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On theta functions and Dedekind's function according to the changes in the value periods

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Abstract: It's a reality that there is a relationship between a sigma function of Weierstrass and a theta function. We know that an elliptic function can be set up using the theta functions just as it can be established with the help of sigma function of Weierstrass.

In this study, we investigate relations between the Dedekind's η-function and θ-theta function by the using characteristic values $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (mod2) for θ-function according to the (u,τ) pair , where u,τ complex numbers satisfying $Im\tau > 0$. Also, we give the transformations among the theta functions according to the quarter periods and obtain a Jacobian style elliptic function by the help of a function we define.

Keywords: characteristic values, Dedekind's η-function, theta function, period pair, complex numbers.

1 Introduction

The $\wp$(u),ζ(u), and σ(u)-functions of Weiestrass which we have so far considered are not suitable to numerical computation. Therefore, it can be favorable to introduce another function defined by θ(u,τ), which is directly connected with the sigma function of Weierstrass. Let τ be a complex variable, where $\tau = \frac{\omega_2}{\omega_1}$ is not real number, ω_1, ω_2 are period points and $Im\tau > 0$, let $\omega = m\omega_1 + n\omega_2$ with $m,n = 0,\pm1,\pm2,...$ and let u be a complex variable. The function $\theta(u,\tau)$ is defined by the series,

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (u,\tau) = \sum_n \exp\left\{ (n+\frac{\varepsilon}{2})^2 \pi i\tau + 2\pi i(n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2}) \right\}, \tag{1}$$

where

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\bmod 2),$$

ε and ε' are integers and n ranges over all the integers (-$\infty$to∞) [1]. The series (1) converges absolutely and uniformly in compact sets of the u-complex plane and therefore represents an entire function of u [2].

If (ω_1, ω_2) is a pair of complex numbers with $Im\tau > 0$ and $\omega = m\omega_1 + n\omega_2$ with $m,n = 0,\pm1,\pm2,...$ then for $|u| \le R$, we find

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (u+\frac{1}{2^r}+\frac{\tau}{2^r},\tau) = \mu\theta \begin{bmatrix} \varepsilon+\frac{1}{2^{r-1}} \\ \varepsilon'+\frac{1}{2^{r-1}} \end{bmatrix}, \tag{2}$$

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where

$$\mu = exp\{-\frac{1}{4^r}(\tau+2)\pi i - \frac{1}{2^r}(2u+\varepsilon')\pi i\}$$

and

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\bmod 2),$$

and ε' are integers in [5].

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (u,\tau) = \sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i \tau + 2\pi i (n+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})\right\}, \tag{3}$$

where

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\bmod 2),$$

ε and ε' are integers n ranges over all the integers (-∞ to ∞) in [2].

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (u,\tau) = \sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2 \pi i \tau + 2i (n+\frac{\varepsilon}{2})(u-\frac{\varepsilon'}{2}\pi)\right\} \tag{4}$$

where

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\bmod 2),$$

ε and ε' are integers n ranges over all the integers (-∞ to ∞) in [4].

If

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\bmod 2)$$

then

$$\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (u,\tau) = -i\sum_n (-1)^n \exp\left\{(n+\frac{1}{2})^2 \pi i \tau + (2n+1)\pi i u\right\}. \tag{5}$$

This the function

$$\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (u,\tau)$$

is alternative formula in [3].

If

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\bmod 2)$$

and $u = 0$ then

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0,\tau) = \sum_n \exp(n^2 \pi i \tau). \tag{6}$$

This is the function

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)$$

that is an alternative formula in [1].

Definition 1. *A period, denoted by*

$$\begin{Bmatrix}a\\b\end{Bmatrix},$$

is $b+a\tau$. A quarter period is quarter of a period, written

$$\frac{1}{4}\begin{Bmatrix}a\\b\end{Bmatrix}=\frac{b}{4}+\frac{a\tau}{4}.$$

A reduced quarter-period is a quarter-period in which a and b equal 0 or 1 [5]. With the help of this alternative formula (2) in [5] above, we can get the following equalities according to quarter-periods.

If

$$\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\1\end{bmatrix}(\bmod 2)$$

then

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\sum_n \exp\left\{(n+\frac{1}{2})^2\pi i\tau+2\pi i(n+\frac{1}{2})(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}+\frac{1}{2}\right\}$$

$$=ie^{-\frac{\pi i\tau}{4}}\sum_n(-1)^n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}. \quad (7)$$

If

$$\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}1\\0\end{bmatrix}(\bmod 2)$$

then

$$\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\sum_n \exp\left\{(n+\frac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\frac{1}{2})(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}$$

$$=e^{-\frac{\pi i\tau}{4}}\sum_n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}. \quad (8)$$

Using the equations (7) and (8) we can get

$$\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}=\frac{ie^{-\frac{\pi i\tau}{4}}\sum_n(-1)^n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}}{e^{-\frac{\pi i\tau}{4}}\sum_n\exp\left\{(n+\frac{1}{2})^2\pi i\tau+(2n+1)\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}+\frac{\pi i\tau}{4}+\frac{\pi i}{4}\right\}}.$$

(i) If n is 0 or even integer then, $\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=i\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$.

(ii) If n is odd integer then $\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$ = -i$\theta\begin{bmatrix}1\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)$.

If

$$\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\1\end{bmatrix}(\bmod 2),$$

then

$$\begin{aligned}\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi i(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}+\tfrac{1}{2}\right\}\\&=\sum_n(-1)^n\exp\left\{n^2\pi i\tau+2n\pi iu+\tfrac{n\pi i}{2}+\tfrac{n\pi i\tau}{2}\right\}.\end{aligned}\tag{9}$$

If $\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}\equiv\begin{bmatrix}0\\0\end{bmatrix}(\bmod 2)$, then

$$\begin{aligned}\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi i(u+\tfrac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix}\right\}\\&=\sum_n\exp\left\{n^2\pi i\tau+2n\pi iu+\tfrac{n\pi i}{2}+\tfrac{n\pi i\tau}{2}\right\}.\end{aligned}\tag{10}$$

From the equations (9) and (10) we obtain

$$\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)}=\frac{\sum_n(-1)^n\exp\left\{n^2\pi i\tau+2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}}{\sum_n\exp\left\{n^2\pi i\tau(2n\pi iu+\frac{n\pi i}{2}+\frac{n\pi i\tau}{2}\right\}}.$$

(iii) If n is 0 or even integer then

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

(iv) If n is odd integer the

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(u+\frac{1}{4}\begin{Bmatrix}1\\1\end{Bmatrix},\tau).$$

Theorem 1. *The function*

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)$$

defined in [3] is odd function of u and it can be expressed by infinite product

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)=ce^{\frac{\pi i\tau}{4}}2\sin\pi u\prod_{n=1}^{\infty}\left\{1-e^{2(n\tau+u)\pi i}\right\}\prod_{n=1}^{\infty}\left\{1-e^{2(n\tau-u)\pi i}\right\}$$

where $c=\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau})$, *Imt* >0 *[2]. We consider the function* $\phi(u,\tau)$ *expressed by product*

$$\varphi(u,\tau)=\prod_{n=1}^{\infty}\left\{1-e^{[(2n-1)\tau+2u]\pi i}\right\}\prod_{n=1}^{\infty}\left\{1-e^{[(2n-1)\tau-2u]\pi i}\right\}$$

Theorem 2. *The function*

$$\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}$$

is a elliptic function with periods 1 *and* τ.

Proof. Let

$$\psi(u,\tau)=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}.$$

We write

$$\psi(u+1,\tau)=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+1,\tau)}{\varphi(u+1,\tau)}=\frac{-\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(-u,\tau)}{\varphi(u,\tau)}$$

where

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)=-\theta\begin{bmatrix}1\\1\end{bmatrix}(-u,\tau).$$

From theorem 2. We write

$$\psi(u+\tau,\tau)=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u+\tau,\tau)}{\varphi(u+\tau,\tau)}=\frac{-e^{-(2u+\tau)\pi i}\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{e^{-(2u+\tau)\pi i}\varphi(u,\tau)}=\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)}{\varphi(u,\tau)}.$$

Since for where $n=m$.

$$\begin{aligned}
&\varphi(u+\tau,\tau)=\prod_{n=1}^{\infty}\left\{1-e^{(2n-1)\tau\pi i+2\pi i(u+\tau)}\right\}\prod_{n=1}^{\infty}\left\{1-e^{(2n-1)\tau\pi i-2\pi i(u+\tau)}\right\}\\
&=\prod_{n=1}^{\infty}\left\{1-e^{(2n+1)\tau\pi i+2\pi iu}\right\}\prod_{n=1}^{\infty}\left\{1-e^{(2n-3)\tau\pi i-2\pi iu}\right\}\\
&=\prod_{n=1}^{\infty}\left\{1-e^{[2(n+1)-1]\tau\pi i+2\pi iu}\right\}\prod_{n=1}^{\infty}\left\{1-e^{[2(n-1)-1]\tau\pi i-2\pi iu}\right\}\\
&=\prod_{m=2}^{\infty}\left\{1-e^{(2m-1)\tau\pi i+2\pi iu}\right\}\prod_{n=0}^{\infty}\left\{1-e^{(2m-1)\tau\pi i-2\pi iu}\right\}\\
&=\prod_{m=1}^{\infty}\left\{1-e^{(2m-1)\tau\pi i+2\pi iu}\right\}\prod_{m=1}^{\infty}\left\{1-e^{(2m-1)\tau\pi i-2\pi iu}\right\}\left\{1-e^{-(\pi i\tau+2\pi iu)}\right\}\left\{1-e^{(\pi i\tau+2\pi iu)}\right\}^{-1}\\
&=-e^{-(2u+\tau)\pi i}\varphi(u,\tau).
\end{aligned}$$

The function $\psi(u,\tau)$ is therefore a doubly periodic with periods 1 and τ having neither zeros nor poles on account of the fact that

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(u,\tau)$$

possesses the same periodicity factors as $\phi(u,\tau)$. Hence the function $\psi(u,\tau)$ is an elliptic function since the set of all meromorphic functions form a field and $\psi(u,\tau)$ is meromorphic and periodic with periods 1 and τ.

The Riemann's θ-function was defined by the series

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}\exp\left\{(n+a)^2\pi i\tau+2\pi i(n+a)(u+b)\right\}$$

with a given complex number u and complex number τ satisfying $\mathrm{Im}(\tau=\frac{\omega_1}{\omega_2}\neq real)>0$ and characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$ where a and b are rational numbers [3].

We want to define an action of an element of

$$M=\begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

on theta function.

We try the special value of $M=\begin{bmatrix} d & c \\ b & a \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (Mod 2),

$$\begin{aligned}
\theta\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}(u,\tau) &= \theta\begin{bmatrix} d\mu-c\mu'+\frac{cd}{2} \\ -b\mu+a\mu'+\frac{ab}{2} \end{bmatrix}\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=\theta\begin{bmatrix} \mu \\ -\mu+\mu'+\frac{1}{2} \end{bmatrix}(z,\tau+1) \\
&= \sum_{n=-\infty}^{\infty}\exp\left\{(n+\mu)^2\pi i(\tau+1)+2\pi i(n+\mu)(z-\mu+\mu'+\frac{1}{2})\right\} \\
&= \sum_{n\to\infty}^{\infty}\exp\left\{(n+\mu)^2\mu i\tau+2\pi i(n+\mu)(z+\mu')+(n+\mu)^2\pi i+2\pi i(n+\mu)(-\mu+\frac{1}{2})\right\} \\
&= \sum_{n=-\infty}^{\infty}\exp\left\{(n+\mu)^2\pi i\tau+2\pi i(n+\mu)(z+\mu')+\pi i(n^2+n+\mu-\mu^2)\right\} \\
&= \exp\pi i(\mu-\mu^2)\theta\begin{bmatrix} \mu \\ \mu' \end{bmatrix}(z,\tau).
\end{aligned}$$

Since $n^2+n=n(n+1)$ is congruent to zero module 2 [$n(n+1)$ if $n\equiv 0(\bmod 2)$ and $|n(n+1)$] from $n+1\equiv 0(\bmod 2)$ if $n\equiv 1(\bmod 2)$).

We use the special value of θ- function with characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}\exp\left\{(n+\tfrac{\varepsilon}{2})^2\pi i\tau+2\pi i(n+\tfrac{\varepsilon}{2})(u+\tfrac{\varepsilon'}{2})\right\}$$

where

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}\equiv\begin{bmatrix} 0 \\ 1 \end{bmatrix},\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\bmod 2)$$

and ε, ε' are integers.

Thus, we have the following relations

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}(\exp(n^2\pi i\tau+2n\pi iu).$$

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}(-1)^n\exp(n^2\pi i\tau+2n\pi iu).$$

Theorem 3. *There is a connection the Weierstrass's σ-function and a theta function as the following*

$$\sigma(u;\omega_1,\omega_2)=\theta(\tfrac{u}{\omega_1},\tau).\frac{\omega_1}{\theta'(0,\tau)}.e^{\frac{u^2}{\omega_1}\eta_1}$$

where $\eta_1=\zeta(\frac{\omega_1}{2})$, and (ω_1,ω_2) is a pair of periods of the Weierstrass's elliptic function $\wp(u;\omega_1,\omega_2)$ [4].

Now, let us observe that

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)=\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)=\sum_n\exp(n^2\pi i\tau+2n\pi iu).$$

Then, we see that the function $\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)$ defined by the series in [1] *is a alternative formula of $\theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(u,\tau)$. The formulas $\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)$ and $\theta\begin{bmatrix}0\\1\end{bmatrix}(u,\tau)$ are used in this article where $u\neq 0$. we see the infinite products at the first*

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)=\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau+2\pi iu}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau-2\pi iu}).$$

which it converges absolutely [2].

Theorem 4. *We have the relations*

(i) $\eta(u)=e^{\frac{\pi iu}{12}}.\theta\begin{bmatrix}0\\0\end{bmatrix}(\frac{u+1}{2},3u+2k).$

(ii) $\theta\begin{bmatrix}0\\1\end{bmatrix}(\frac{u+4}{4},\frac{3}{2}u)=e^{-\frac{\pi iu}{12}}\eta(u)\prod_{n=1}^{\infty}(1-e^{(2n-1)\pi iu}),$

where $\tau=\frac{3}{2}u$ and $u=\frac{u+4}{4}$ between the functions $\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau),\theta\begin{bmatrix}0\\1\end{bmatrix}(u,\tau)$ and Dedekind's η-function which defined by the infinite product

$$\eta(u)=e^{\frac{\pi iu}{12}}.\prod_{n=1}^{\infty}(1-e^{2n\pi iu})$$

where $\mathrm{Im}\tau>0$ and k is a integer.

Proof.

(a) Let us recall the formula

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)=\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau+2\pi iu}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau-2\pi iu}).$$

If k integer, then we have

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2},3u+2k)$$
$$=\prod_{n=1}^{\infty}(1-e^{2n\pi i(3u+2k)})\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i(3u+2k)+2\pi i(\frac{u+1}{2})})\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i(3u+2k)-2\pi i(\frac{u+1}{2})})$$

$$=\prod_{n=1}^{\infty}(1-e^{6n\pi iu}).\prod_{n=1}^{\infty}(1+e^{6n\pi iu-2\pi iu-(2k-1)\pi i}).\prod_{n=1}^{\infty}(1+e^{6n\pi iu-4\pi iu-(2k+1)\pi i})$$

$$=\prod_{n=1}^{\infty}(1-e^{6n\pi iu}).\prod_{n=1}^{\infty}(1-e^{6n\pi iu-2\pi iu}).\prod_{n=1}^{\infty}(1-e^{6n\pi iu-4\pi iu}) \quad \text{for} \quad \tau=3u+2k \quad and \quad u=\tfrac{u+1}{2}.$$

If we set R= $e^{2\pi iu}$, then we obtain

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2},3u+2k)=\prod_{n=1}^{\infty}(1-R^{3n})\prod_{n=1}^{\infty}(1-R^{3n-1})\prod_{n=1}^{\infty}(1-R^{3n-2}).$$

On the other hand, we may set $n=n'+1$, then

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2},3u+2k)=\prod_{n'=1}^{\infty}(1-R^{3n'+3})\prod_{n'=1}^{\infty}(1-R^{3n'+2})\prod_{n'=1}^{\infty}(1-R^{3n'+1})$$
$$=(1-R)(1-R^2)(1-R^3)(1-R^4)...=\prod_{m=1}^{\infty}(1-R^m)=\prod_{m=1}^{\infty}(1-e^{2m\pi iu}).$$

According to above, we have

$$\eta(u)=e^{\frac{\pi iu}{12}}.\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2},3u+2k)$$

from the Dedekind's η-function defined by the infinite product

$$\eta(u)=e^{\frac{\pi iu}{12}}.\prod_{n=1}^{\infty}(1-e^{2n\pi iu})$$

where m = n'.

(b) We have

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u,\tau)=\sum_{n=-\infty}^{\infty}(-1)^n\exp(n^2\pi i\tau+2n\pi iu)$$

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(\tfrac{u+4}{4},\tfrac{3}{2}u)=\sum_{n=-\infty}^{\infty}(-1)^n\exp\left[\tfrac{1}{2}n(3n+1)\pi iu\right]$$
$$=1+\sum_{n=1}^{\infty}(-1)^n\left\{\exp\left[\tfrac{1}{2}n(3n-1)\pi iu\right]+\exp\left[\tfrac{1}{2}n(3n+1)\pi iu\right]\right\}$$

$$=1+\sum_{n=1}^{\infty}(-1)^n\left[x^{\frac{1}{2}n(3n-1)}+x^{\frac{1}{2}n(3n+1)}\right]=1-x-x^2+x^5+x^7-x^{12}-x^{15}+...$$

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(\tfrac{u+4}{4},\tfrac{3}{2}u)=1-x)(1-x^2)(1-x^3)...=\prod_{n=1}^{\infty}(1-x^n)$$

where $x = e^{\pi iu}$ for $|x| < 1$ and $\frac{1}{2}n(3n+1)$ are known as the pentagonal numbers $n = -1, -2, -3, \ldots$

This results play an important role in the workings concerning relations between the theta function and Dedekind's η-function.

$$\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(\frac{u+4}{4}, \frac{3}{2}u)}{\prod\limits_{n=1}^{\infty}(1-e^{(2n-1)\pi iu})} = \frac{\sum\limits_{n=-\infty}^{\infty}(-1)^n \exp(\frac{1}{2}n(3n+1)\pi iu)}{\prod\limits_{n=1}^{\infty}(1-e^{(2n-1)\pi iu})}$$

$$= \frac{\prod\limits_{n=1}^{\infty}(1-e^{n\pi iu})}{\prod\limits_{n=1}^{\infty}(1-e^{(2n-1)\pi iu})} = \prod_{n=1}^{\infty}(1-e^{2n\pi iu}) = e^{-\frac{\pi iu}{12}}\eta(u).$$

2 Conclusion

As a result, it has been obtained the relation between theta function and Dedkind's function by using the characteristic

$$\begin{bmatrix}0\\1\end{bmatrix}$$

and the variable $\tau = \frac{u+4}{4}$ instead of the characteristic

$$\begin{bmatrix}0\\0\end{bmatrix}$$

and the variable $\tau = \frac{u+1}{2}$ which were previously used by Jaccobi.

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