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Relations characteristic of theta functions according to quarter periods

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Abstract: In this study, using the characteristic values $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \equiv \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix} (\mathrm{mod}\,2)$ a theorem on the $\frac{1}{2^r}$ coefficients of periods of first order theta function according to the $(1,\tau)$ period pair (for $r \in N^+$) is established . The following equalities are also obtained.

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau = i\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$$

and

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau) = \theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

Keywords: Characteristic values, Theta function, period pair, elliptic function.

1 Introduction

Let $\Gamma = SL_2(Z)$, we define Γ_N (or $\Gamma(N)$) for each positive integer N to be subgroup of the modular group Γ consisting of those matrices satisfying the condition $\begin{bmatrix}a & b\\ c & d\end{bmatrix} \equiv I(modN)$

For unit matrix $I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ in other words, $a \equiv d \equiv 1(\,\mathrm{mod}\,N)$ and $c \equiv b \equiv 0(modN)$ [2].

We first define a theta characteristic to be a two by one matrix of integers, written $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$. Next, given a complex number u, and another complex number $\tau(\mathrm{Im}\tau) > 0, \Im$, to denote the upper half-plane). Z for the set of rational integers and $\Gamma(1)$ for the group. Let $N \geq 1$ be an integer and put

$$\Gamma_0(N) = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma(1):\ c \equiv 0(modN)\right\}.$$

Let be

$$\mathrm{U} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \mathrm{V} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \mathrm{W} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \mathrm{P} = \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}.$$

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$\Gamma_U(2)$, $\Gamma_V(2)$ and $\Gamma_W(2)$ are defined by

$$\begin{aligned} \Gamma_U(2) &= \{S \in \Gamma(1): \quad S \equiv I \text{ or } \quad S \equiv U(Mod2)\} \\ \Gamma_V(2) &= \{S \in \Gamma(1): \quad S \equiv I \text{ or } \quad S \equiv V(Mod2)\} \\ \Gamma_W(2) &= \{S \in \Gamma(1): \quad S \equiv I \text{ or } \quad S \equiv W(Mod2)\} \end{aligned}$$

were I is the unit matrix. The three subgroups $\Gamma_U(2)$, $\Gamma_V(2)$ and $\Gamma_W(2)$ are conjugate. The subgroup θ of $\Gamma(1)$ is generated by U and V. For an odd positive integer n, the set of elements in θ of the from

$$\begin{pmatrix} a & b \\ nc & d \end{pmatrix}$$

is a subgroup of which will be denoted $\theta(n)$ [3].

Definition 1. *For $u \in C$, $\tau \in \Im$ and characteristic value $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$, the function defined as*

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{ (n+\frac{\varepsilon}{2})^2 \pi i \tau + 2\pi i (n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) \right\} \tag{1}$$

is called first order theta function [1].

Definition 2. *A half-period is half of a period (in particular a complex vector), written*

$$\begin{bmatrix} \mu \\ \mu' \end{bmatrix} \equiv \frac{1}{2}\begin{Bmatrix} \mu \\ \mu' \end{Bmatrix} = \frac{\mu'}{2} + \frac{\mu\tau}{2}.$$

A reduced half-period is half period in which μ and μ' equal 0 *or* 1 *where μ and μ' are integers* [1].

In the present paper, whenever the integers μ and μ' will be as $\mu = 1$ and $\mu' = 1$, unless otherwise stated. In this study,

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\bmod 2)$$

values of characteristic are used. When the periodicity of the function $\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u,\tau)$ for $(1,\tau)$ period pair is examined.

$$\begin{aligned} \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u+1,\tau) &= \sum_{n=-\infty}^{\infty} \exp\left\{ (n+\tfrac{\varepsilon}{2})^2 \pi i \tau + 2\pi i (n+\tfrac{\varepsilon}{2})(u+1+\tfrac{\varepsilon'}{2}) \right\} \\ &= \sum_{n=-\infty}^{\infty} \exp\left\{ (n+\tfrac{\varepsilon}{2})^2 \pi i \tau + 2\pi i (n+\tfrac{\varepsilon}{2})(u+\tfrac{\varepsilon'}{2}) + n2\pi i + \pi i \varepsilon \right\} \\ &= \left((-1)^{\varepsilon} \cdot \theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u,\tau) \right), \end{aligned}$$

also

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\tau,\tau)=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\tfrac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\tfrac{\varepsilon}{2})(u+\tau+\tfrac{\varepsilon'}{2})\right\}$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\tfrac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\tfrac{\varepsilon}{2})(u+\tfrac{\varepsilon'}{2})+n2\pi i\tau+\pi i\tau\varepsilon\right\}$$
$$=(-1)^{\varepsilon'}\exp(-\pi i\tau-2\pi iu)\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau).$$

and

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\tau+1,\tau)=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\tfrac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\tfrac{\varepsilon}{2})(u+\tau+1+\tfrac{\varepsilon'}{2})\right\}$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\tfrac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\tfrac{\varepsilon}{2})(u+\tfrac{\varepsilon'}{2})+n2\pi i\tau+\pi i\tau\varepsilon\right\}$$
$$=(-1)^{\varepsilon}\exp(-\pi i\tau-2\pi iu-\pi i\varepsilon')\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau).$$

By using $\eta_1=(-1)^{\varepsilon},\eta_2=(-1)^{\varepsilon'}\exp(-\pi i\tau-2\pi iu)$ and $\eta_3=(-1)^{\varepsilon}\exp(-\pi i\tau-2\pi iu-\pi i\varepsilon')$ we obtain

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\tau+1,\tau)=\eta_3.\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau).$$

As it is seen here, for $\eta_3=1$, because $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,\tau)$ is doubly periodic, it would be an elliptic function.

Theorem 1.
$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}(2u+\varepsilon')\pi i\right\}.\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau)$$

where $r\in N^+$.

Proof.

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u+\tfrac{1}{2^r}+\tfrac{\tau}{2^r},\tau)=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\tfrac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\tfrac{\varepsilon}{2})(u+\tfrac{1}{2^r}+\tfrac{\tau}{2^r}++\tfrac{\varepsilon'}{2})\right\}$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\tfrac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\tfrac{\varepsilon}{2})(u+\tfrac{\varepsilon'}{2})+\tfrac{n\pi i\tau}{2^{r-1}}+\tfrac{\pi i\tau}{2^{r-1}}+\tfrac{\pi i\tau\varepsilon'}{2^r}+\tfrac{\pi i\varepsilon'}{2^r}\right\}. \quad (2)$$

On the other hand, the reduced representative of an arbitrary characteristic

$$\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$$

to be that reduced characteristic whose entries are the least nonnegative residues (mod2) of ε and ε'.

There are four reduced characteristics $\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}$. But $\theta\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)\equiv 0$.

$$\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\tfrac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\tfrac{\varepsilon}{2})(u+\tfrac{\varepsilon'}{2})+\tfrac{n\pi i\tau}{2^{r-1}}+\tfrac{2\pi iu}{2^r}+\tfrac{2\pi i\varepsilon'}{2^r}+\tfrac{\pi i\tau\varepsilon}{2^r}+\tfrac{n\pi i}{2^{r-1}}+\tfrac{\pi i\tau}{4^r}+\tfrac{2\pi i}{4^r}+\tfrac{\pi i\varepsilon}{2^r}\right\}$$

and

$$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}(2u+\varepsilon')\pi i\right\}\theta\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\ \varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}(u,\tau)$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})+\frac{n\pi i\tau}{2^{r-1}}+\frac{\pi i\tau\varepsilon}{2^r}+\frac{n\pi i}{2^{r-1}}+\frac{\pi i\varepsilon}{2^r}\right\}. \tag{3}$$

By the theorem given above we can obtain the following characteristic equalities for $u=0$ value of the complex variable

(a)

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}\pi i\right\}.\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2}+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2}+\frac{1}{2^r})(0+\frac{1}{2}+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}+\frac{\pi i}{2^{r-1}}+\frac{\pi i}{2^r}\right\}$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2})^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+\frac{\pi i\tau}{2^r}+\frac{n\pi i}{2^{r-1}}+\frac{\pi i}{2^r}+\frac{\pi i}{2}+n\pi i\right\} \tag{4}$$

and

$$\theta\begin{bmatrix}1\\0\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}.\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2}+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2}+\frac{1}{2^r})(0+\frac{1}{2}+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}-\frac{\pi i}{2^{r-1}}\right\}$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2})^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+\frac{\pi i\tau}{2^r}+\frac{n\pi i}{2^{r-1}}+\frac{\pi i}{2^r}+\frac{\pi i}{2}+n\pi i\right\}. \tag{5}$$

From the equations (4) and (5), we can get the following equality

$$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}\pi i\right\}\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}\theta\begin{bmatrix}1+\frac{1}{2^{r-1}}\\ 0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau).$$

(b)

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}.\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\ 0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2^r})(0+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}+\frac{\pi i}{2^{r-1}}\right\}$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{n^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+\frac{n\pi i}{2^{r-1}}\right\} \tag{6}$$

and

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(0+\frac{1}{2^r}+\frac{\tau}{2^r},\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{\pi i}{2^r}\right\}.\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\frac{1}{2^r})^2\pi i\tau+2\pi i(n+\frac{1}{2^r})(0+\frac{1}{2}+\frac{1}{2^r})-\frac{\pi i\tau}{4^r}-\frac{\pi i}{2^{r-1}}-\frac{\pi i}{2^r}\right\}$$
$$=\sum_{n=-\infty}^{\infty}\exp\left\{n^2\pi i\tau+\frac{n\pi i\tau}{2^{r-1}}+n\pi i+\frac{n\pi i}{2^{r-1}}\right\} \tag{7}$$

If $n=2k\in Z$, then from the equalities (6) and (7) the following is obtained

$$\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i-\frac{1}{2^r}\pi i\right\}\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\1+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)=\exp\left\{-\frac{1}{4^r}(\tau+2)\pi i\right\}\theta\begin{bmatrix}0+\frac{1}{2^{r-1}}\\0+\frac{1}{2^{r-1}}\end{bmatrix}(0,\tau)$$

With the help of this theorem proved, transformations among theta functions can be found for characteristic value $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}$ according to all multiples $\frac{1}{2^r}$ of the periods. The subject that should be discussed here is; characteristic values

$$\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(mod2)$$

of first order theta function can be expressed as characteristic values

$$\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\\varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}.$$

This situation has proved that theta functions are generalized as characteristic values

$$\begin{bmatrix}\varepsilon+\frac{1}{2^{r-1}}\\\varepsilon'+\frac{1}{2^{r-1}}\end{bmatrix}.$$

With the help of this alternative formula above, we can get the following equalities according to quarter-periods. If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix}(mod2)$ then

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\sum_n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+2\pi i(n+\frac{1}{2})(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}+\frac{1}{2}\right\}$$
$$=ie^{-\frac{\pi i\tau}{4}}\sum_n(-1)^n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}. \tag{8}$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\0\end{bmatrix}(mod2)$ then

$$\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\sum_n\exp\left\{(n+\tfrac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\tfrac{1}{2})(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}$$
$$=e^{-\frac{\pi i\tau}{4}}\sum_n\exp\left\{(n+\tfrac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\tfrac{n\pi i}{2}+\tfrac{n\pi i\tau}{2}+\tfrac{\pi i\tau}{4}+\tfrac{\pi i}{4}\right\}. \tag{9}$$

Using the equations (8) and (9) we can get

$$\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}=\frac{ie^{-\frac{\pi i\tau}{4}}\sum_n(-1)^n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}}{e^{-\frac{\pi i\tau}{4}}\sum_n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}}.$$

(i) If n is 0 or even integer then,

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=i\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

(ii) If n is odd integer then

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=-i\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\1\end{bmatrix}(mod2)$ then

$$\begin{aligned}\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi i(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}+\tfrac{1}{2}\right\}\\&=\sum_n(-1)^n\exp\left\{n^2\pi i\tau+2n\pi iu+\tfrac{n\pi i}{2}+\tfrac{n\pi i\tau}{2}\right\}.\end{aligned}\tag{10}$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\0\end{bmatrix}(mod2)$ then

$$\begin{aligned}\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi i(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}\\&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi iu+\tfrac{n\pi i}{2}+\tfrac{n\pi i\tau}{2}\right\}\end{aligned}\tag{11}$$

From the equations (10) and (11) we obtain

$$\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}=\frac{\sum_n(-1)^n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}}{\sum_n\exp\left\{n^2\pi i\tau(2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}}.$$

(iii) If n is 0 or even integer then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

(iv) If n is odd integer then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=-\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

2 Conclusion

A general relation has been obtained by using $(1,r)$ periodic couples of first order theta functions according to the multiples of $1/2^r$ of these couples for $r \in N$. Furthermore an example has been given by using these relation for $r = 2$ according to the multiples of $1/4$.

References

[1] Gregory L. Wilson, A family of modular functions Arising from the theta function London Math. Soc. (3),55,1987, London.
[2] Harry E. Rauch., Elliptic functions, theta functions and Riemann surfaces. The Williams and Wilkins Company Baltimore. 75-79.(1973). Maryland.
[3] S. Raghavan., Cups Forms of Degre 2 and weight. Jour. of. Math. Ann.(224), 14156.(1976).
[4] Yıldız İsmet., On extension of modular transformations over the modular group by reflection. Journal of Applied Mathematics and Computation. Volume 153,Issue 1, p.111-116. (2004).
[5] Zhi-Guo Liu1. An addition formula for the Jacobian theta function and its applications. Advances in Mathematics Volume 212, Issue 1, 20 June 2007, Pages 389–40.
[6] István Mező. Several special values of Jacobi theta functions. (Submitted on 14 Jun 2011 (v1), last revised 24 Sep 2013 (this version, v3)). Cornell Universty Library.
[7] Noam D. Elkies, Lattices, Linear Codes and Invariants. Notices of the AMS, Volume 47, Number 10 (November 2000).
[8] Don Zagier. Ramanujan's Mock Theta Functions And Their Applications. Seminaire Bourbaki Novembre 2007 .60'eme ann'ee, 2006-2007, no 986.
[9] Theta Functions. Irish Math. Soc. Bulletin 60 (2007), 91–112.