, \times \vec{g_1} \right\rangle \\ &= \left(\tfrac{ds}{ds_1}\right)^3 \left(-k_n (1+\lambda k_g)^2 - \lambda^2 \tau_g^2 k_n\right) + \left(\tfrac{ds}{ds_1}\right)^2 \left(\lambda \dot{\tau}_g (1+\lambda k_g) - \lambda^2 \tau_g \dot{k}_g\right) \end{aligned} \quad (30)$$

Then the relations between the geodesic curvature k_{g_1} of $x_1(s_1)$ and the geodesic curvature k_g, the normal curvature $k_n$and the geodesic torsion $\tau_g$of $x(s)$are given as follows:
(1) If $k_g = 0$ then from (30) the geodesic curvature k_{g_1} of $x_1(s_1)$ is

$$k_{g_1} = -\left(\frac{ds}{ds_1}\right)^3 (1+\lambda^2 \tau_g^2) k_n + \left(\frac{ds}{ds_1}\right)^2 \lambda \dot{\tau}_g \quad (31)$$

(2) If $k_n = 0$ then the relation between k_g, τ_g and k_{g_1} is

$$k_{g_1} = \lambda \left(\frac{ds}{ds_1}\right)^2 \left(\dot{\tau}_g (1+\lambda k_g) - \lambda \tau_g \dot{k}_g\right) \quad (32)$$

(3) If $\tau_g = 0$ then, for the geodesic curvature k_{g_1}, we have

$$k_{g_1} = -\left(\frac{ds}{ds_1}\right)^3 (1+\lambda k_g)^2 k_n \quad (33)$$

From (31), (32) and (33) we give the following corollary.

Corollary 1. Let x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x. Then the relations between the geodesic curvature k_{g_1} of $x_1(s_1)$ and the geodesic curvature k_g, the normal curvature k_n and the geodesic torsion $\tau_g$of $x(s)$are given as follows,

(i) If x is a geodesic curve, then the geodesic curvature k_{g_1} of $x_1(s_1)$ is

$$k_{g_1} = -\left(\frac{ds}{ds_1}\right)^3 (1+\lambda^2\tau_g^2)k_n + \left(\frac{ds}{ds_1}\right)^2 \lambda\dot{\tau}_g.$$

(ii) If x is an asymptotic line, then the equation of k_{g_1} is

$$k_{g_1} = \lambda\left(\frac{ds}{ds_1}\right)^2 \left(\dot{\tau}_g(1+\lambda k_g) - \lambda\tau_g\dot{k}_g\right).$$

(iii) If x is a principal line, then the geodesic curvature k_{g_1} of $x_1(s_1)$ is

$$k_{g_1} = -\left(\frac{ds}{ds_1}\right)^3 (1+\lambda k_g)^2 k_n.$$

Similarly, From the second equation of (3) and by using the fact that g is coincident with n_1, the relation between the geodesic torsion τ_{g_1} of $x_1(s_1)$ and the geodesic torsion τ_g of $x(s)$ is given by

$$\tau_{g_1} = \left(\frac{ds}{ds_1}\right)^2 \tau_g \tag{34}$$

Furthermore, by using (12), from (34) we have

$$\tau_g\tau_{g_1} = \frac{\sin^2\theta}{\lambda^2} \tag{35}$$

Then, from (34) and (35) we can give the following corollary.

Corollary 2. Let x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x. Then the relation between the geodesic torsion τ_{g_1} of $x_1(s_1)$ and the geodesic torsion τ_g of $x(s)$is given by one of the followings,

$$\tau_{g_1} = \left(\frac{ds}{ds_1}\right)^2 \tau_g \ \text{ or } \ \tau_g\tau_{g_1} = \frac{\sin^2\theta}{\lambda^2}$$

and so, the Mannheim partner D-curve x_1 is a principal line when the Mannheim D-curve x is a principal line.

Similarly, from (12) and (34) we get

$$\frac{\tau_g}{\tau_{g_1}} = \frac{\cos^2\theta}{(1-\lambda k_{n_1})^2}$$

Then, if $x_1(s_1)$ is an asymptotic curve, i.e.,$k_{n_1} = 0$, we have

$$\tau_g = \cos^2\theta\,\tau_{g_1} \tag{36}$$

From (36) we have the following corollary.

Corollary 3. Let x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x. If $x_1(s_1)$ is an asymptotic curve then the relation between the geodesic torsion τ_g of $x(s)$ and the geodesic torsion τ_{g_1} of $x_1(s_1)$ is given as follows,

$$\tau_g = \cos^2\theta\,\tau_{g_1}$$

where θ is the angle between the tangent vectors T and T_1 at the corresponding points of x and x_1.

Example 1. Let consider the great circle $x(\theta) = (\cos\theta, \sin\theta, 0)$ on the unit sphere $S(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$. The Mannheim partner D-curve of $x(\theta)$ is the curve $x_1(\theta) = (\cos\theta, \sin\theta, \lambda)$, where λ is a non-zero constant, and the curve $x_1(\theta)$ lies on a ruled surface given by $S_1(\theta,v) = (\cos\theta, \sin\theta, \lambda) + v(-\sin\theta, \cos\theta, 0)$ which is the surface of the tangents of the curve $x_1(\theta)$. Then the pair $\{x, x_1\}$ is a Mannheim D-pair (Fig. 2).

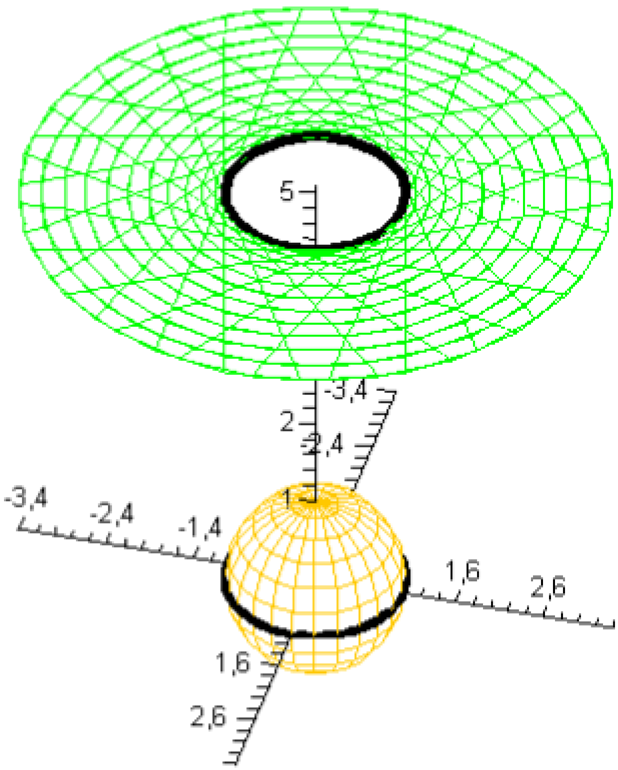

Fig. 2

Example 2. Let consider the helix curve given by $x(\theta) = (\cos\theta, \sin\theta, \theta)$ on the right cylinder $S(\theta,\varphi) = (\cos\theta, \sin\theta, \varphi)$. The Mannheim partner D-curve of $x(\theta)$ is the curve

$$x_1(\theta) = \left(\cos\theta + \frac{\lambda}{\sqrt{2}}\sin\theta,\ \sin\theta - \frac{\lambda}{\sqrt{2}}\cos\theta,\ \theta + \frac{\lambda}{\sqrt{2}}\right),$$

where λ is a non-zero constant, and the curve $x_1(\theta)$ lies on a helicoid surface given by

$$S_1(\theta,v) = \left(\cos\theta + \frac{\lambda}{\sqrt{2}}\sin\theta + av\cos\theta,\ \sin\theta - \frac{\lambda}{\sqrt{2}}\cos\theta + av\sin\theta,\ \theta + \frac{\lambda}{\sqrt{2}}\right)$$

where a is a non-zero constant. Then the pair $\{x, x_1\}$ is Mannheim D-pair (Fig. 3).

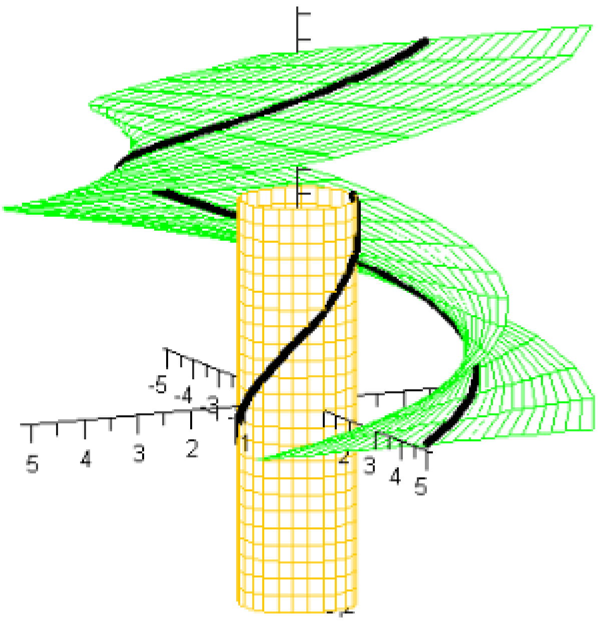

Fig. 3

4. Conclusions

In this paper, the definition and characterizations of Mannheim partner D-curves are given which is a new study of associated curves lying on surfaces. It is shown that the definition and the characterizations of Mannheim partner D-curves include those of Mannheim partner curves in some special cases. Furthermore, the relations between the geodesic curvature, the normal curvature and the geodesic torsion of these curves are given.

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