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Lie derivatives of almost contact structure and almost paracontact structure with respect to X^C and X^V on tangent bundle $T(M)$

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Abstract: The differential geometry of tangent bundles was studied by several authors, for example: D. E. Blair [1], V. Oproiu [3], A. Salimov [5], Yano and Ishihara [8] and among others. It is well known that differant structures deffined on a manifold M can be lifted to the same type of structures on its tangent bundle. Our goal is to study Lie derivatives of almost contact structure and almost paracontact structure with respect to X^C and X^V on tangent bundle $T(M)$. In addition, this Lie derivatives which obtained shall be studied for some special values.

Keywords: Lie derivative, almost contact structure, almost paracontact structure, tangent bundle.

1 Introduction

Let M be an $n-$dimensional differentiable manifold of class C^∞ and let $T_p(M)$ be the tangent space of M at a point p of M. Then the set [8]

$$T(M) = \bigcup_{p\in M} T_p(M) \tag{1}$$

is called the tangent bundle over the manifold M. For any point $\tilde{p}$ of $T(M)$, the correspondence $\tilde{p} \to p$ determines the bundle projection $\pi : T(M) \to M$, thus $\pi(\tilde{p}) = p$, where $\pi : T(M) \to M$ defines the bundle projection of $T(M)$ over M. The set $\pi^{-1}(p)$ is called the fibre over $p \in M$ and M the base space.

Suppose that the base space M is covered by a system of coordinate neighbour-hoods $\{U; x^h\}$, where (x^h) is a system of local coordinates defined in the neighbour-hood U of M. The open set $\pi^{-1}(U) \subset T(M)$ is naturally differentiably homeomorphic to the direct product $U \times R^n$, R^n being the $n-$dimensional vector space over the real field R, in such a way that a point $\tilde{p} \in T_p(M)(p \in U)$ is represented by an ordered pair (P,X) of the point $p \in U$, and a vector $X \in R^n$,whose components are given by the cartesian coordinates (y^h) of $\tilde{p}$ in the tangent space $T_p(M)$ with respect to the natural base $\{\partial_h\}$, where $\partial_h = \frac{\partial}{\partial x^h}$. Denoting by (x^h) the coordinates of $p = \pi(\tilde{p})$ in U and establishing the correspondence $(x^h, y^h) \to \tilde{p} \in \pi^{-1}(U)$, we can introduce a system of local coordinates (x^h, y^h) in the open set $\pi^{-1}(U) \subset T(M)$. Here we cal (x^h, y^h) the coordinates in $\pi^{-1}(U)$ induced from (x^h) or simply, the induced coordinates in $\pi^{-1}(U)$.

We denote by $\Im^r_s(M)$ the set of all tensor fields of class $C^\infty$and of type (r,s) in M. We now put $\Im(M) = \sum_{r,s=0}^{\infty} \Im^r_s(M)$, which is the set of all tensor fields in M. Similarly, we denote by $\Im^r_s(T(M))$ and $\Im(T(M))$ respectively the corresponding sets of tensor fields in the tangent bundle $T(M)$.

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2 Vertical lifts

If f is a function in M, we write f^v for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \to M$ and $f : M \to R$, so that

$$f^v = fo\pi. \tag{2}$$

Thus, if a point $\tilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

$$f^v(\tilde{p}) = f^v(x,y) = fo\pi(\tilde{p}) = f(p) = f(x). \tag{3}$$

Thus, the value of $f^v(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^v the vertical lift of the function f [8].

Let $\tilde{X} \in \Im_0^1(T(M))$ be such that $\tilde{X} f^v = 0$ for all $f \in \Im_0^0(M)$. Then we say that $\tilde{X}$ is a vertical vector field. Let $\begin{bmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{bmatrix}$ be components of $\tilde{X}$ with respect to the induced coordinates. Then $\tilde{X}$ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

$$\begin{bmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{bmatrix} = \begin{bmatrix} 0 \\ X^{\bar{h}} \end{bmatrix}. \tag{4}$$

Suppose that $X \in \Im_0^1(M)$, so that is a vector field in M. We define a vector field X^v in $T(M)$ by

$$X^v(\iota\ \omega) = (\omega X)^v \tag{5}$$

ω being an arbitrary $1-$form in M. We cal X^v the vertical lift of X [8].

Let $\tilde{\omega} \in \Im_1^0(T(M))$ be such that $\tilde{\omega}(X)^v = 0$ for all $X \in \Im_0^1(M)$. Then we say that $\tilde{\omega}$ is a vertical $1-$form in $T(M)$. We define the vertical lift ω^v of the $1-$form ω by

$$\omega^v = (\omega_i)^v (dx^i)^v \tag{6}$$

in each open set $\pi^{-1}(U)$, where $(U; x^h)$ is coordinate neighbourhood in M and ω is given by $\omega = \omega_i dx^i$ in U. The vertical lift ω^v of ω with lokal expression $\omega = \omega_i dx^i$ has components of the form

$$\omega^v : (\omega^i, 0) \tag{7}$$

with respect to the induced coordinates in $T(M)$.

Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions

$$(P \otimes Q)^V = P^V \otimes Q^V,\ (P+R)^V = P^V + R^V \tag{8}$$

P, Q and R being arbitrary elements of $\Im(M)$. The vertical lifts F^V of an element $F \in \Im_1^1(M)$ with lokal components F_i^h has components of the form [8]

$$F^V : \begin{pmatrix} 0 & 0 \\ F_i^h & 0 \end{pmatrix}. \tag{9}$$

Vertical lift has the following formulas ([4],[8]):

$$\begin{aligned}&(fX)^{v}=f^{v}X^{v},I^{v}X^{v}=0,\eta^{v}(X^{v})=0\\&(f\eta)^{v}=f^{v}\eta^{v},[X^{v},Y^{v}]=0,\varphi^{v}X^{v}=0\\&X^{v}f^{v}=0,X^{v}f^{v}=0\end{aligned} \tag{10}$$

hold good, where $f\in\Im_0^0(M_n)$, $X,Y\in\Im_0^1(M_n)$, $\eta\in\Im_1^0(M_n)$, $\varphi\in\Im_1^1(M_n)$, $I=id_{M_n}$.

3 Complete lifts

If f is a function in M, we write f^c for the function in $T(M)$ defined by

$$f^c=\imath(df)$$

and call f^c the comple lift of the function f. The complete lift f^c of a function f has the lokal expression

$$f^c=y^i\partial_i f=\partial f \tag{11}$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i\partial_i f$.

Suppose that $X\in\Im_0^1(M)$. then we define a vector field X^c in $T(M)$ by

$$X^c f^c=(Xf)^c, \tag{12}$$

f being an arbitrary function in M and call X^c the complete lift of X in $T(M)$ ([2],[8]). The complete lift X^c of X with components x^h in M has components

$$X^c=\begin{pmatrix}X^h\\ \partial X^h\end{pmatrix} \tag{13}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\omega\in\Im_1^0(M)$, then a $1-$form ω^c in $T(M)$ defined by

$$\omega^c(X^c)=(\omega X)^c. \tag{14}$$

X being an arbitrary vector field in M. We call ω^c the complete lift of ω. The complete lift ω^c of ω with components ω_i in M has components of the form

$$\omega^c:(\partial\omega_{i,}\omega_i) \tag{15}$$

with respect to the induced coordinates in $T(M)$ [2].

The complete lifts to a unique algebra isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients, is given by the conditions

$$(P\otimes Q)^C=P^C\otimes Q^V+P^V\otimes Q^C,\ (P+R)^C=P^C+R^C, \tag{16}$$

where P,Q and R being arbitrary elements of $\Im(M)$. The complete lifts F^C of an element $F\in\Im_1^1(M)$ with lokal components F_i^h has components of the form

$$F^C:\begin{pmatrix} F_i^h & 0\\ \partial F_i^h & F_i^h\end{pmatrix}. \tag{17}$$

In addition, we know that the complete lifts are defined by ([4],[8]):

$$\begin{aligned}
&(fX)^c = f^cX^v + f^vX^c = (Xf)^c,\\
&X^cf^v = (Xf)^v, \eta^v(x^c) = (\eta(x))^v,\\
&X^vf^c = (Xf)^v, \varphi^vX^c = (\varphi X)^v,\\
&\varphi^cX^v = (\varphi X)^v, (\varphi X)^c = \varphi^cX^c,\\
&\eta^v(X^c) = (\eta(X))^c, \eta^c(X^v) = (\eta(X))^v,\\
&[X^v,Y^c] = [X,Y]^v, I^c = I, I^vX^c = X^v, [X^c,Y^c] = [X,Y]^c.
\end{aligned} \tag{18}$$

Let M be an $n-$dimensional diferentiable manifold. Differantial transformation $D=\mathscr{L}_X$ is called Lie derivation with respect to vector field $X\in\Im_0^1(M)$ if

$$\begin{aligned}
&\mathscr{L}_Xf = Xf, \forall f\in\Im_0^0(M),\\
&\mathscr{L}_XY = [X,Y], \forall X,Y\in\Im_0^1(M).
\end{aligned} \tag{19}$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $\mathscr{L}_XF$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by ([8])

$$(\mathscr{L}_XF)Y = [X,FY] - F[X,Y].$$

Proposition 1. *For any $X\in\Im_0^1(M_n)$, $f\in\Im_0^0(M_n)$ and $\mathscr{L}_X$ is the Lie derivation with respect to vector field X [8]*

(i) $\mathscr{L}_{X^v}f^v = 0$,
(ii) $\mathscr{L}_{X^v}f^c = (\mathscr{L}_Xf)^v$,
(iii) $\mathscr{L}_{X^c}f^v = (\mathscr{L}_Xf)^v$,
(iv) $\mathscr{L}_{X^c}f^c = (\mathscr{L}_Xf)^c$.

Proposition 2. *For any $X,Y\in\Im_0^1(M_n)$ and $\mathscr{L}_X$ is the Lie derivation with respect to vector field X [8]*

(i) $\mathscr{L}_{X^v}Y^v = 0$,
(ii) $\mathscr{L}_{X^v}Y^c = (\nabla_XY)^v$,
(iii) $\mathscr{L}_{X^c}Y^v = (\nabla_XY)^v$,
(iv) $\mathscr{L}_{X^c}Y^c = (\nabla_XY)^c$.

4 Main results

Definition 1. *Let an $n-$dimensional differentiable manifold M be endowed with a tensor field φ of type $(1,1)$, a vector field ξ, a $1-$form η, I the identity and let them satisfy*

$$\varphi^2 = -I + \eta\otimes\xi, \qquad \varphi(\xi) = 0, \qquad \eta o\varphi = 0, \qquad \eta(\xi) = 1. \tag{20}$$

Then (φ,ξ,η) define almost contact structure on M ([4],[7],[8]). From (20), we get on taking complete and vertical lifts

$$\begin{aligned}(\varphi^c)^2 &= -I+\eta^v\otimes\xi^c+\eta^c\otimes\xi^v, \\ \varphi^c\xi^v &= 0, \varphi^c\xi^c = 0, \eta^v o\varphi^c = 0, \\ \eta^c o\varphi^c &= 0, \eta^v(\xi^v) = 0, \eta^v(\xi^c) = 1, \\ \eta^c(\xi^v) &= 1, \eta^c(\xi^c) = 0.\end{aligned} \tag{21}$$

We now define a $(1,1)$ tensor field J on $\Im(M)$ by

$$J = \varphi^c - \xi^v\otimes\eta^v + \xi^c\otimes\eta^c. \tag{22}$$

Then it is easy to show that $J^2X^v = -X^v$ and $J^2X^c = -X^c$, which give that J is an almost contact structure on $\Im(M)$. We get from (22)

$$\begin{aligned}JX^v &= (\varphi X)^v + (\eta(X))^v\xi^c, \\ JX^c &= (\varphi X)^c - (\eta(X))^v\xi^v + (\eta(X))^c\xi^c\end{aligned}$$

for any $X\in\Im_0^1(M)$ [4].

Theorem 1. *For $\mathscr{L}_X$ the operator Lie derivation with respect to X, $J\in\Im_1^1(\Im(M))$ defined by (22) and $\eta(Y)=0$, we have*

(i) $(\mathscr{L}_{X^v}J)Y^v = 0$,
(ii) $(\mathscr{L}_{X^v}J)Y^c = ((\mathscr{L}_X\varphi)Y)^v + ((\mathscr{L}_X\eta)Y)^v\xi^c$,
(iii) $(\mathscr{L}_{X^c}J)Y^v = ((\mathscr{L}_X\varphi)Y)^v + ((\mathscr{L}_X\eta)Y)^v\xi^c$,
(iv) $(\mathscr{L}^c_{X^c}J)Y^c = ((\mathscr{L}_X\varphi)Y)^c - ((\mathscr{L}_X\eta)Y)^v\xi^v + ((\mathscr{L}_X\eta)Y)^c\xi^c$,

where $X,Y\in\Im_0^1(M)$, a tensor field $\varphi\in\Im_1^1(M)$, a vector field ξ and a $1-$form $\eta\in\Im_1^0(M)$.

Proof. For $J=\varphi^c-\xi^v\otimes\eta^v+\xi^c\otimes\eta^c$ and $\eta(Y)=0$, we get

(i)
$$\begin{aligned}(\mathscr{L}_{X^v}J)Y^v &= \mathscr{L}_{X^v}(\varphi^c-\xi^v\otimes\eta^v+\xi^c\otimes\eta^c)Y^v - (\varphi^c-\xi^v\otimes\eta^v+\xi^c\otimes\eta^c)\mathscr{L}_{X^v}Y^v \\ &= \mathscr{L}_{X^v}(\varphi Y)^v - \mathscr{L}_{X^v}(\eta^v(Y)^v)\xi^v + \mathscr{L}_{X^v}(\eta(Y))^v\xi^c \\ &= 0,\end{aligned}$$

(ii)
$$\begin{aligned}(\mathscr{L}_{X^v}J)Y^c &= \mathscr{L}_{X^v}(\varphi^c-\xi^v\otimes\eta^v+\xi^c\otimes\eta^c)Y^c - (\varphi^c-\xi^v\otimes\eta^v+\xi^c\otimes\eta^c)\mathscr{L}_{X^v}Y^c \\ &= \mathscr{L}_{X^v}\varphi^cY^c - \mathscr{L}_{X^v}(\eta Y)^v\xi^v + \mathscr{L}_{X^v}(\eta(Y))^c\xi^c - \varphi^c\mathscr{L}_{X^v}Y^c \\ &\quad + \eta^v(\mathscr{L}_XY)^v\xi^v - (\eta(\mathscr{L}_XY))^v\xi^c \\ &= (\mathscr{L}_{X^v}\varphi^c)Y^c + \varphi^c(\mathscr{L}_{X^v}Y^c) - \varphi^c\mathscr{L}_{X^v}Y^c - (\mathscr{L}_X(\eta(Y)))^v\xi^c + ((\mathscr{L}_X\eta)Y)^v\xi^c \\ &= (\mathscr{L}_X\varphi)Y)^v + ((\mathscr{L}_X\eta)Y)^v\xi^c,\end{aligned}$$

(iii)
$$\begin{aligned}(\mathscr{L}_{X^c}J)Y^v &= \mathscr{L}_{X^c}(\varphi^c-\xi^v\otimes\eta^v+\xi^c\otimes\eta^c)Y^v - (\varphi^c-\xi^v\otimes\eta^v+\xi^c\otimes\eta^c)\mathscr{L}_{X^c}Y^v \\ &= \mathscr{L}_{X^c}\varphi^cY^v - \mathscr{L}_{X^c}(\eta^v(Y)^v)\xi^v + \mathscr{L}_{X^c}(\eta(Y))^v\xi^c - \varphi^c\mathscr{L}_{X^c}Y^v \\ &\quad + \eta^v(\mathscr{L}_XY)^v\xi^v - (\eta(\mathscr{L}_XY))^v\xi^c \\ &= (\mathscr{L}_{X^c}\varphi^c)Y^v + \varphi^c(\mathscr{L}_{X^c}Y^v) - \varphi^c\mathscr{L}_{X^c}Y^v - (\mathscr{L}_X(\eta(Y)))^v\xi^c + (\mathscr{L}_X\eta)Y)^v\xi^c \\ &= (\mathscr{L}_X\varphi)Y)^v + ((\mathscr{L}_X\eta)Y)^v\xi^c,\end{aligned}$$

(iv)
$$\begin{aligned}(\mathscr{L}_{X^c}J)Y^c &= \mathscr{L}_{X^c}(\varphi^c-\xi^v\otimes\eta^v+\xi^c\otimes\eta^c)Y^c - (\varphi^c-\xi^v\otimes\eta^v+\xi^c\otimes\eta^c)\mathscr{L}_{X^c}Y^c \\ &= \mathscr{L}_{X^c}\varphi^cY^c - \mathscr{L}_{X^c}((\eta Y)^v)\xi^v + \mathscr{L}_{X^c}(\eta(Y))^c\xi^c - \varphi^c\mathscr{L}_{X^c}Y^c \\ &\quad + (\eta(\mathscr{L}_XY))^v\xi^v - (\eta(\mathscr{L}_XY))^c\xi^c \\ &= (\mathscr{L}_{X^c}\varphi^c)Y^c + \varphi^c(\mathscr{L}_{X^c}Y^c) - \varphi^c\mathscr{L}_{X^c}Y^c + (\mathscr{L}_X(\eta(Y)))^v\xi^v - ((\mathscr{L}_X\eta)Y)^v\xi^v\end{aligned}$$

$$
\begin{aligned}
&-(\mathscr{L}_X(\eta(Y)))^c\xi^c+((\mathscr{L}_X\eta)Y)^c\xi^c \\
&= (\mathscr{L}_X\varphi)Y)^c-((\mathscr{L}_X\eta)Y)^v\xi^v+((\mathscr{L}_X\eta)Y)^c\xi^c.
\end{aligned}
$$

Corollary 1. *If we put $Y=\xi$, i.e. $\eta(\xi)=1$ and ξ has the conditions of (20), then we get different results*

(i) $(\mathscr{L}_{X^v}J)\xi^v=(\mathscr{L}_X\xi)^v$,
(ii) $(\mathscr{L}_{X^v}J)\xi^c=((\mathscr{L}_X\varphi)\xi)^v+(((\mathscr{L}_X\eta))\xi)^v\xi^c$,
(iii) $(\mathscr{L}_{X^c}J)\xi^v=((\mathscr{L}_X\varphi)\xi)^v+(\mathscr{L}_X\xi)^c+((\mathscr{L}_X\eta)\xi)^v\xi^c$,
(iv) $(\mathscr{L}_{X^c}J)\xi^c=(\mathscr{L}_X\varphi)\xi)^c-(\mathscr{L}_X\xi)^v-((\mathscr{L}_X\eta)\xi)^v\xi^v+((\mathscr{L}_X\eta)\xi)^c\xi^c$.

Definition 2. *Let an $n-$dimensional differentiable manifold M be endowed with a tensor field φ of type $(1,1)$, a vector field ξ, a $1-$form η, I the identity and let them satisfy*

$$\varphi^2=I-\eta\otimes\xi,\qquad \varphi(\xi)=0,\qquad \eta o\varphi=0,\qquad \eta(\xi)=1. \tag{23}$$

Then (φ,ξ,η) define almost paracontact structure on M ([4],[7]). From (23), we get on taking complete and vertical lifts

$$
\begin{aligned}
&(\varphi^c)^2=I-\eta^v\otimes\xi^c-\eta^c\otimes\xi^v, \\
&\varphi^c\xi^v=0,\varphi^c\xi^c=0,\eta^v o\varphi^c=0, \\
&\eta^c o\varphi^c=0,\eta^v(\xi^v)=0,\eta^v(\xi^c)=1, \\
&\eta^c(\xi^v)=1,\eta^c(\xi^c)=0.
\end{aligned} \tag{24}
$$

We now define a $(1,1)$ tensor field $\widetilde{J}$ on $\Im(M)$ by

$$\widetilde{J}=\varphi^c-\xi^v\otimes\eta^v-\xi^c\otimes\eta^c. \tag{25}$$

Then it is easy to show that $\widetilde{J}^2X^v=X^v$ and $\widetilde{J}^2X^c=X^c$, which give that $\widetilde{J}$ is an almost product structure on $\Im(M)$. We get from (25) for any $X\in\Im_0^1(M)$.

$$
\begin{aligned}
&\widetilde{J}X^v=(\varphi X)^v-(\eta(X))^v\xi^c, \\
&\widetilde{J}X^c=(\varphi X)^v-(\eta(X))^v\xi^v-(\eta(X))^c\xi^c.
\end{aligned}
$$

Theorem 2. *For $\mathscr{L}_X$ the operator Lie derivation with respect to X, $\widetilde{J}\in\Im_1^1(\Im(M))$ defined by (25)and $\eta(Y)=0$, we have*

(i) $(\mathscr{L}_{X^v}\widetilde{J})Y^v=0$,
(ii) $(\mathscr{L}_{X^v}\widetilde{J})Y^c=((\mathscr{L}_X\varphi)Y)^v-((\mathscr{L}_X\eta)Y)^v\xi^c$,
(iii) $(\mathscr{L}_{X^c}\widetilde{J})Y^v=((\mathscr{L}_X\varphi)Y)^v-((\mathscr{L}_X\eta)Y)^v\xi^c$,
(iv) $(\mathscr{L}_{X^c}\widetilde{J})Y^c=((\mathscr{L}_X\varphi)Y)^c-((\mathscr{L}_X\eta)Y)^v\xi^v-((\mathscr{L}_X\eta)Y)^c\xi^c$,

where $X,Y\in\Im_0^1(M)$, a tensor field $\varphi\in\Im_1^1(M)$, a vector field $\xi\in\Im_0^1(M)$ and a $1-$form $\eta\in\Im_1^0(M)$.

Proof. For $\widetilde{J}=\varphi^c-\xi^v\otimes\eta^v-\xi^c\otimes\eta^c$ and $\eta(Y)=0$, we get

(i)
$$
\begin{aligned}
(\mathscr{L}_{X^v}\widetilde{J})Y^v&=\mathscr{L}_{X^v}(\varphi^c-\xi^v\otimes\eta^v-\xi^c\otimes\eta^c)Y^v-(\varphi^c-\xi^v\otimes\eta^v-\xi^c\otimes\eta^c)\mathscr{L}_{X^v}Y^v \\
&=\mathscr{L}_{X^v}(\varphi Y)^v-\mathscr{L}_{X^v}(\eta^v(Y)^v)\xi^v-\mathscr{L}_{X^v}(\eta(Y))^v\xi^c \\
&=0,
\end{aligned}
$$

(ii)
$$
\begin{aligned}
(\mathscr{L}_{X^v}\widetilde{J})Y^c&=\mathscr{L}_{X^v}(\varphi^c-\xi^v\otimes\eta^v-\xi^c\otimes\eta^c)Y^c-(\varphi^c-\xi^v\otimes\eta^v-\xi^c\otimes\eta^c)\mathscr{L}_{X^v}Y^c \\
&=\mathscr{L}_{X^v}\varphi^cY^c-\mathscr{L}_{X^v}(\eta Y)^v\xi^v-\mathscr{L}_{X^v}(\eta(Y))^c\xi^c-\varphi^c(\mathscr{L}_XY)^v
\end{aligned}
$$

$$
\begin{aligned}
&+\eta^{v}(\mathscr{L}_{X}Y)^{v}\xi^{v}+(\eta(\mathscr{L}_{X}Y))^{v}\xi^{c}\\
&=(\mathscr{L}_{X^{v}}\varphi^{c})Y^{c}+\varphi^{c}(\mathscr{L}_{X^{v}}Y^{c})-\varphi^{c}(\mathscr{L}_{X}Y)^{v}+(\mathscr{L}_{X}(\eta(Y)))^{v}\xi^{c}-((\mathscr{L}_{X}\eta)Y)^{v}\xi^{c}\\
&=(\mathscr{L}_{X}\varphi)Y)^{v}-((\mathscr{L}_{X}\eta)Y)^{v}\xi^{c},
\end{aligned}
$$

$$
\begin{aligned}
\text{(iii)}\ \ (\mathscr{L}_{X^{c}}\widetilde{J})Y^{v}&=\mathscr{L}_{X^{c}}(\varphi^{c}-\xi^{v}\otimes\eta^{v}-\xi^{c}\otimes\eta^{c})Y^{v}-(\varphi^{c}-\xi^{v}\otimes\eta^{v}-\xi^{c}\otimes\eta^{c})\mathscr{L}_{X^{c}}Y^{v}\\
&=\mathscr{L}_{X^{c}}\varphi^{c}Y^{v}-\mathscr{L}_{X^{c}}(\eta^{v}(Y)^{v})\xi^{v}-\mathscr{L}_{X^{c}}(\eta(Y))^{v}\xi^{c}-\varphi^{c}\mathscr{L}_{X^{c}}Y^{v}\\
&+\eta^{v}(\mathscr{L}_{X}Y)^{v}\xi^{v}+(\eta(\mathscr{L}_{X}Y))^{v}\xi^{c}\\
&=(\mathscr{L}_{X^{c}}\varphi^{c})Y^{v}+\varphi^{c}(\mathscr{L}_{X^{c}}Y^{v})-\varphi^{c}\mathscr{L}_{X^{c}}Y^{v}+(\mathscr{L}_{X}(\eta(Y)))^{v}\xi^{c}-(\mathscr{L}_{X}\eta)Y)^{v}\xi^{c}\\
&=(\mathscr{L}_{X}\varphi)Y)^{v}-((\mathscr{L}_{X}\eta)Y)^{v}\xi^{c},
\end{aligned}
$$

$$
\begin{aligned}
\text{(iv)}\ \ (\mathscr{L}_{X^{c}}\widetilde{J})Y^{c}&=\mathscr{L}_{X^{c}}(\varphi^{c}-\xi^{v}\otimes\eta^{v}-\xi^{c}\otimes\eta^{c})Y^{c}-(\varphi^{c}-\xi^{v}\otimes\eta^{v}-\xi^{c}\otimes\eta^{c})\mathscr{L}_{X^{c}}Y^{c}\\
&=\mathscr{L}_{X^{c}}\varphi^{c}Y^{c}-\mathscr{L}_{X^{c}}((\eta Y)^{v})\xi^{v}-\mathscr{L}_{X^{c}}(\eta(Y))^{c}\xi^{c}-\varphi^{c}\mathscr{L}_{X^{c}}Y^{c}\\
&+(\eta(\mathscr{L}_{X}Y))^{v}\xi^{v}+(\eta(\mathscr{L}_{X}Y))^{c}\xi^{c}\\
&=(\mathscr{L}_{X^{c}}\varphi^{c})Y^{c}+\varphi^{c}(\mathscr{L}_{X^{c}}Y^{c})-\varphi^{c}\mathscr{L}_{X^{c}}Y^{c}+(\mathscr{L}_{X}(\eta(Y)))^{v}\xi^{v}-((\mathscr{L}_{X}\eta)Y)^{v}\xi^{v}\\
&+(\mathscr{L}_{X}(\eta(Y)))^{c}\xi^{c}-((\mathscr{L}_{X}\eta)Y)^{c}\xi^{c}\\
&=(\mathscr{L}_{X}\varphi)Y)^{c}-((\mathscr{L}_{X}\eta)Y)^{v}\xi^{v}-((\mathscr{L}_{X}\eta)Y)^{c}\xi^{c}.
\end{aligned}
$$

Corollary 2. *If we put $Y=\xi$, i.e. $\eta(\xi)=1$ and ξ has the conditions of (23), then we have*

(i) $(\mathscr{L}_{X^{v}}\widetilde{J})\xi^{v}=-(\mathscr{L}_{X}\xi)^{v}$,

(ii) $(\mathscr{L}_{X^{v}}\widetilde{J})\xi^{c}=((\mathscr{L}_{X}\varphi)\xi)^{v}-(((\mathscr{L}_{X}\eta))\xi)^{v}\xi^{c}$,

(iii) $(\mathscr{L}_{X^{c}}\widetilde{J})\xi^{v}=((\mathscr{L}_{X}\varphi)\xi)^{v}-(\mathscr{L}_{X}\xi)^{c}-((\mathscr{L}_{X}\eta)\xi)^{v}\xi^{c}$,

(iv) $(\mathscr{L}_{X^{c}}\widetilde{J})\xi^{c}=(\mathscr{L}_{X}\varphi)\xi)^{c}-(\mathscr{L}_{X}\xi)^{v}-((\mathscr{L}_{X}\eta)\xi)^{v}\xi^{v}-((\mathscr{L}_{X}\eta)\xi)^{c}\xi^{c}$.

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