New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2016115659

On the Blaschke trihedrons of a line congruence

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Received: 20 October 2015, Revised: 21 October 2015, Accepted: 6 December 2015
Published online: 30 January 2016.

Abstract: In this study, it is aimed to find relation between the Blaschke vectors of parameter ruled surfaces of a line congruence which are not principle ruled surfaces. By this relation, we can find some basic formulae of the line space. (e.g.the Mannheim's and Liouville's formulae.

Keywords: Dual Space, Blaschke Trihedron, Dual Curvature.

1 Introduction

A set of one-parameter of lines is called a ruled surface. Ruled surfaces, especially, developable ruled surfaces are used and applied several areas in mathematics and engineering,[1].

Dual number is a useful tool for line trajectories. Indeed, lines in Euclidean 3-space can be expressed by unit dual vectors and there is one to one correspondence between points of dual unit sphere S^2 and lines of Euclidean 3-space due to Study theorem. For detail, see [6,7,8,9].

On the other hand, if we take two parameters in unit dual vectors, we have a line congruence. In practices, the line congruence defines a family of ruled surfaces. The study of line congruence was started by E.Kummer [5] ,in which he gave a classificasion of those of order one. The applications of the line geometry and dual number representations of line trajectories have been developed by Blaschke [4], and Muller [10].

In [2] and [3], Caliskan gave a formulae between the Blaschke vectors of any ruled surface $\overrightarrow{R_1}$ and the parameter ruled surfaces $\overrightarrow{R_{11}}$, $\overrightarrow{R_{21}}$. Here, he used the parameter ruled surfaces by choosing as principle ruled surfaces.

In this study, it is aimed to find relation between the Blaschke vectors of parameter ruled surfaces of a line congruence which are not principle ruled surfaces. By this relation, we can find some basic formulae of the line space. (e.g.the Manheim's and Liouville's formulae).

2 Preliminaries

Let A=a+εa_0 be a dual number, $A \in ID = \{(a,a_0)|a,a_0 \in IR\}$ and ID be a commutative ring with a unit element. We call dual number $\varepsilon = (0,1) \in ID$ as dual unit which satisfies $\varepsilon^2 = (0,0).(D^3,+)$ is a module on the dual number ring. We call

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it *ID*-module, and dual vectors are the elements of this modul. We denote a unit vector $\vec{A}$ as

$$\vec{A} = (\vec{a}, \vec{a_0}) = \vec{a} + \varepsilon\vec{a_0}, < \vec{a}, \vec{a} >= 1, < \vec{a}, \vec{a}_0 >= 0, \tag{1}$$

where $\vec{a}, \vec{a_0} \in R^3$.

Definition 1. *The scalar product of two dual vectors* $\vec{A} = \vec{a} + \varepsilon\vec{a_0}$ *and* $\vec{B} = \vec{b} + \varepsilon\vec{b_0}$ *is given by*

$$< \vec{A}, \vec{B} >=< \vec{a}, \vec{b} > +\varepsilon(< \vec{a}, \vec{b_0} > + < \vec{a_0}, \vec{b} >). \tag{2}$$

Definition 2. [4] *The vectoral product between two dual vectors* $\vec{A} = \vec{a} + \varepsilon\vec{a_0}$ *and* $\vec{B} = \vec{b} + \varepsilon\vec{b_0}$ *is defined by*

$$\vec{A} \wedge \vec{B} = \vec{a} \wedge \vec{b} + \varepsilon(\vec{a} \wedge \vec{b_0} + \vec{a_0} \wedge \vec{b}). \tag{3}$$

3 The ruled surface and the line congruence

The Blaschke trihedron $\{\vec{R_1}, \vec{R_2}, \vec{R_3}\}$ depends on the striction point of the ruled surface $\vec{R_1}(t)$ in dual space ID^3,[4]. According to this, the first axis $\vec{R_1}(t)$ of the trihedron is the generator which passes from the striction point of the ruled surface , the second axis $\vec{R_2}(t)$ is normal of the surface at this point and finally the third axis $\vec{R_3}(t)$ is the tangent of the striction line at this point. The derivative formulae of the Blaschke trihedron $\{\vec{R_1}, \vec{R_2}, \vec{R_3}\}$ with respect to dual arc parameter S of the striction curve are written

$$\vec{R_1'} = P\vec{R_2}, \ \ \vec{R_2'} = -P\vec{R_1} + Q\vec{R_3}, \ \ \vec{R_3'} = -Q\vec{R_2}, \tag{4}$$

where $P = \| \vec{R_1'} \|$, $Q = det\frac{(R_1, R_1', R_1'')}{P^2}$.

A ruled surface is given as dual vectorial function by

$$\vec{R_1}(t) = \vec{r}(t) + \varepsilon\vec{r_0}(t), \ \ \vec{R_1}^2 = 1. \tag{5}$$

Definition 3. [3] *The dual spherical curvature of the ruled surface* $\vec{R_1}(t)$ *is defined as follows:*

$$\Sigma = \frac{Q}{P}. \tag{6}$$

On the other hand, the line congruence in ID^3 can be represented by a unit dual vector which depends on two real parameters u and v as follows:

$$\vec{R}(u,v) = \vec{r}(u,v) + \varepsilon\vec{r_0}(u,v), \ \ \vec{R}^2 = 1. \tag{7}$$

The dual arc element of a ruled surface of the line congruence can be given as

$$dS^2 = d\vec{R}^2 = (\vec{R}_u du + \vec{R}_v dv)^2 = Edu^2 + 2Fdudv + Gdv^2, \tag{8}$$

where $E = e + \varepsilon e_0 =< \vec{R}_u, \vec{R}_u >$, $F = f + \varepsilon f_0 =< \vec{R}_u, \vec{R}_v >$ and $G = g + \varepsilon g_0 =< \vec{R}_v, \vec{R}_v >$. Moreover, the differential form I and II of the line congruence are

$$I = edu^2 + 2fdudv + gdv^2 \tag{9}$$

$$II = e_0 du^2 + 2f_0 dudv + g_0 dv^2, \tag{10}$$

respectively. Thus, we have

$$dS^2 = I + \varepsilon II. \tag{11}$$

Definition 4. [4] *The drall of a ruled surface of a line congruence can be written as follows:*

$$\frac{1}{d} = \frac{I}{2II}. \tag{12}$$

4 The relations among the magnitudes of the ruled surface $\vec{R_1}$, $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$

Let us consider a ruled surface $\vec{R} = \vec{R_1}(t)$ of the line congruence $\vec{R} = \vec{R}(u,v)$ where u and v are the functions of t. Let us write the parameter ruled surfaces of the line congruence as

$$\overrightarrow{R_1 1} = \overrightarrow{R_1 1}(u, v_0), \quad \overrightarrow{R_1 1}^2 = 1 \tag{13}$$

and

$$\overrightarrow{R_2 1} = \overrightarrow{R_2 1}(u_0, v), \quad \overrightarrow{R_2 1}^2 = 1, \tag{14}$$

where the ruled surfaces $\vec{R_1}$,$\overrightarrow{R_1 1}$ and $\overrightarrow{R_2 1}$ have common line defined as

$$\vec{R_0} = \vec{R}(u_0, v_0) = \overrightarrow{R_1 1}(u_0, v_0) = \overrightarrow{R_2 1}(u_0, v_0) \tag{15}$$

The Blaschke trihedrons of these ruled surfaces are given by

$$\{\vec{R_0}, \vec{R_2}, \vec{R_3}\}, \{\vec{R_0}, \overrightarrow{R_1 2}, \overrightarrow{R_1 3}\}, \quad \{\vec{R_0}, \overrightarrow{R_2 2}, \overrightarrow{R_2 3}\} \tag{16}$$

and hence , one can get

$$\vec{R_1}' = P\vec{R_2}, \quad \vec{R_2}' = -P\vec{R_1} + Q\vec{R_3}, \quad \vec{R_3}' = -Q\vec{R_2}, \tag{17}$$

$$\overrightarrow{R_1 1}' = P_1 \overrightarrow{R_1 2}, \quad \overrightarrow{R_1 2}' = -P_1 \overrightarrow{R_1 1} + Q_1 \overrightarrow{R_1 3}, \quad \overrightarrow{R_1 3}' = -Q_1 \overrightarrow{R_1 2}, \tag{18}$$

$$\overrightarrow{R_2 1}' = P_2 \overrightarrow{R_2 2}, \quad \overrightarrow{R_{22}}' = -P_2 \overrightarrow{R_{21}} + Q_2 \overrightarrow{R_{23}}, \quad \overrightarrow{R_{23}}' = -Q_2 \overrightarrow{R_{22}}, \tag{19}$$

where $P_1 = \sqrt{\overrightarrow{R_u}^2} = \sqrt{E}$, $P_2 = \sqrt{\overrightarrow{R_v}^2} = \sqrt{G}$, and Q_1, Q_2 are dual curvatures of parameter ruled surfaces. The dual arc elements of these ruled surfaces are $\overrightarrow{R_1}$, $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$ can be given respectively as

$$dS = Pdt,\ dS_1 = P_1 du = \sqrt{E}du,\ dS_2 = P_2 dv = \sqrt{G}dv. \tag{20}$$

Moreover, the Blaschke vectors of the Blaschke trihedrons are given by

$$\overrightarrow{B} = Q\overrightarrow{R_0} + P\overrightarrow{R_3}, \overrightarrow{B_1} = Q_1\overrightarrow{R_0} + P_1\overrightarrow{R_{13}}, \overrightarrow{B_2} = Q_2\overrightarrow{R_0} + P_2\overrightarrow{R_{23}}. \tag{21}$$

If we choose the parameter ruled surfaces as principle ruled surfaces, we may write $F = \overrightarrow{R_u}\overrightarrow{R_v} = 0$, [2]. In this study we suppose that $F \neq 0$. Let us consider any parameter ruled surfaces of the line congruence $\overrightarrow{R}=\overrightarrow{R}(u,v)$, $\overrightarrow{R}^2 = 1$.

Remark. The second edges of the parameter ruled surfaces can be written by

$$\overrightarrow{R_{12}} = \frac{\overrightarrow{R_{11}}'}{P_1} = \frac{\overrightarrow{R}_u}{\sqrt{E}},\ \overrightarrow{R_{22}} = \frac{\overrightarrow{R_{21}}'}{P_2} = \frac{\overrightarrow{R}_v}{\sqrt{G}}. \tag{22}$$

From (4.10), we have

$$\overrightarrow{R_0} = \frac{\overrightarrow{R_u} \wedge \overrightarrow{R_v}}{\|\overrightarrow{R_u} \wedge \overrightarrow{R_v}\|} = \frac{\overrightarrow{R_u} \wedge \overrightarrow{R_v}}{\sqrt{EG} sin\Theta} \tag{23}$$

and

$$\overrightarrow{R_2} = \frac{\overrightarrow{R_1}'}{P} = \sqrt{E}\frac{du}{dS}\overrightarrow{R_{12}} + \sqrt{G}\frac{dv}{dS}\overrightarrow{R_{22}}. \tag{24}$$

On the other hand, let us consider the dual angle between the edges $\overrightarrow{R_{12}}$ and $\overrightarrow{R_{22}}$ as Θ, and the dual angle between the edges $\overrightarrow{R_2}$ and $\overrightarrow{R_{12}}$ as Φ. If we apply dot product both sides (4.12) with $\overrightarrow{R_{12}}$ and $\overrightarrow{R_{22}}$, we have

$$\overrightarrow{R_2}\overrightarrow{R_{12}} = \cos\Phi = \sqrt{E}\frac{du}{dS} + \cos\Theta\sqrt{G}\frac{dv}{dS}, \tag{25}$$

$$\overrightarrow{R_2}\overrightarrow{R_{22}} = \cos(\Theta - \Phi) = \cos\Theta\sqrt{E}\frac{du}{dS} + \sqrt{G}\frac{dv}{dS}. \tag{26}$$

Thus, from (4.13) and (4.14)

$$\frac{\sin(\Theta - \Phi)}{sin\Theta} = \sqrt{E}\frac{du}{dS}, \tag{27}$$

$$\frac{\sin\Phi}{sin\Theta} = \sqrt{G}\frac{dv}{dS} \tag{28}$$

are written. Finally, putting (4.15) and (4.16) into (4.12), we have the following equation between the second edges of the Blaschke trihedrons of the ruled surfaces $\overrightarrow{R_2}$, $\overrightarrow{R_{12}}$ and $\overrightarrow{R_{22}}$ by

$$\overrightarrow{R_2} = \frac{\sin(\Theta - \Phi)}{sin\Theta}\overrightarrow{R_{12}} + \frac{\sin\Phi}{sin\Theta}\overrightarrow{R_{22}}. \tag{29}$$

Moreover, by the definition of the angle Θ between $\overrightarrow{R_{12}}$ and $\overrightarrow{R_{22}}$ we write

$$\overrightarrow{R_{12}}\overrightarrow{R_{22}} = \cos\Theta. \tag{30}$$

Then, from (4.18) we have

$$\frac{\sin(\Theta-\Phi)}{sin\Theta} = \frac{dS_1}{dS} = \sqrt{E}\frac{du}{dS}, \tag{31}$$

$$\frac{\sin\Phi}{sin\Theta} = \frac{dS_2}{dS} = \sqrt{G}\frac{dv}{dS}. \tag{32}$$

Corollary 1. *The third elements $\overrightarrow{R_3}$, $\overrightarrow{R_{13}}$ and $\overrightarrow{R_{23}}$ of the Blaschke trihedrons of the ruled surfaces $\overrightarrow{R_1}$, $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$ are lineer dependent as follows:*

$$\overrightarrow{R_3} = \frac{\sin(\Theta-\Phi)}{sin\Theta}\overrightarrow{R_{13}} + \frac{\sin\Phi}{sin\Theta}\overrightarrow{R_{23}}. \tag{33}$$

Proof. If we substitute the equation (4.17) in $\overrightarrow{R_3} = \overrightarrow{R_0} \wedge \overrightarrow{R_2}$ and consider the Blaschke trihedrons of the parameter ruled surfaces $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$ we obtain (4.21).

Theorem 1. *The third elements $\overrightarrow{R_{13}}$ and $\overrightarrow{R_{23}}$ of the parameter ruled surfaces $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$ can be expressed by dual vectors $\overrightarrow{R_{12}}$ and $\overrightarrow{R_{21}}$ as follows:*

$$\overrightarrow{R_{13}} = \frac{1}{sin\Theta}(-cos\Theta\overrightarrow{R_{12}} + \overrightarrow{R_{22}}), \tag{34}$$

$$\overrightarrow{R_{23}} = \frac{1}{sin\Theta}(-\overrightarrow{R_{12}} + cos\Theta\overrightarrow{R_{22}}) \tag{35}$$

Proof. From the equations (4.4) and (4.11) we write

$$\overrightarrow{R_{12}} \wedge \overrightarrow{R_{13}} = \overrightarrow{R_0}, \overrightarrow{R_{22}} \wedge \overrightarrow{R_{23}} = \overrightarrow{R_0}, \overrightarrow{R_{12}} \wedge \overrightarrow{R_{22}} = \sin\Theta\overrightarrow{R_0}, \tag{36}$$

$$\overrightarrow{R_12} \wedge (\overrightarrow{R_{13}} - \frac{\overrightarrow{R_{22}}}{sin\Theta}) = \overrightarrow{0} \Rightarrow \sin\Theta\overrightarrow{R_{13}} - \overrightarrow{R_22} = M\overrightarrow{R_{12}}, \tag{37}$$

$$\overrightarrow{R_{22}} \wedge (\overrightarrow{R_{23}} + \frac{\overrightarrow{R_{12}}}{sin\Theta}) = \overrightarrow{0} \Rightarrow \sin\Theta\overrightarrow{R_{23}} + \overrightarrow{R_{12}} = N\overrightarrow{R_{22}}. \tag{38}$$

Here, M and N are dual scalars. And if we apply dot product of (4.25) and (4.26) by the dual vectors $\overrightarrow{R_{12}}$ and $\overrightarrow{R_{22}}$ respectively, and also with the relation (4.4) and (4.21), we have $M = -\cos\Theta$ and $N = \cos\Theta$. Finally, substuting the dual vectors M and N in (4.25) and (4.26), we obtain (4.22) and (4.23).

Corollary 2. *The Blaschke vectors of the ruled surfaces $\overrightarrow{R_1}$, $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$, which are defined in (4.9), can be written according to unit dual vectors $\overrightarrow{R_{12}}$, $\overrightarrow{R_{22}}$ and $\overrightarrow{R_0}$ as follows:*

$$\overrightarrow{B} = Q\overrightarrow{R_0} - P(\frac{\cos(\Theta-\Phi)}{sinh\Theta}\overrightarrow{R_{12}} - \frac{\cos\Phi}{sin\Theta}\overrightarrow{R_{2_2}}), \tag{39}$$

$$\vec{B_1} = Q_1\vec{R_0} + \frac{P_1}{sin\Theta}(-\cos\Theta\overrightarrow{R_{12}} + \overrightarrow{R_{22}}), \tag{40}$$

$$\vec{B_2} = Q_2\vec{R_0} + \frac{P_2}{sin\Theta}(-\overrightarrow{R_{12}} + \cos\Theta\overrightarrow{R_{22}}). \tag{41}$$

Theorem 2. *Let $\vec{R_1}$ and $\overrightarrow{R_{11}}$, $\overrightarrow{R_{21}}$ be a ruled surface and parameter ruled surfaces of congruence $\overrightarrow{R(u,v)}$, respectively. Then, we have*

$$\overrightarrow{R_{12}}\frac{\partial\overrightarrow{R_{22}}}{\partial S_1} = -\overrightarrow{R_{22}}.\frac{\partial\overrightarrow{R_{12}}}{\partial S_1} = \frac{(\sqrt{E})_v - \cos\Theta(\sqrt{G})_u}{\sqrt{EG}}, \tag{42}$$

$$\overrightarrow{R_{22}}\frac{\partial\overrightarrow{R_{12}}}{\partial S_2} = -\overrightarrow{R_{12}}.\frac{\partial\overrightarrow{R_{22}}}{\partial S_2} = \frac{(\sqrt{G})_u - \cos\Theta(\sqrt{E})_v}{\sqrt{EG}}. \tag{43}$$

where Θ is dual angle between $\overrightarrow{R_{12}}$ and $\overrightarrow{R_{22}}$.

Proof. Differentiating (4.10) according to the parameters v and u we have

$$\frac{\partial\overrightarrow{R_{12}}}{\partial v} = \frac{\overrightarrow{R_{uv}}\sqrt{E} - (\sqrt{E})_v\vec{R_u}}{E} \tag{44}$$

and

$$\frac{\partial\overrightarrow{R_{22}}}{\partial u} = \frac{\overrightarrow{R_{uv}}\sqrt{G} - (\sqrt{G})_u\vec{R_v}}{G} \tag{45}$$

Then,we get

$$\overrightarrow{R_{12}}\frac{\partial\overrightarrow{R_{22}}}{\partial u} = \frac{\vec{R_u}}{\sqrt{E}}(\frac{\overrightarrow{R_{uv}}\sqrt{G} - (\sqrt{G})_u\vec{R_v}}{G}) = \frac{(\sqrt{E})_v - \cos\Theta(\sqrt{G})_u}{\sqrt{G}}, \tag{46}$$

$$\overrightarrow{R_{22}}\frac{\partial\overrightarrow{R_{12}}}{\partial v} = \frac{\vec{R_v}}{\sqrt{G}}.(\frac{\overrightarrow{R_{uv}}\sqrt{E} - (\sqrt{E})_v\vec{R_u}}{E}) = \frac{(\sqrt{G})_u - \cos\Theta(\sqrt{E})_v}{\sqrt{E}}. \tag{47}$$

On the other hand, considering (4.8) in (4.34) and (4.35) we obtain

$$\overrightarrow{R_{12}}.\frac{\partial\overrightarrow{R_{22}}}{\partial S_1} = \frac{1}{\sqrt{E}}\overrightarrow{R_{12}}\frac{\partial\overrightarrow{R_{22}}}{\partial u} = \frac{(\sqrt{E})_v - \cos\Theta(\sqrt{G})_u}{\sqrt{EG}}, \tag{48}$$

$$\overrightarrow{R_{22}}.\frac{\partial\overrightarrow{R_{12}}}{\partial S_2} = \frac{1}{\sqrt{G}}\overrightarrow{R_{22}}\frac{\partial\overrightarrow{R_{12}}}{\partial v} = \frac{(\sqrt{G})_u - \cos\Theta(\sqrt{E})_v}{\sqrt{EG}}. \tag{49}$$

Finally, differentiating dual arcs S_1 and S_2, we obtain

$$\overrightarrow{R_{12}}.\frac{\partial\overrightarrow{R_{22}}}{\partial S_1} = -\overrightarrow{R_{22}}\frac{\overrightarrow{R_{12}}}{\partial S_1}, \tag{50}$$

$$\overrightarrow{R_{22}}.\frac{\partial\overrightarrow{R_{12}}}{\partial S_2}=-\overrightarrow{R_{12}}\frac{\partial\overrightarrow{R_{22}}}{\partial S_2}. \quad (51)$$

Thus,from (4.36) and (4.38),(4.30) is obtained.In a similar way one can obtain (4.31).

Proposition 1. *The Blaschke trihedrons* $\{\overrightarrow{R_0},\overrightarrow{R_{12}},\overrightarrow{R_{13}}\}$ *and* $\{\overrightarrow{R_0},\overrightarrow{R_{22}},\overrightarrow{R_{23}}\}$ *of the parameter ruled surfaces of the congruence* $\overrightarrow{R(u,v)}$ *always coincide such that the dual unit vectors between* $\overrightarrow{R_{12}}$ *and* $\overrightarrow{R_{22}}$ *are orthogonal. Moreover, the Blaschke derivative formulae are given by*

$$\frac{d\overrightarrow{R_{12}}}{dS_2}=\overrightarrow{B_2}\wedge\overrightarrow{R_{12}},\ \ \frac{d\overrightarrow{R_{22}}}{dS_1}=\overrightarrow{B_1}\wedge\overrightarrow{R_{22}},\ \ \frac{d\overrightarrow{R_0}}{dS}=\overrightarrow{B}\wedge\overrightarrow{R_0}, \quad (52)$$

$$\frac{d\overrightarrow{R_{12}}}{dS_1}=\overrightarrow{B_1}\wedge\overrightarrow{R_{12}},\ \ \frac{d\overrightarrow{R_{22}}}{dS_2}=\overrightarrow{B_2}\wedge\overrightarrow{R_{22}}. \quad (53)$$

Proof. If we take $\Theta=\frac{\pi}{2}$ in (4.22) and (4.23), we have first assertion.

$$\overrightarrow{R_{13}}=\overrightarrow{R_{22}},\overrightarrow{R_{23}}=-\overrightarrow{R_{12}}. \quad (54)$$

Differentiating (4.22) according to the dual arc paramater S_1, we get

$$\frac{\partial\overrightarrow{R_{13}}}{\partial S_1}=\frac{1}{\sin\Theta}(-\cos\Theta\frac{\partial\overrightarrow{R_{12}}}{\partial S_1}+\frac{\partial\overrightarrow{R_{22}}}{\partial S_1}), \quad (55)$$

$$=\frac{1}{\sin\Theta}(-\cos\Theta(\vec{B}_1\wedge\overrightarrow{R_{12}})+\frac{\overrightarrow{R_{22}}}{\partial S_1}).$$

Moreover, from (4.41) and (4.22) we know

$$\frac{\partial\overrightarrow{R_{13}}}{\partial S_1}=\vec{B}_1\wedge\overrightarrow{R_{13}}=\vec{B}_1\wedge(\frac{1}{\sin\Theta}(-\cos\Theta(\vec{B}_1\wedge\overrightarrow{R_{12}})+(\vec{B}_1\wedge\overrightarrow{R_{22}}) \quad (56)$$

Thus, from the second relation of (4.43) and (4.44), we get the second relation of (4.40). Similarly, it can be shown that other relations are satisfied.

Theorem 3. *The dual curvatures* Q_1 *and* Q_2 *of the parameter ruled surfaces* $\overrightarrow{R_{11}}$ *and* $\overrightarrow{R_{21}}$ *are given by*

$$Q_1=-\frac{1}{\sin\Theta\sqrt{G}}((\sqrt{E})_v-\cos\Theta(\sqrt{G})_u), \quad (57)$$

$$Q_2=\frac{1}{\sin\Theta\sqrt{E}}((\sqrt{G})_u-\cos\Theta(\sqrt{E})_v). \quad (58)$$

Proof. From (4.36) and (4.8) we have

$$\overrightarrow{R_{12}}.\frac{\partial\overrightarrow{R_{22}}}{\partial S_1}=\frac{(\sqrt{E})_v-\cos\Theta(\sqrt{G})_u}{\sqrt{EG}}, \quad (59)$$

$$\frac{\partial \overrightarrow{R_{22}}}{\partial S_1} = \frac{1}{\sqrt{E}} \frac{\partial \overrightarrow{R_{22}}}{\partial u} = \frac{1}{\sqrt{E}} \overrightarrow{B_1} \wedge \overrightarrow{R_{22}}. \tag{60}$$

Then, we obtain

$$\frac{(\sqrt{E})_v - \cos\Theta(\sqrt{G})_u}{\sqrt{EG}} = \frac{1}{\sqrt{E}} \overrightarrow{R_{12}} \frac{\partial \overrightarrow{R_{22}}}{\partial u} \tag{61}$$

$$= \frac{1}{\sqrt{E}} \overrightarrow{R_{12}} (\overrightarrow{B_1} \wedge \overrightarrow{R_{22}}) = \frac{-\sin\Theta \overrightarrow{R_0} \overrightarrow{B_1}}{\sqrt{E}}$$

On the other hand , from (4.9), taking dot product $\overrightarrow{B_1}$ with $-\sin\Theta\overrightarrow{R_0}$ we have

$$\frac{-\sin\Theta \overrightarrow{R_0} \overrightarrow{B_1}}{\sqrt{E}} = -\frac{Q_1 \sin\Theta}{\sqrt{E}}. \tag{62}$$

Thus, from(4.49) and (4.50) we get Q_1. Similarly, (4.46) can be obtained.

Theorem 4. *Let us consider any ruled surface $\overrightarrow{R_1}$ of a line congruence $\overrightarrow{R(u,v)}$ and the dual arc elements of the ruled surfaces $\overrightarrow{R_1}$, $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{22}}$ as S,S_1 and S_2, respectively. Let $\overrightarrow{B_1}$ and $\overrightarrow{B_2}$ be Blaschke vectors of $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$. We also suppose that the dual angle between the line $\overrightarrow{R_2}$ and $\overrightarrow{R_{12}}$ is Φ, then we have the following relations*

$$\overrightarrow{C} = \frac{1}{P_1} \frac{\sin(\Theta - \Phi)}{sin\Theta} \overrightarrow{B}_1 + \frac{1}{P_2} \frac{\sin\Phi}{sin\Theta} \overrightarrow{B}_2 \tag{63}$$

and

$$\frac{d\overrightarrow{R_{12}}}{dS} = \overrightarrow{C} \wedge \overrightarrow{R_{12}}, \frac{d\overrightarrow{R_{22}}}{dS} = \overrightarrow{C} \wedge \overrightarrow{R_{22}}, \frac{d\overrightarrow{R_0}}{dS} = \overrightarrow{C} \wedge \overrightarrow{R_0} \tag{64}$$

Proof. We know that the dual vector $\overrightarrow{R_{12}}$ is the function of S_1 and S_2. Hence we have

$$\frac{d\overrightarrow{R_{12}}}{dS} = \frac{\partial \overrightarrow{R_{12}}}{\partial S_1} \frac{dS_1}{dS} + \frac{\partial \overrightarrow{R_{12}}}{dS_2} \frac{dS_2}{dS}. \tag{65}$$

From the equalities (4.19), (4.20), (4.40) and (4.41) we obtain

$$\frac{dR_{12}}{dS} = \frac{\sin(\Theta - \Phi)}{sin\Theta} \frac{1}{\sqrt{E}} (\overrightarrow{B_1} \wedge \overrightarrow{R_{12}}) + \frac{\sin\Phi}{sin\Theta} \frac{1}{\sqrt{E}} (\overrightarrow{B_2} \wedge \overrightarrow{R_{12}})$$

$$= (\frac{\sin(\Theta - \Phi)}{sin\Theta} \frac{1}{P_1} \overrightarrow{B_1} + \frac{\sin\Phi}{sin\Theta} \frac{1}{P_2} \overrightarrow{B_2}) \wedge \overrightarrow{R_{12}}$$

$$= \overrightarrow{C} \wedge \overrightarrow{R_{12}}.$$

Thus, we have the first relation of (4.52). Other assertion can be obtained.

Theorem 5. *Let us consider the ruled surfaces $\overrightarrow{R_1}$, $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{22}}$ which have common line $\overrightarrow{R_0}$ on the line congruence $\overrightarrow{R(u,v)}$. Let $\overrightarrow{B_1}$ and $\overrightarrow{B_2}$ be Blaschke vectors of $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$. We obtain following equality among the Blaschke vectors*

$$\vec{B} = P\left(\frac{1}{P_1}\frac{\sin(\Theta-\Phi)}{sin\Theta}\vec{B}_1 + \frac{1}{P_2}\frac{\sin\Phi}{sin\Theta}\vec{B}_2 + \frac{d\Phi}{dS}\overrightarrow{R_0}\right). \tag{66}$$

Proof. From (4.17), we have

$$\overrightarrow{R_2} = \frac{\sin(\Theta-\Phi)}{sin\Theta}\overrightarrow{R_{12}} + \frac{\sin\Phi}{sin\Theta}\overrightarrow{R_{22}}. \tag{67}$$

Then, by taking derivative with respect to dual arc S, from equation (4.17), we obtain

$$\frac{d\overrightarrow{R_2}}{dS} = \frac{d\overrightarrow{R_{12}}}{dS}\frac{\sin(\Theta-\Phi)}{sin\Theta} + \frac{d\overrightarrow{R_{22}}}{dS}\frac{\sin\Phi}{sin\Theta} - \tag{68}$$

$$\frac{\cos(\Theta-\Phi)}{\sin\Theta}\frac{d\Phi}{dS}\overrightarrow{R_{12}} + \frac{\cos\Phi}{sin\Theta}\frac{d\Phi}{dS}\overrightarrow{R_{22}}.$$

On the other hand, considering the Blaschke trihedrons $\{\overrightarrow{R_0},\overrightarrow{R_{12}},\overrightarrow{R_{13}}\}$ and $\{\overrightarrow{R_0},\overrightarrow{R_{22}},\overrightarrow{R_{23}}\}$, we write

$$\overrightarrow{R_{13}} = \overrightarrow{R_{12}} \wedge \overrightarrow{R_0} = \overrightarrow{R_{12}} \wedge \left(\frac{\overrightarrow{R_{12}} \wedge \overrightarrow{R_{22}}}{sin\Theta}\right) \tag{69}$$

$$= \frac{(<\overrightarrow{R_{12}},\overrightarrow{R_{22}}>\overrightarrow{R_{12}} - <\overrightarrow{R_{12}},\overrightarrow{R_{12}}>\overrightarrow{R_{22}})}{sin\Theta}$$

$$= \frac{1}{sin\Theta}(cos\Theta\overrightarrow{R_{12}} - \overrightarrow{R_{22}})$$

and

$$\overrightarrow{R_{23}} = \overrightarrow{R_{22}} \wedge \overrightarrow{R_0} = \overrightarrow{R_{22}} \wedge \left(\frac{\overrightarrow{R_{12}} \wedge \overrightarrow{R_{22}}}{\sin\Theta}\right) \tag{70}$$

$$= \frac{<\overrightarrow{R_{22}},\overrightarrow{R_{22}}>\overrightarrow{R_{12}} - <\overrightarrow{R_{22}},\overrightarrow{R_{12}}>\overrightarrow{R_{22}}}{sin\Theta}$$

$$= \frac{1}{sin\Theta}(\overrightarrow{R_{12}} - \cos\Theta\overrightarrow{R_{22}}).$$

And then, substituting the equations (4.57) and (4.58) in -$\overrightarrow{R_{12}} = \overrightarrow{R_0} \wedge \overrightarrow{R_{13}}$ and $-\overrightarrow{R_{22}} = \overrightarrow{R_0} \wedge \overrightarrow{R_{23}}$ we have (4.55) as follows

$$\frac{d\overrightarrow{R_2}}{dS} = \frac{d\overrightarrow{R_{12}}}{dS}\frac{\sin(\Theta-\Phi)}{sin\Theta} + \frac{d\overrightarrow{R_{22}}}{dS}\frac{\sin\Phi}{sin\Theta} \tag{71}$$

$$+\left[-\overrightarrow{R_0} \wedge \frac{(cos\Theta\overrightarrow{R_{12}} - \overrightarrow{R_{22}})}{\sin\Theta}\frac{\cos(\Theta-\Phi)}{sin\Theta} + \overrightarrow{R_0} \wedge \frac{(\overrightarrow{R_{12}} - \cos\Theta\overrightarrow{R_{22}})}{\sin\Theta}\frac{\cos\Phi}{sin\Theta}\right]\frac{d\Phi}{dS}.$$

According to theorem 4, we have

$$\frac{d\overrightarrow{R_{12}}}{dS}=\vec{C}\wedge\overrightarrow{R_{12}},\frac{d\overrightarrow{R_{22}}}{dS}=\vec{C}\wedge\overrightarrow{R_{22}}, \tag{72}$$

$$\vec{C}=\frac{1}{P_1}\frac{\sin(\Theta-\Phi)}{sin\Theta}\vec{B}_1+\frac{1}{P_2}\frac{\sin\Phi}{sin\Theta}\vec{B}_2. \tag{73}$$

By using the dual trigonometric expression, we find

$$\frac{d\overrightarrow{R_2}}{dS}=\frac{\sin(\Theta-\Phi)}{sin\Theta}\vec{C}\wedge\overrightarrow{R_{12}}+\frac{\sin\Phi}{sin\Theta}\vec{C}\wedge\overrightarrow{R_{22}}+ \tag{74}$$

$$\overrightarrow{R_0}\wedge[\frac{\sin(\Theta-\Phi)}{sin\Theta}\overrightarrow{R_{12}}+\frac{\sin\Phi}{sin\Theta}\overrightarrow{R_{22}}]\frac{d\Phi}{dS}$$

$$=(\vec{C}+\overrightarrow{R_0}\frac{d\Phi}{dS})\wedge\overrightarrow{R_2}.$$

Since we have

$$\vec{M}=\vec{C}+\overrightarrow{R_0}\frac{d\Phi}{dS} \tag{75}$$

and

$$\vec{C}=\vec{M}-\overrightarrow{R_0}\frac{d\Phi}{dS}. \tag{76}$$

Thus, (4.62) is written by

$$\frac{d\overrightarrow{R_2}}{dS}=\vec{M}\wedge\overrightarrow{R_2}. \tag{77}$$

From the last equation of (4.52) we have

$$\frac{d\overrightarrow{R_0}}{dS}=\vec{C}\wedge\overrightarrow{R_0} \tag{78}$$

$$=(\vec{M}-\overrightarrow{R_0}\frac{d\Theta}{dS})\wedge\overrightarrow{R_0}$$

$$=(\vec{M}\wedge\overrightarrow{R_0})$$

and

$$\frac{d\overrightarrow{R_0}}{dS}=\vec{M}\wedge\overrightarrow{R_0}.$$

By using the Blaschke vector of $\frac{d\vec{R_2}}{dS}$ and (4.65) we obtain

$$\vec{M} \wedge \vec{R_2} - \vec{B} \wedge \vec{R_2} = 0$$

$$(\vec{M} - \vec{B}) \wedge \vec{R_2} = 0$$

$$\vec{M} - \vec{B} = \Lambda \, \vec{R_2}; \;\; \Lambda \varepsilon D \tag{79}$$

and

$$\vec{M} \wedge \vec{R_2} - \vec{B} \wedge \vec{R_0} = 0$$

$$(\vec{M} - \vec{B}) \wedge \vec{R_0} = 0$$

$$\vec{M} - \vec{B} = \Omega \, \vec{R_0}, \Omega \varepsilon D. \tag{80}$$

From (4.67) and (4.68) we have

$$\Lambda \vec{R_2} = \Omega \vec{R_0}, \tag{81}$$

$$\Lambda = \Omega = 0. \tag{82}$$

Consequently, we have obtained

$$\vec{M} = \vec{B}. \tag{83}$$

and hence we conclude

$$\vec{B} = \vec{C} + \vec{R_0}\frac{d\Theta}{dS}. \tag{84}$$

Corollary 3. *Let $\vec{R}$ and $\overrightarrow{R_{11}}$, $\overrightarrow{R_{21}}$ be a ruled surface and parameter ruled surface of the line congruence $R(u,v)$, respectively.Then, we have*

$$\Sigma = \Sigma_1 \frac{\sin(\Theta - \Phi)}{sin\Theta} + \Sigma_2 \frac{\sinh \Phi}{sin\Theta} + \frac{d\Phi}{dS} \tag{85}$$

where Θ is dual angle between $\overrightarrow{R_{12}}$ and $\overrightarrow{R_{22}}$, Φ is dual angle between $\vec{R_2}$ and $\overrightarrow{R_{12}}$, Σ is dual spherical curvature of the ruled surface $\vec{R_1}$, Σ_1 is dual spherical curvature of the ruled surface $\overrightarrow{R_{11}}$,Σ_2 is dual spherical curvature of the ruled surface $\overrightarrow{R_{22}}$.

Proof. If we substitute the Blaschke vectors $\vec{B}$, $\vec{B_1}$ and $\vec{B_2}$ in (4.54) and then taking dot product of both sides by $\vec{R_0}$, we have

$$Q = P(\frac{Q_1}{P_1}\frac{\sin(\Theta - \Phi)}{sin\Theta} + \frac{Q_2}{P_2}\frac{\sin \Phi}{sin\Theta} + \frac{d\Phi}{dS}). \tag{86}$$

Thus, considering (3.3) we get assertion.

Theorem 6.*(Mannheim's Formula) The following relation*

$$P = P_1 \frac{\sin(\Theta - \Phi)}{sin\Theta} + P_2 \frac{\sinh \Phi}{sin\Theta}] \tag{87}$$

is satisfied among the dual curvatures of ruled surfaces $\overrightarrow{R_1}$, $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$ of the line congruence $\overrightarrow{R(u,v)}$.

*Proof.*Substituting the Blaschke vectors $\overrightarrow{B}, \overrightarrow{B_1}$ *and*$\overrightarrow{B_2}$ in (4.54) and then taking dot product both of sides by $\overrightarrow{R_3}$, desired equation is obtained.

Theorem 7. *(Liouville's Formula) There are following relation among the dual torsions of the ruled surfaces $\overrightarrow{R_1}$, $\overrightarrow{R_{11}}$ and $\overrightarrow{R_{21}}$ of the spacelike line congruence $\overrightarrow{R(u,v)}$ as*

$$Q = Q_1 \frac{\sin(\Theta - \Phi)}{sin\Theta} + Q_2 \frac{\sin \Phi}{sin\Theta} + \frac{d\Phi}{dS}. \tag{88}$$

Proof. If we substitute the Blaschke vectors $\overrightarrow{B}, \overrightarrow{B_1}$ and $\overrightarrow{B_2}$ in (4.54) and then taking dot product both of sides by $\overrightarrow{R_0}$, desired equation is obtained.

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