New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2016115617

On the Darboux ruled surface of a general helix in the Nil space Nil_3

Seyda Kilicoglu

Baskent University, Department of Mathematical Teaching, Ankara, Turkey

Received: 17 September 2015, Revised: 11 November 2015, Accepted: 25 December 2015
Published online: 1 January 2016.

Abstract: Nil geometry is one of the eight geometries of Thurston's conjecture. In this paper we study in Nil 3-space and the Nil metric with respect to the standard coordinates (x,y,z) is $g_{Nil_3} = (dx)^2 + (dy)^2 + (dz - xdy)^2$ in IR^3. In [8] we have already find out the explicit parametric equation of a general helix and Frenet vector fields, with first and second curvatures κ and τ, respectively, in Nil 3-Space. Here we find out the parametric equations of the Darboux ruled surface of the general helix in Nil Space Nil_3.

Keywords: Nil space, general helix, ruled surface.

1 Introduction and preliminaries

In mathematics, Thurston's conjecture proposed a complete characterization of geometric structures on three-dimensional manifolds. The conjecture was proposed by William Thurston (1982), and implies several other conjectures, such as the Poincaré conjecture and Thurston's elliptization conjecture. Thurston's geometrization conjecture states that certain three-dimensional topological spaces each have a unique geometric structure that can be associated with them. It is an analogue of the uniformization theorem for two-dimensional surfaces, which states that every simply-connected Riemann surface can be given one of three geometries (Euclidean, spherical, or hyperbolic).

In three dimensions, it is not always possible to assign a single geometry to a whole topological space. Instead, the geometrization conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure. Thurston's conjecture is that, after you split a three-manifold into its connected sum and the Jaco-Shalen-Johannson torus decomposition, the remaining components each admit exactly one of the following geometries:

(1) Euclidean geometry,
(2) Hyperbolic geometry,
(3) Spherical geometry,
(4) The geometry of $S^2 \times \mathbf{R}$,
(5) The geometry of $H^2 \times \mathbf{R}$,
(6) The geometry of the universal cover $SL\_2\mathbf{R}^{\tilde{}}$ of the Lie group $SL\_2\mathbf{R}$,
(7) Nil geometry,
(8) Sol geometry.

For more detail see [15].

* Corresponding author e-mail: seyda@baskent.edu.tr

Nilmanifolds are important geometric objects. A nilmanifold is a differentiable manifold which has a transitive nilpotent group of diffeomorphisms acting on it. In the Riemannian category, there is also a good notion of a nilmanifold. A Riemannian manifold is called a homogeneous nilmanifold if there exist a nilpotent group of isometries acting transitively on it. The requirement that the transitive nilpotent group acts by isometries leads to the following rigid characterization: every homogeneous nilmanifold is isometric to a nilpotent Lie group with left-invariant metric (see [16]).

The two-parameter family of metrics first appeared in the works of Bianchi, Cartan and Vranceanu, these spaces are often referred to as Bianchi-Cartan-Vranceanu spaces, or $BCV-spaces$ for short. Some well-known examples of $BCV-spaces$ are the Riemannian product spaces $S^2\times\mathbf{R}$, $H^2\times\mathbf{R}$ and the 3-dimensional Heisenberg group [4].

Definition 1. *Let κ and τ be real numbers, with $\tau\geq 0$. The Bianchi-Cartan-Vranceanu spaces, $(BCV-spaces)$ $M^3(\kappa,\tau)$ is defined as the set*

$$\left\{(x,y,z)\in\mathbf{R}^3 : 1+\frac{\kappa}{4}\left(x^2+y^2\right)>0\right\}$$

equipped with metric

$$ds^2=\frac{dx^2+dy^2}{\left(1+\frac{\kappa}{4}\left(x^2+y^2\right)\right)^2}+\left(dz+\tau\frac{ydx-xdy}{1+\frac{\kappa}{4}\left(x^2+y^2\right)}\right)^2.$$

(i) if $\kappa=\tau=0$, then $M^3(\kappa,\tau)\cong IE^3$

(ii) if $\kappa=0$ and $\tau\neq 0$, then $M^3(\kappa,\tau)\cong Nil_3$.

More details can be found in [4] and [17].

In [4] it is restricted to the 3-dimensional Heisenberg group coming from R^2 with the canonical symplectic form $\psi((x,y),(x_1,y_1))=xy_1-x_1y$, i.e., they consider R^3 with the group operation

$$(x,y,z)*(x_1,y_1,z_1)=\left(x+x_1,y+y_1,z+z_1+\frac{xy_1}{2}-\frac{x_1y}{2}\right)$$

For every non-zero number τ the following Riemannian metric on $\left(\mathbf{R}^3,*\right)$ is left invariant:

$$ds^2=dx^2+dy^2+4\tau^2\left(dz+\frac{ydx-xdy}{2}\right)^2.$$

After the change of coordinates $(x,y,2\tau z)\to(x,y,z)$, this metric is expressed as

$$ds^2=dx^2+dy^2+(dz+\tau(ydx-xdy))^2.$$

By some authors the notation Nil_3 is only used if $\tau=\frac{1}{2}$.

1.1 Riemannian structure of Nil space Nil_3

The Riemannian Structure of Sol, Nil and Heisenberg Spaces are examined in [11]. Linear biharmonic maps into Sol, Nil and Heisenberg Spaces are examined with three metric too. It is well known that Nil space is isometric to Heisenberg space. The geometry of Nil is the three dimensional Lie group of all real 3 triangular matrices of the form

$$\begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}.$$

Let $(\mathbf{R}^3, g_{Nil_3})$ denote Nil space, where the metric with respect to the standard coordinates (x,y,z) in $\mathbf{R}^3$ can be written ([11]) as

$$g_{Nil_3} = (dx)^2 + (dy)^2 + (dz - xdy)^2.$$

Hence we get the symetric tensor field g_{Nil_3} on Nil_3 by components

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & -x \\ 0 & -x & 1 \end{bmatrix}.$$

Note that the Nil metric can also be written as: $ds^2 = \sum_{i=1}^{3} \omega i \otimes \omega i$, where $\omega^1 = dx$, $\omega^2 = dy$, $\omega^3 = dz - xdy$, and the orthonormal basis dual to the 1-forms is $E_1 = \frac{\partial}{\partial x}$, $E_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ $E_3 = \frac{\partial}{\partial z}$. With respect to this orthonormal basis, the Levi-Civita connection and the Liebrackets can be easily computed as:

$$\begin{aligned}
\nabla_{E_1}E_1 &= 0, & \nabla_{E_1}E_2 &= \frac{1}{2}E_3, & \nabla_{E_1}E_3 &= \frac{-1}{2}E_2 \\
\nabla_{E_2}E_1 &= \frac{-1}{2}E_3, & \nabla_{E_2}E_2 &= 0, & \nabla_{E_2}E_3 &= \frac{1}{2}E_1 \\
\nabla_{E_3}E_1 &= \frac{-1}{2}E_2, & \nabla_{E_3}E_2 &= \frac{1}{2}E_1, & \nabla_{E_3}E_3 &= 0.
\end{aligned}$$

$$[E_1,E_2] = E_3,\ [E_2,\ E_3] = 0,\ [E_1,E_3] = 0.$$

Hence

$$\nabla = \begin{bmatrix} \nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 \\ \nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 \\ \nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}E_3 & \frac{-1}{2}E_2 \\ \frac{-1}{2}E_3 & 0 & \frac{1}{2}E_1 \\ \frac{-1}{2}E_2 & \frac{1}{2}E_1 & 0 \end{bmatrix}$$

is the matrix with $(i,j)-element$ in the table equals $\nabla_{E_i}E_i$ for the basis $\{E_1,E_2,E_3\}$.

2 The parametric equation of general helix in Nil space Nil_3

Helix is one of the fascinating curve in science and nature. In this section, we study on the general helices in Nil_3. We characterize the general helices in terms of their curvature and torsion. A curve of constant slope or general helix is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [13]. for details) is: A necessary and sufficient condition that a curve be a helix is that the ratio of curvature to torsion be constant. Helices are examined in [2], [10].

It is well-known that, if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret apparatus of the curves. Let Frenet vector fields be T,N,B of the curve α and let the first and second curvatures of the curve α be κ and τ, respectively. The quantities $\{T,N,B,\tilde{D},\kappa,\tau\}$ are collectively Frenet-Serret apparatus of the curves. Also the Darboux vector provides a concise way of interpreting curvature κ and torsion τ geometrically; curvature is the measure of the rotation of the Frenet frame about the binormal unit vector, and torsion is the measure of the rotation of the Frenet frame about the tangent unit vector. For any unit speed curve α,in terms of the Frenet-Serret apparatus, the Darboux vector D can be expressed as $D(s) = \tau(s)T(s) + \kappa(s)B(s)$.Let a vector field be $\tilde{D}(s) = \frac{\tau}{\kappa}(s)T(s) + B(s)$ along $\alpha(s)$ under the condition that $\kappa(s) \neq 0$ and it is called the modified Darboux vector field of α (see [6]).Let α be a helix that lies on the cylinder. A helix which lies on the cylinder is called cylindrical helix or general helix. Assume that $\{T, N, B, \tilde{D}, \kappa, \tau\}$ be the Frenet apparatus along the curve α. It has been known that the curve α is a cylindrical helix if and only if $\left(\frac{\kappa}{\tau}\right)$ is constant, then $\left(\frac{\kappa}{\tau}\right)' = 0$ where κ and τ are the curvatures of α. If the curve is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be constant. We call a curve a circular helix if both $\tau \neq 0$ and κ are constant. Then, the Frenet frame satisfies the following Frenet–Serret equations

$$\begin{aligned}\nabla_T T &= \kappa N,\\ \nabla_T N &= -\kappa T + \tau B,\\ \nabla_T B &= -\tau N.\end{aligned}$$

With respect to the orthonormal basis $\{E_1, E_2, E_3\}$, we can write

$$\begin{aligned}T &= T_1E_1 + T_2E_2 + T_3E_3,\\ N &= N_1E_1 + N_2E_2 + N_3E_3,\\ B &= T \times N = B_1E_1 + B_2E_2 + B_3E_3.\end{aligned}$$

Parametric equations of general helices in the sol space Sol_3 are examined in [14]. From the Riemannian Structure of Nil space 3, parametric equations of general helices in in Nil Space are examined in the following theorem.

Theorem 1. *Let $\alpha : I \to Nil_3$ be a unit speed non-geodesic general helix.Then, the equation of a unit speed non-geodesic general helix α, with respect to the orthonormal basis $\{E_1, E_2, E_3\}$*

$$\begin{aligned}\alpha(s) = {}&\left(\frac{\sin\beta}{C_1}\sin D + C_3\right)E_1 + \left(\frac{-\sin\beta}{C_1}\cos D + C_4\right)E_2\\ &+ \left(\frac{\sin^2\beta}{4C_1^2}\sin 2D - \frac{C_4\sin\beta}{C_1}\sin D + \left(\frac{\sin^2\beta}{2C_1} + \cos\beta\right)s - C_3C_4 + C_5\right)E_3,\end{aligned}$$

where we take $D = C_1s + C_2$ and $C_1, C_2 \in \mathbf{IR}$.

Proof. Assume that $\alpha : I \to Nil_3$ be a unit speed non-geodesic general helix. So,without loss of generality, we take its axis as parallel to the vectore E_3. Then, $g_{nil_3}(\mathbf{T}, E_3) = T_3 = \cos\beta$, where β is constant angle. On the other hand the tangent vector $\mathbf{T}$ is an unit vector, so the following condition is satisfied $T_1^2 + T_2^2 = 1 - \cos^2\beta$.Since $\cos^2\beta + \sin^2\beta = 1$,we have the general solution of $T_1^2 + T_2^2 = \sin^2\beta$ can be written in the following form

$$T_1 = \sin\beta\cos D, T_2 = \sin\beta\sin D, T_3 = \cos\beta$$

Also, without loss of generality, where we take $D = C_1s + C_2$, $C_1, C_2 \in \mathbf{IR}$. So, substituting the components T_1, T_2 and T_3 in the equation, we have the following equation

$$\mathbf{T} = \sin\beta\cos DE_1 + \sin\beta\sin DE_2 + \cos\beta E_3. \tag{1}$$

Using $E_1 = \frac{\partial}{\partial x}$, $E_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, and $E_3 = \frac{\partial}{\partial z}$ in (1), we obtain

$$\begin{aligned} T &= \sin\beta\cos(C_1 s + C_2)\frac{\partial}{\partial x} + \sin\beta\sin(C_1 s + C_2)\left[\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right] + \cos\beta\frac{\partial}{\partial z} \\ &= \left(\sin\beta\cos D,\ \sin\beta\sin D,\ \frac{\sin^2\beta}{C_1}\sin^2 D + C_3\sin\beta\sin D + cos\beta\right). \end{aligned}$$

From the definition of tangent vector field T of the curve α, we get the following equations;

$$\begin{aligned} \frac{dx}{ds} &= \sin\beta\cos(C_1 s + C_2) \\ \frac{dy}{ds} &= \sin\beta\sin(C_1 s + C_2) \\ \frac{dz}{ds} &= x(s)\sin\beta\sin(C_1 s + C_2) + \cos\beta \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} &\Longrightarrow x(s) = \frac{\sin\beta}{C_1}\sin(C_1 s + C_2) + C_3 \\ &\Longrightarrow y(s) = \frac{-\sin\beta}{C_1}\cos(C_1 s + C_2) + C_4 \\ &\Longrightarrow z(s) = \left(-\frac{\sin^2\beta}{4C_1^2}\sin 2D - \frac{C_3\sin\beta}{C_1}\cos D + \left(\frac{\sin^2\beta}{2C_1}\cos\beta\right)s\right) + C_5 \end{aligned}$$

where C_3, C_4, C_5 are constant of integration. By substituting all them

$$\begin{aligned} \alpha(s) = &\left(\frac{\sin\beta}{C_1}\sin(C_1 s + C_2) + C_3\right)\frac{\partial}{\partial x} \\ &+ \left(\frac{-\sin\beta}{C_1}\cos(C_1 s + C_2) + C_4\right)\frac{\partial}{\partial y} \\ &+ \left(\frac{-\sin^2\beta}{4C_1^2}\sin 2D + \left(\frac{\sin^2\beta}{2C_1} + \cos\beta\right)s - \frac{\sin\beta}{C_1}\cos DC_3 + C_5\right)\frac{\partial}{\partial z}. \end{aligned}$$

Using of equalities $E_1 = \frac{\partial}{\partial x}$, $E_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, *and* $E_3 = \frac{\partial}{\partial z}$ proves our assertion.

2.1 Frenet apparatus of general helices in Nil space Nil_3

First, to calculate the normal vector field of the general helix we need to know the curvature (first curvature) of the general helix in Nil Space Nil_3. It can be given by the following theorem.

Theorem 2. *The curvature (first curvature) of the general helix in Nil Space Nil_3 is*

$$\kappa = \sin\beta(\cos\beta - C_1); \quad (\cos\beta - C_1) > 0. \tag{2}$$

Proof. Assume that $\alpha : I \to Nil_3$ be a unit speed non-geodesic general helix with tangent vector field

$$\mathbf{T} = \sin\beta\cos DE_1 + \sin\beta\sin DE_2 + \cos\beta E_3.$$

The Levi-Civita connection and Lie brackets can be easily computed as:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \left(\dot{T}_1 + T_2T_3\right)E_1 + \left(\dot{T}_2 - T_1T_3\right)E_2 + \left(\dot{T}_3\right)E_3 \\ &= (\cos\beta - C_1)(\sin\beta\sin DE_1 - \sin\beta\cos DE_2).\end{aligned}$$

By the use of Frenet formula $\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}$ and from the equation, $g_{nil_3}(\kappa\mathbf{N},\kappa\mathbf{N}) = \kappa^2 g_{nil_3}(N,N)$ it is achieved the curvature $\kappa = \sin\beta(\cos\beta - C_1)$; $(\cos\beta - C_1) > 0$.

Theorem 3. *Let $\alpha : I \to Nil_3$ be a unit speed non-geodesic general helix. Then, the tangent vector field of the general helix in terms of E_1, E_2, E_3 is $\mathbf{T} = \sin\beta\cos DE_1 + \sin\beta\sin DE_2 + \cos\beta E_3$.*

Theorem 4. *Let $\alpha : I \to Nil_3$ be a unit speed non-geodesic general helix. Then, the normal vector field of the general helix is $\mathbf{N} = \sin DE_1 - \cos DE_2$. Where we take $D = C_1 s + C_2$ and $C_1, C_2 \in IR$.*

Proof. By the use of Frenet formulae $\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}$ and from the above equation, it is achieved the normal vector field $\mathbf{N} = (\sin D, -\cos D, 0)$.

Theorem 5. *The torsion (second curvature) of the general helix in Nil Space Nil_3 is*

$$\tau = \sqrt{C_1^2 - C_1\cos\beta + \frac{1}{4}} \tag{3}$$

Proof. With the Levi-Civita connection and Lie brackets can be easily computed as:

$$\begin{aligned}\nabla_T\mathbf{N} &= \left(\dot{N}_1 + \frac{1}{2}N_2T_3 + \frac{1}{2}N_3T_2\right)E_1 + \left(\dot{N}_2 + \frac{-1}{2}N_1T_3 + \frac{-1}{2}N_3T_1\right)E_2 + \left(\dot{N}_3 + \frac{1}{2}N_2T_1 + \frac{-1}{2}N_1T_2\right)E_3 \\ \nabla_T\mathbf{N} &= \left(C_1 - \frac{\cos\beta}{2}\right)\cos DE_1 + \left(C_1 - \frac{\cos\beta}{2}\right)\sin DE_2 - \frac{1}{2}\sin\beta E_3.\end{aligned}$$

Also for $\mathbf{N}=\sin DE_1 - \cos DE_2$ we know that

$$N_1 = \sin D;\ \dot{N}_1 = C_1\cos D\ N_2 = -\cos D;\ \dot{N}_2 = C_1\sin D,\ N_3 = 0,\ \dot{N}_3 = 0.$$

Now it is easy to say that for $\nabla_T\mathbf{N} = \frac{1}{2}((2C_1 - \cos\beta)\cos DE_1 + (2C_1 - \cos\beta)\sin DE_2 - \sin\beta E_3)$. It is well known that $\tau = g_{Nil_3}\langle\nabla_{\mathbf{T}}\mathbf{N},\mathbf{B}\rangle$, hence $\tau = \sqrt{C_1^2 - C_1\cos\beta + \frac{1}{4}}$ or $\tau = \frac{1}{2}\sqrt{(2C_1 - \cos\beta)^2 + \sin^2\beta}$.

Theorem 6. *Let $\alpha : I \to Nil_3$ be a unit speed non-geodesic general helix. Then, the binormal vector field of the general helix is*

$$\mathbf{B} = \frac{1}{\tau}\left[\begin{array}{c}\left(C_1 - \frac{1}{2}\cos\beta + \sin^2\beta\cos\beta - \sin^2\beta C_1\right)\cos DE_1 \\ +\left(C_1 - \frac{1}{2}\cos\beta + \sin^2\beta\cos\beta - \sin^2\beta C_1\right)\sin DE_2 \\ +\left(\cos^2\beta - \cos\beta C_1 - \frac{1}{2}\right)\sin\beta E_3\end{array}\right] \tag{4}$$

where we take $D = C_1 s + C_2$ and $\tau = \frac{1}{2}\sqrt{(2C_1 - \cos\beta)^2 + \sin^2\beta}$, for $C_1, C_2 \in IR$.

Proof. By using the Frenet-Serret equation $\nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}$, we have

$$\begin{aligned}\mathbf{B} &= \frac{1}{\tau}(\nabla_{\mathbf{T}}\mathbf{N} + \kappa\mathbf{T}) \\ &= \frac{1}{\tau}\left(\begin{array}{c}\left(\left(C_1 - \frac{1}{2}\cos\beta\right)\cos D, \left(C_1 - \frac{1}{2}\cos\beta\right)\sin D, \frac{-1}{2}\sin\beta\right) \\ +\kappa(\sin\beta\cos D, \sin\beta\sin D, \cos\beta)\end{array}\right).\end{aligned}$$

This proves our assertion. Thus, the proof of theorem is completed.

3 Darboux ruled surface of the general helix in Nil space Nil_3

3.1 Ruled surface and Frenet ruled surfaces

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line in Euclidean $3-space$. Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines, and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α' and v satisfy $\left\langle \alpha', v \right\rangle = 0$. To illustrate the current situation, we bring here the famous example of L. K. Graves , so called the $B-scroll$. The special ruled surfaces $B-scroll$ over null curves with null rulings in 3-dimensional Lorentzian space form has been introduced by L. K. Graves in [5]. In the paper [6] the following special two ruled surfaces associated to a space curve α with $k_1 \neq 0$ which are respectively related to cylindrical helices and Bertrand curves has been considered.

Definition 2. *The ruled surface $\varphi(s,u) = \alpha(s) + uN(s)$ is called the normal surface of α.*

Definition 3. *The ruled surface $\varphi(s,u) = \alpha(s) + uB(s)$ is called the binormal ruled surface of α. The ruled surface $\varphi(s,u) = \alpha(s) + u\tilde{D}(s)$is the parametrization of the ruled surface which is called rectifying developable surface of the curve α in [6]. Here, it is referred to as $\tilde{D}-scroll$ cause of generator (directrix) vector is modified Darboux vector field.*

The parametrization of the ruled surface which is called *involutive* $B-scroll$ (binormal scroll) of the curve α [7]. Before, we find out the explicit parametric equation of helix with curvatures in the Nil_3 (see [8]). Also ruled surface with Bishop Frame is studied in [9]. Frenet ruled surface is one which can be generated by the motion of a Frenet vector of any curve in Euclidean $3-space$. Tangent, Normal, Binormal,Darboux ruled surfaces of any curve are collectively named Frenet ruled surfaces. In the paper [6], the following special ruled surfaces associated to a space curve α with $k_1 \neq 0$ which are respectively related to cylindrical helix and Bertrant curves has been considered.

We have already examined the parametric equation of normal and binormal ruled surface of general helix by the following way in Nil_3.

Theorem 7. *Let $\alpha : I \to Nil_3$ be a unit speed non-geodesic general helix and* **N** *be its normal vector field. Then, the parametric equation of* ***normal ruled surface****, in terms of E_1, E_2, E_3, is given by*

$$\begin{aligned}\varphi(s,u) = &\left(\frac{\sin\beta}{C_1}\sin D + u\sin D + C_3\right)E_1 + \left(\frac{-\sin\beta}{C_1}\cos D - u\cos D + C_4\right)E_2 \\ &+ \left(\frac{\sin^2\beta}{4C_1^2}\sin 2D - \frac{C_4\sin\beta}{C_1}\sin D + \left(\frac{\sin^2\beta}{2C_1} + \cos\beta\right)s - C_3C_4 + C_5\right)E_3\end{aligned}$$

where we take $D = C_1 s + C_2$ where $C_1, C_2 \in$ **IR**.

Theorem 8. *Let $\alpha : I \to Nil_3$ be a unit speed non-geodesic general helix and* **B** *its binormal. Then, the parametric equation of* ***binormal ruled surface****, in terms of E_1, E_2, E_3 is given by*

$$\begin{aligned}\varphi(s,u) = &\left[\left(\frac{\sin\beta}{C_1}\sin D + C_3\right) + \frac{u}{\tau}\left(C_1 - \frac{1}{2}\cos\beta + \sin^2\beta\cos\beta - \sin^2\beta C_1\right)\cos D\right]E_1 \\ &+ \left[\left(\frac{-\sin\beta}{C_1}\cos D + C_4\right) + \frac{u}{\tau}\left(C_1 - \frac{1}{2}\cos\beta + \sin^2\beta\cos\beta - \sin^2\beta C_1\right)\sin D\right]E_2 \\ &+ \left[\begin{array}{c}\left(\frac{\sin^2\beta}{4C_1^2}\sin 2D - \frac{C_4\sin\beta}{C_1}\sin D + \left(\frac{\sin^2\beta}{2C_1} + \cos\beta\right)s - C_3C_4 + C_5\right) \\ + \frac{u}{\tau}\left(\cos^2\beta - \cos\beta C_1 - \frac{1}{2}\right)\sin\beta\end{array}\right]E_3.\end{aligned}$$

4 Darboux ruled surface along general helix in Nil space Nil_3

In this section Darboux ruled surfaces of a general helix is examined. The ruled surface

$$\varphi(s,u) = \alpha(s) + u\tilde{D}(s)$$

is the parametrization of the ruled surface which is called rectifying developable surface of the curve α in [6]. It is referred to as $\tilde{D}-scroll$ cause of generator vector is modified Darboux vector field $\tilde{D}$ in [12].

Theorem 9. *Let $\alpha : I \to Nil_3$ be a unit speed non-geodesic general helix and* **B** *its binormal. Then, the parametric equation of* ***Darboux ruled surface****, in terms of E_1, E_2, E_3 is given by*

$$\varphi(s,u) = \begin{bmatrix} a\sin D + u\frac{a^2bC_1 - b + \frac{2}{b} + C_1 - 1}{\sqrt{1-4bC_1}}\cos D + C_3 \\ \\ -a\cos D + u\frac{a^2bC_1 - b + \frac{2}{b} + C_1 - 1}{\sqrt{1-4bC_1}}\sin D + C_4 \\ \\ \frac{a^2\sin 2D}{4} - aC_4\sin D + \left(\frac{a}{2} + b + C_1\right)s + u\frac{\left(a^2bC_1 - 1 + \frac{2}{b}\right)\cot\beta - aC_1}{\sqrt{1-4C_1b}} + e \end{bmatrix}$$

Proof. Since $\varphi(s,u) = \alpha(s) + u\tilde{D}(s)$ we have the parametrization $\varphi(s,u) = \alpha(s) + u\left(\frac{\tau^2+\kappa^2}{\kappa\tau}\right)\mathbf{T} + u\frac{1}{\tau}\nabla_{\mathbf{T}}\mathbf{N}$. Substituing $\frac{\sin\beta}{C_1} = a$; and $-C_3C_4 + C_5 = e$ in $\mathbf{T} = \sin\beta\cos DE_1 + \sin\beta\sin DE_2 + \cos\beta E_3$,

$$\nabla_T\mathbf{N} = \frac{1}{2}\left((2C_1 - \cos\beta)\cos DE_1 + (2C_1 - \cos\beta)\sin DE_2 - \sin\beta E_3\right),$$

and $\alpha(s)$, we have $\mathbf{T} = aC_1\cos DE_1 + aC_1\sin DE_2 + \cos\beta E_3$,

$$\nabla_T\mathbf{N} = \left(\frac{1}{2}(C_1 - b)\cos D, \frac{1}{2}(C_1 - b)\sin D, -\frac{1}{2}\sin\beta,\right)$$

and

$$\alpha(s) = (a\sin D + C_3)E_1 + (-a\cos D + C_4)E_2$$

$$+\left(\frac{a^2\sin 2D}{4} - aC_4\sin D + \left(\frac{a}{2} + \cos\beta\right)s + e\right)E_3.$$

Also we can find curvatures as

$$\tau = \sqrt{C_1^2 - C_1\cos\beta + \frac{1}{4}} = \frac{1}{2}\sqrt{1 - 4bC_1}$$

$$\kappa = \sin\beta(\cos\beta - C_1) = abC_1.$$

Also

$$\frac{\tau^2 + \kappa^2}{\kappa\tau} = \frac{2ab^2\sin\beta - 2b + \frac{1}{2C_1}}{ab\sqrt{1 - 4C_1b}}.$$

Hence

$$\varphi(s,u)=\begin{bmatrix}(a\sin D+C_3)\\(-a\cos D+C_4)\\\left(\frac{a^2\sin 2D}{4}-aC_4\sin D+\left(\frac{a}{2}+\cos\beta\right)s+e\right)\end{bmatrix}+u\frac{2ab^2\sin\beta-2b+\frac{1}{2C_1}}{ab\sqrt{1-4C_1b}}\begin{bmatrix}aC_1\cos D\\aC_1\sin D\\\cos\beta\end{bmatrix}$$
$$+u\frac{1}{\frac{1}{2}\sqrt{1-4bC_1}}\begin{bmatrix}\frac{1}{2}(C_1-b)\cos D\\\frac{1}{2}(C_1-b)\sin D\\\frac{-1}{2}\sin\beta\end{bmatrix}$$

$$\varphi(s,u)=\begin{bmatrix}a\sin D+C_3\\-a\cos D+C_4\\\frac{a^2\sin 2D}{4}-aC_4\sin D+\left(\frac{a}{2}+\cos\beta\right)s+e\end{bmatrix}$$
$$+\frac{u}{\sqrt{1-4C_1b}}\left(\frac{ab^2\sin\beta-b+2}{C_1ab}\begin{bmatrix}aC_1\cos D\\aC_1\sin D\\\cos\beta\end{bmatrix}+\begin{bmatrix}(C_1-b)\cos D\\(C_1-b)\sin D\\-\sin\beta\end{bmatrix}\right)$$

$$\varphi(s,u)=\begin{bmatrix}a\sin D+C_3\\-a\cos D+C_4\\\frac{a^2\sin 2D}{4}-aC_4\sin D+\left(\frac{a}{2}+\cos\beta\right)s+e\end{bmatrix}$$
$$+\frac{u}{b\sqrt{1-4C_1b}}\left(\begin{bmatrix}\left(a^2b^2C_1-b+2\right)\cos D+b(C_1-b)\cos D\\\left(a^2b^2C_1-b+2\right)\sin D+b(C_1-b)\sin D\\\left(a^2b^2C_1-b+2\right)\cot\beta-abC_1\end{bmatrix}\right)$$

$$=\begin{bmatrix}a\sin D+C_3\\-a\cos D+C_4\\\frac{a^2\sin 2D}{4}-aC_4\sin D+\left(\frac{a}{2}+\cos\beta\right)s+e\end{bmatrix}$$

$$+\frac{u}{b\sqrt{1-4C_1b}}\left(\begin{bmatrix}\left[\left(a^2b^2C_1-b+2\right)+b(C_1-b)\right]\cos D\\\left[\left(a^2b^2C_1-b+2\right)+b(C_1-b)\right]\sin D\\\left(a^2b^2C_1-b+2\right)\cot\beta-abC_1\end{bmatrix}\right)$$

$$=\begin{bmatrix}a\sin D+u\frac{\left[a^2b^2C_1-b^2+2+bC_1-b\right]}{b\sqrt{1-4C_1b}}\cos D+C_3\\-a\cos D+u\frac{\left[a^2b^2C_1-b^2+2+bC_1-b\right]}{b\sqrt{1-4C_1b}}\sin D+C_4\\\frac{a^2\sin 2D}{4}-aC_4\sin D+\left(\frac{a}{2}+\cos\beta\right)s+\frac{u\left(a^2b^2C_1-b+2\right)\cot\beta-abC_1}{b\sqrt{1-4C_1b}}+e\end{bmatrix}$$

Corollary 1. *Darboux ruled surface, cant be defined.under the condition* $\cos\beta=\frac{1+4C_1^2}{4C_1}$

Proof. It is trivial since

$$\sqrt{1-4C_1 b} = 0,$$
$$4C_1(\cos\beta - C_1) = 1.$$

References

[1] D.E. Blair, Contact Manifolds in Riemannian Geometry. Lecture Notes in Mathematics, Springer-Verlag, 509, Berlin-New York, (1976).9.

[2] N. Ekmekci, K.,Ilarslan, "Null general helices and submanifolds",. Bol. Soc. Mat. Mexicana, 9(2) (2003).279-286.

[3] M. Ergüt, E., Turhan, and T.,Körpınar, "On the Normal ruled surfaces of general helices in the Sol space Sol*3*",. TWMS J. Pure Appl. Math.V.4 , N.2 (2013) 125-130.

[4] J. Fastenakels, M.I. Munteanu, and J. van der Veken, Constant angle surfaces in the Heisenberg Group, Acta Mathematica Sinica, English Series Apr.,March 15 Vol. 27 No . 4, pp.(2011) 747–756.

[5] L.K. Graves, "Codimension one isometric immersions between Lorentz spaces", Trans. Amer. Math. Soc. 252 (1979) 367–392.

[6] S. Izumiya, N. Takeuchi, "Special curves and Ruled surfaces", Beitrage zur Algebra und Geometrie Contributions to Algebra and Geometry. Volume 44 No. 1 (2003) 203-212.

[7] Ş. Kılıçoğlu, "On the Involutive B-scrolls in the Euclidean three-space E 3", XIII^{th} . Geometry Integrability and Quantization.Varna, Bulgaria: Sofia , pp (2012) 205-214.

[8] Ş. Kılıçoğlu, "On the explicit parametric equation of a general helix with first and second curvature in Nil 3-space", Pure and applied Mathematics Journal. Submitted.

[9] Ş. Kılıçoğlu, H. H. Hacısalihoğlu, "On the ruled surface whose frame is the Bishop Frame in the Euclidean 3-space", International Electronic Journal of Geometry, , 6(2) (2013)110-117.

[10] L. Kula, Y. Yayli, "On slant helix and its spherical indicatrix",. Applied Mathematics and Computation, 169,(2005) 600-607.

[11] Y. Ou,and Z. Wang, "Linear biharmonic maps into Sol, Nil and Heisenberg Spaces", Mediterr. j. math. 5 (2008), 379-394.

[12] S. Senyurt, and S. Kilicoglu "On the differential geometric elements of the *involute* $\tilde{D}$ *scroll*", Adv. Appl. Clifford Algebras 2015 Springer Basel, doi:10.1007/s00006-015-0535-z.

[13] D. J. Struik, Lectures on Classical Differential Geometry. Dover, New-York, 1988.

[14] E. Turhan, T. Körpınar, "Parametric equations of general helices in the sol space Sol3", Bol. Soc. Paran. Mat., 31(1) (2013) 99-104.

[15] W. P. Thurston, "Three-Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry" Bull. Amer. Math. Soc. 6 (1982) 357-381.

[16] E. Wilson, "Isometry groups on homogeneous nilmanifolds", Geometriae Dedicata 12 (1982) 337-346.

[17] A.Yildirim, H.H. Hacısalihoğlu, "On BCV-Sasakian Manifold", International Mathematical Forum. Vol. 6, no 34 (2011) 1669 - 1684.