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Some multiordered difference sequence spaces of fuzzy real numbers defined by modulus function

Manmohan Das[1] *and Bipul Sarma*[2]

[1]Department of Mathematics, Bajali College, Gauhati University, Assam, India
[2]Department of Mathematics, MC College, Gauhati University, Assam, India

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Abstract: In this article we introduce some new multi ordered difference operator on sequence spaces of fuzzy real numbers by using modulus function and study their some algebraic and topological properties. Also we study some statistical convergent sequence space of fuzzy real numbers defined by modulus function.

Keywords: Fuzzy real numbers, difference sequence, statistical convergence, modulus function.

1 Introduction

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [15] and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [7] where it was shown that every convergent sequence is bounded. Nanda [9] studied the spaces of bounded and convergent sequence of fuzzy numbers and showed that they are complete metric spaces. In [13] Savaş studied the space $m(\Delta)$, which we call the space of Δ-bounded sequence of fuzzy numbers and showed that this is a complete metric space.

A modulus function f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x) = 0$ iff $x = 0$,
(ii) $f(x+y) \leq f(x) + f(y)$ for all $xy \geq$,
(iii) f is increasing,
(iv) f is continuous from the right at 0 .

It follows that f must be continous everywhere on $[0,\infty)$ and a modulus function may be bounded or unbounded.

Let X be a linear metric space. A function $p : X \to R$ is called paranorm if

(i) $p(x) \geq 0$ for all $x \in X$,
(ii) $p(-x) = p(x)$ for all $x \in X$,
(iii) $p(x+y) \leq p(x) + p(y)$,
(iv) If (λ_n) be a sequence of scalars such that $\lambda_n \to 0$ as $n \to \infty$ and (x_n) be a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

* Corresponding author e-mail: mdas.bajali@gmail.com

A paranorm p for which $p(x) = 0 \Rightarrow x = 0$ is called a total paranorm and the pair (X, p) is called a total paranormed space.

Kizmaz [6] defined the difference Sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ for crisp sets as follows

$$Z(\Delta) = X = (X_k) : (\Delta X_k) \in Z$$

where $Z = \ell_\infty, c$ and c_0.

2 Definitions and background

Let D denote the set of all closed and bounded intervals $X = [a_1, b_1]$ on the real line R. For $X = [a_1, b_1], Y = [a_2, b_2] \in D$ define $d(X,Y)$ by

$$d(X,Y) = \max(|a_1 - b_1|, |a_2 - b_2|)$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on R *i.e.* A mapping $X : R \to L(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

The α- level set $[X]^\alpha$ of a fuzzy real number X for $0 < \alpha \leq 1$, defined as

$$X^\alpha : \{t \in R : X(t) \geq \alpha\}$$

A fuzzy real number X is called *convex*, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper-semi continuous* if, for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$ is open for all $a \in I$ is open in the usual topology of R.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $L(R)$ and throughout the article, by a fuzzy real number we mean that the number belongs to $L(R)$.

The *absolute value*, $|X|$ of $X \in L(R)$ is defined by (see for instance Kaleva and Seikkala [2]),

$$\begin{aligned} |X|(t) =& \max\{X(t), X(-t)\}, \text{ if } t \geq 0, \\ =& 0, \text{ if } t < 0. \end{aligned}$$

Let $\overline{d} : L(R) \times L(R) \to R$ be defined by

$$\overline{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha).$$

Then $\overline{d}$ defines a metric on $L(R)$.

A sequence (X_k) of fuzzy real numbers is said to be *convergent* to the fuzzy real number X_0 if, for every $\varepsilon > 0$, there exists $k_0 \in N$ such that $\overline{d}(X_k, X_0) < \varepsilon$, for all $k \geq k_0$. The set of convergent sequences is denoted by c^F.

Recently Das and Sarma [18] discussed some properties of the operator $\Delta^s_{(v,r)}$ which is generalizes all previous studied difference operators.

A sequence (X_k) of fuzzy real numbers is raid to be $\Delta^s_{(v,r)}$ convergent to the fuzzy real number X_0, if for every $\varepsilon > 0$, there exists $k_0 \in N$ such that $\overline{d}(\Delta^s_{(v,r)}X_k, X_0)\varepsilon$ for all $k \geq k_0$, where r and s be two non-negative integers and $v = (v_k)$ be a sequence of non-zero reals and $(\Delta^s_{(v,r)}X_k) = (\Delta^{s-1}_{(v,r)}X_k - \Delta^{s-1}_{(v,r)}X_{k+r})$ and $\Delta^0_{(v,r)}X_k = v_k X_k$ for all $k \in N$, which is equivalent to the following binomial representation

$$\Delta^s_{(v,r)}X_k = \sum_{i=0}^{s}(-1)^i \begin{pmatrix} s \\ i \end{pmatrix} v_{k+ri}X_{k+ri}$$

Let f be a modulus function. Let r and s be two non-negative integers and $v = (v_k)$ be a sequence of non-zero reals. Then for a sequence $p = (p_k)$ of strictly positive real numbers, we define the classes of sequences as follows

$$w^F\left(\Delta^s_{(v,r)}, f, p\right) = \left\{X \in w^F : \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\overline{d}\left(\Delta^s_{(v,r)}X_k, X_0\right)\right)\right]^{p_k} = 0\right\} \text{ for some } X_0 \in w^F,$$

$$w_0^F\left(\Delta^s_{(v,r)}, f, p\right) = \left\{X \in w^F : \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\overline{d}\left(\Delta^s_{(v,r)}X_k, \overline{0}\right)\right)\right]^{p_k} = 0\right\},$$

$$w_\infty^F\left(\Delta^s_{(v,r)}, f, p\right) = \left\{X \in w^F : \sup_n\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\overline{d}\left(\Delta^s_{(v,r)}X_k, \overline{0}\right)\right)\right]^{p_k} < \infty\right\}.$$

Lemma 1. *Let (α_k) and (β_k) be sequences of real or complex numbers and (p_k) be a bounded sequence of positive real numbers, then*

$$|\alpha_k + \beta_k|^{p_k} \leq C\left(|\alpha_k|^{p_k} + |\beta_k|^{p_k}\right)$$

and $|\lambda|^{p_k} \leq \max\left(1, |\lambda|^H\right)$, *where* $C = \max\left(1, |\lambda|^{H-1}\right)$, $H = \sup p_k$, λ *is any real or complex number.*

Lemma 2. *If $\bar{d}$ is translation invariant then*

(a) $\bar{d}\left(\Delta^s_{(v,r)}X_k + \Delta^s_{(v,r)}Y_k, 0\right) \leq \bar{d}\left(\Delta^s_{(v,r)}X_k, 0\right) + \bar{d}\left(\Delta^s_{(v,r)}Y_k, 0\right)$

(b) $\bar{d}\left(\alpha\Delta^s_{(v,r)}X_k, 0\right) \leq |\alpha|\bar{d}\left(\Delta^s_{(v,r)}X_k, 0\right)$

3 Main results

Theorem 1. *Let f be a modulus function and $p = (p_k)$ be a sequence of strictly positive real numbers, then the classes of sequences $w^F\left(\Delta^s_{(v,r)}, f, p\right)$, $w_0^F\left(\Delta^s_{(v,r)}, f, p\right)$ and $w_\infty^F\left(\Delta^s_{(v,r)}, f, p\right)$ are closed under addition and scalar multiplication of fuzzy real numbers.*

Proof. We shall give the proof for $w_0^F\left(\Delta^s_{(v,r)}, f, p\right)$ and others are similar.

Let $X = (X_k)$, $Y = (Y_k) \in w_0^F\left(\Delta^s_{(v,r)}, f, p\right)$. For scalars a and b there exists integers M_a, N_b such that $|a| < M_a$ and

$|b| < N_b$. By properties of f we have,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}(aX_k+bY_k),\bar{0}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[f\left(|a|\bar{d}\left(\Delta^s_{(v,r)}X_k,\bar{0}\right)\right)+f\left(|b|\bar{d}\left(\Delta^s_{(v,r)}Y_k,\bar{0}\right)\right)\right]^{p_k}$$
$$\le C(M_a)^H\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}X_k,\bar{0}\right)\right)\right]^{p_k}+C(N_b)^H\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}Y_k,\bar{0}\right)\right)\right]^{p_k}, \to 0 \text{ as } n\to\infty.$$

This completes the proof.

Theorem 2. *The space* $w_0^F\left(\Delta^s_{(v,r)},f,p\right)$ *is a paranormed space w.r.to the paranorm defined by*

$$g(X)=\sup_n\left\{\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}(\Delta^s_{(v,r)}X_k,\bar{0}\right)\right]^{p_k}\right\}^{\frac{1}{M}},$$

where $\sup_k p_k<\infty$ *and* $M=\max(1,H)$.

Proof. Obviously $g(X)=g(-X)$ for all $X\in w_0^F\left(\Delta^s_{(v,r)},f,p\right)$

It is trivial that $v_kX_k=\bar{0}$ for $X_k=\bar{0}$. Since $\frac{p_k}{M}\le 1$ and $M\ge 1$ by using Minkowski's inequality, we have,

$$\left\{\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}(\Delta^s_{(v,r)}X_k+\Delta^s_{(v,r)}Y_k,\bar{0}\right)\right]^{p_k}\right\}^{\frac{1}{M}} \le \left\{\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}(\Delta^s_{(v,r)}X_k,\bar{0}\right)+f\left(\bar{d}(\Delta^s_{(v,r)}Y_k,\bar{0}\right)\right]^{p_k}\right\}^{\frac{1}{M}}$$
$$\le\left\{\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}(\Delta^s_{(v,r)}X_k,\bar{0}\right)\right]^{p_k}\right\}^{\frac{1}{M}}+\left\{\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}(\Delta^s_{(v,r)}Y_k,\bar{0}\right)\right]^{p_k}\right\}^{\frac{1}{M}}.$$

It follows that $g(X+Y)\le g(X)+g(Y)$.

Finally to check the continuity of scalar multiplication, let λ be any scaler, by definition we have,

$$g(\lambda X)=\sup_n\left\{\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}\left(\lambda\Delta^s_{(v,r)}X_k,\bar{0}\right)\right)\right]^{p_k}\right\}^{\frac{1}{M}}\le K_\lambda^{\frac{H}{M}}g(X),$$

where k_λ is an integer such that $|\lambda|<K_\lambda$.

Now let $\lambda\to 0$ for fixed X with $g(X)\neq 0$. By properties of f for $|\lambda|<1$ we have,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}\left(\lambda\Delta^s_{(v,r)}X_k,\bar{0}\right)\right)\right]^{p_k}<\varepsilon \text{ for } n\ge N(\varepsilon). \tag{1}$$

Also for $1\le n\le N$, taking λ small enough, f is continuous we have,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}\left(\lambda\Delta^s_{(v,r)}X_k,\bar{0}\right)\right)\right]^{p_k}<\varepsilon. \tag{2}$$

Eq (1) and (2) together follow that $g(\lambda X)\to 0$ as $\lambda\to 0$. This completes the proof.

Theorem 3. *If* $0 < p_k \leq q_k$ *and* $\frac{q_k}{p_k}$ *is bounded, then* $w^F\left(\Delta^s_{(v,r)}, f, q\right) \subseteq w^F\left(\Delta^s_{(v,r)}, f, p\right)$

Proof. Let $X = (X_k) \in w^F\left(\Delta^s_{(v,r)}, f, q\right)$. Define $w_k = \left[f\left(\overline{d}\left(\Delta^s_{(v,r)} X_k, X_0\right)\right)\right]^{q_k}$ and $\lambda_k = \frac{p_k}{q_k}$ for all $k \in N$ so that $0 < \lambda \leq \lambda_k \leq 1$.

Consider the sequences (u_k) and (v_k) given by as follow.

For $w_k \geq 1$, let $u_k = w_k$, $v_k = 0$ and $w_k < 1$, let $u_k = 0$, $v_k = w_k$. Then for all $k \in N$, we have, $w_k = u_k + v_k$, $w_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, $u_k^{\lambda_k} \leq u_k \leq w_k$ and $v_k^{\lambda_k} \leq v_k^{\lambda}$.

$$\frac{1}{n}\sum_{k=1}^{n} w_k^{\lambda_k} \leq \frac{1}{n}\sum_{k=1}^{n} w_k + \left[\frac{1}{n}\sum_{k=1}^{n} v_k\right]^{\lambda}.$$

Hence

$$X = (X_k) \in w^F\left(\Delta^s_{(v,r)}, f, p\right).$$

This completes the proof.

Theorem 4. *The following results hold:*

(i) $w_0^F\left(\Delta^{s-1}_{(v,r)}, f, p\right) \subseteq w_0^F\left(\Delta^s_{(v,r)}, f, p\right)$,
(ii) $w^F\left(\Delta^{s-1}_{(v,r)}, f, p\right) \subseteq w^F\left(\Delta^s_{(v,r)}, f, p\right)$,
(iii) $w_\infty^F\left(\Delta^{s-1}_{(v,r)}, f, p\right) \subseteq w_\infty^F\left(\Delta^s_{(v,r)}, f, p\right)$.

Proof. We prove the first one, others are similar.

Let

$$X = (X_k) \in w_0^F\left(\Delta^{s-1}_{(v,r)}, f, p\right).$$

Then we have,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\overline{d}\left(\Delta^s_{(v,r)} X_k, \overline{0}\right)\right)\right]^{p_k} = 0.$$

The result follows from the following inequality,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\overline{d}\left(\Delta^s_{(v,r)} X_k, \overline{0}\right)\right)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n}\left[f\left(\overline{d}\left(\Delta^{s-1}_{(v,r)} X_k - \Delta^{s-1}_{(v,r)} X_{k+1}, \overline{0}\right)\right)\right]^{p_k}$$
$$\leq C\left\{\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\overline{d}\left(\Delta^{s-1}_{(v,r)} X_k, \overline{0}\right)\right)\right]^{p_k} + \frac{1}{n}\sum_{k=1}^{n}\left[f\left(\overline{d}\left(\Delta^{s-1}_{(v,r)} X_{k+1}, \overline{0}\right)\right)\right]^{p_k}\right\}.$$

Corollary 1. *Let* f *be a modulus function, then*

(i) $w_0^F\left(\Delta^s_{(v,r)}, p\right) \subseteq w_0^F\left(\Delta^s_{(v,r)}, f, p\right)$,
(ii) $w^F\left(\Delta^s_{(v,r)}, p\right) \subseteq w^F\left(\Delta^s_{(v,r)}, f, p\right)$,
(iii) $w_\infty^F\left(\Delta^s_{(v,r)}, p\right) \subseteq w_\infty^F\left(\Delta^s_{(v,r)}, f, p\right)$.

Theorem 5. *Let* f *be a modulus function and* $\sup_k p_k = H < \infty$. *Then* $w^F\left(\Delta^s_{(v,r)}, f, p\right) \subset S^F\left(\Delta^s_{(v,r)}\right)$.

Proof. Let $X=(X_k)\in w^F\left(\Delta^s_{(v,r)},f,p\right)$. and $\varepsilon>0$. Then,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\right)\right]^{p_k}=\frac{1}{n}\sum_{\substack{k\le n\\ \Delta_\varepsilon}}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\right)\right]^{p_k}+\frac{1}{n}\sum_{\substack{k\le n\\ \Delta_\varepsilon}}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\right)\right]^{p_k}$$

$$\ge\frac{1}{n}\sum_{\substack{k\le n\\ \Delta_\varepsilon}}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\right)\right]^{p_k}\ge\frac{1}{n}\sum_{\substack{k\le n\\ \Delta_\varepsilon}}[f(\varepsilon)]^{p_k}\ge\frac{1}{n}\sum_{\substack{k\le n\\ \Delta_\varepsilon}}\min\left([f(\varepsilon)]^h,[f(\varepsilon)]^H\right)$$

$$=\frac{1}{n}card\left\{k\le n:\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\ge\varepsilon\right\}\min\left([f(\varepsilon)]^h,[f(\varepsilon)]^H\right)$$

where $\Delta_\varepsilon=\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\ge\varepsilon$ and $h=\inf p_k$. Hence $X=(X_k)\in S^F\left(\Delta^s_{(v,r)}\right)$.

Theorem 6. *Let f be bounded modulus function and $0<h=\inf p_k\le p_k\le\sup_k p_k=H<\infty$. Then $S^F\left(\Delta^s_{(v,r)}\right)\subset w^F\left(\Delta^s_{(v,r)},f,p\right)$.*

Proof. Let $X=(X_k)\in w^F\left(\Delta^s_{(v,r)},f,p\right)$ and $\varepsilon>0$. Since f is bounded therefore there exists an integer K such that $|f(x)|\le K$. Then,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\right)\right]^{p_k}\le\frac{1}{n}\sum_{\substack{k\le n\\ \Delta_\varepsilon}}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\right)\right]^{p_k}+\frac{1}{n}\sum_{\substack{k\le n\\ \Delta_\varepsilon}}\left[f\left(\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\right)\right]^{p_k}$$

$$\le\frac{1}{n}\sum_{\substack{k\le n\\ \Delta_\varepsilon}}\max\left(K^h,K^H\right)+\frac{1}{n}\sum_{\substack{k\le n\\ \Delta_\varepsilon}}[f(\varepsilon)]^{p_k}$$

$$\le\max\left(K^h,K^H\right)\frac{1}{n}card\left\{k\le n:\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\ge\varepsilon\right\}+\max\left([f(\varepsilon)]^h,[f(\varepsilon)]^H\right)$$

where $\Delta_\varepsilon=\bar{d}\left(\Delta^s_{(v,r)}X_k,X_0\right)\ge\varepsilon$. Hence $X=(X_k)\in w^F\left(\Delta^s_{(v,r)},f,p\right)$.

Theorem 7. *If f is bounded then $w^F\left(\Delta^s_{(v,r)},f,p\right)=S^F\left(\Delta^s_{(v,r)}\right)$.*

Proof. If f is bounded then by Theorem 3.5 and Theorem 3.6 , we have $w^F\left(\Delta^s_{(v,r)},f,p\right)=S^F\left(\Delta^s_{(v,r)}\right)$.

References

[1] P. Diamond and P. Kloeden, *Metric spaces of fuzzy sets*, Fuzzy Sets and Systems 35, 241-249, (1990).

[2] M. Et and R. Colak, *On generalized difference sequence spaces*, Soochow Jour. Math. 21, 377-386, (1995).

[3] H. Fast, *Sur la convergence statistique*, Colloq. Math., 241-244, (1951).

[4] J.A. Fridy and C. Orhan, *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. 125(12), 3625-3631, (1997).

[5] A.R. Freedman, J.J. Sember and M. Raphael, *Some Cesàro-type summability spaces*, Proc. Lond. Math. Soc. 37(3), 508-520, (1978).

[6] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull. 24(2), 168-176, (1981).

[7] M. Matloka, *Sequences of fuzzy numbers*, BUSEFAL 28, 28-37, (1986).
[8] I.J. Maddox, *A Tauberian condition for statistical convergence*, Math. Proc. Camb. Phil. Soc. 106, 277-280, (1989).
[9] S. Nanda, *On sequence of fuzzy numbers*, Fuzzy Sets and Systems 33, 28-37, (1989).
[10] F. Nuray and E. Savas, *Statistical convergence of sequences of fuzzy numbers*, Math. Slovaca 45(3), 269-273, (1995).
[11] D. Rath and B.C. Tripathy, *Matrix maps on sequence spaces associated with sets of integers*, Indian J. Pure and Appl. Math. 27(2), 197-206, (1996).
[12] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly 66, 3621-4375, (1959).
[13] E. Savas, *A note on sequence of fuzzy numbers*, Inform. Sciences 124, 297-300, (2000).
[14] E. Savas, *On strongly λ-summable sequences of fuzzy numbers*, Inform. Sciences 125, 181-186, (2000).
[15] L.A. Zadeh, *Fuzzy sets*, Inform and Control 8, 338-353, (1965).
[16] B.C. Tripathy and A. Esi, *A new type of difference sequence spaces*, Int. Jour. of Sci. and Tech. 1, 11-14, (2006).
[17] B.C. Tripathy, A. Esi and B.K. Tripathy, *On a new type of generalized difference Cesàro Sequence spaces*, Soochow J. Math. 31(3), 333-340, (2005).
[18] M. Das and B. Sarma, *I-convergence sequence space of fuzzy real numbers defined by sequence modulus functions*, Eng. Math. Lett., 2014:8 (2014).
[19] A. Esi, *On some new paranormed sequence spaces of fuzzy numbers defined by Orlicz functions and statistical convergence*, Math. Model. Anal., 11(4), 379-388, (2006).
[20] A. Zygmund, *Trigonometric series*, Vol. 2, Cambridge, (1993).