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Exact travelling wave solutions for some nonlinear partial differential equations by using the G'/G expansion method

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Abstract: In this paper, the $\left(\frac{G'}{G}\right)$ expansion method with the aid of computer algebraic system Maple is used for constructing exact travelling wave solutions and new kinds of solutions for the modified dispersive water wave equations, the Abrahams-Tsuneto reaction diffusion system, and for a class of reaction diffusion models. The method is straightforward and concise, and it be also applied to other nonlinear partial differential equations.

Keywords: The $\left(\frac{G'}{G}\right)$ expansion method, travelling wave solution, modified dispersive water wave equations, Abrahams-Tsuneto reaction diffusion system, reaction diffusion.

1 Introduction

Nonlinear partial differential equations arise in a large number of physics, mathematics and engineering problems. In the soliton theory, the study of exact solutions to these nonlinear equations plays a very significant role, as they provide much information about the physical models they describe. Various powerful methods have been employed to construct exact travelling wave solutions to nonlinear partial differential equations. These methods include the inverse scattering transform [1], the Backlund transform [2,3], the Darboux transform [4], the Hirota bilinear method [5], the tanh-function method [6,7], the sine-cosine method [8], the exp-function method [9], the generalized Riccati equation [10], the homogeneous balance method [11], the first integral method [12,13], the $\left(\frac{G'}{G}\right)$ expansion method [14,15], and the modified simple equation method [16,18].

In recent years, direct methods to construct exact solutions of NLPDEs have become more and more attractive partly due to the availability of symbolic computation packages like Maple and Mathematica, which enable us to perform the tedious and complex computation on computer. The objective of this paper is to use a powerful method called the $\left(\frac{G'}{G}\right)$ expansion method to obtain travelling wave solution for the modified dispersive water wave equations [19,20] used to model nonlinear and dispersive long gravity waves travelling in shallow water with uniform depth. Furthermore, it will be used to find more and different solutions that require less effort to apply for a class of reaction diffusion models [21,23]a mathematical model describing how the concentration of one or more substances vary over time and space

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under the influence of two terms, and the Abrahams-Tsuneto reaction diffusion system [24,25] arising in superconductivity. The method, first introduced by Wang and Zhang [26], has been widely used to obtain exact solutions of nonlinear equations [27,32].

The main ideas are that the travelling wave solutions of nonlinear equation can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation: $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x - ct$ and λ, μ, c are constants. The degree of this polynomial can be determined by considering the homogenous balance between the highest order derivative and nonlinear terms appearing in the given nonlinear equations. The coefficients of the polynomial λ, μ and c can be obtained by solving a set of algebraic equations resulting from the process of using the proposed method. Moreover, the travelling wave solutions obtained via this method are expressed by the hyperbolic functions, the trigonometric functions and the rational functions.

2 Description of the $\left(\frac{G'}{G}\right)$ expansion method

In this section, we describe the $\left(\frac{G'}{G}\right)$ expansion method for finding travelling wave solutions of nonlinear partial differential equations. Suppose that a nonlinear partial differential equation(PDE), say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0 \tag{1}$$

where $u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which highest order derivatives and nonlinear terms are involved.

The summary of the $\left(\frac{G'}{G}\right)$ expansion method, can be presented in the following six steps:

Step 1. To find the travelling wave solutions of Eq. (1) we introduce the wave variable

$$u(x,t) = u(\xi), \xi = x - ct, \tag{2}$$

where the constant c is generally termed the wave velocity. Substituting Eq. (2) into Eq. (1), we obtain the following ordinary differential equations(ODE) in ξ (which illustrates a principal advantage of a travelling wave solution, i.e., a PDE is reduced to an ODE).

$$P(U, cU', U', cU'', c^2U'', U'', \ldots) = 0 \tag{3}$$

Step 2. If necessary we integrate Eq. (3) as many times as possible and set the constants of integration to be zero for simplicity.

Step 3. We suppose the solution of nonlinear partial differential equation can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{4}$$

where $G = G(\xi)$ satisfies the second-order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{5}$$

where $G' = \frac{dG}{d\xi}, G'' = \frac{d^2G}{d\xi^2}$, and a_i, λ and μ are real constants with $a_m \neq 0$. Here the prime denotes the derivative with respect to ξ. Using the general solutions of Eq. (5), we have

$$\left(\frac{G'}{G}\right) = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right), & \lambda^2 - 4\mu > 0 \\ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + c_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + c_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right), & \lambda^2 - 4\mu < 0 \\ \left(\frac{c_2}{c_1 + c_2\xi}\right) - \frac{\lambda}{2}, \quad \lambda^2 - 4\mu = 0 \end{cases} \tag{6}$$

where c_1 and c_2 are arbitrary constants.

Moreover, it follows from Eq. (4) and (5) that

$$\begin{cases} U' = -\sum_{i=1}^{m} i a_i \left(\left(\frac{G'}{G}\right)^{i+1} + \lambda\left(\frac{G'}{G}\right)^{i} + \mu\left(\frac{G'}{G}\right)^{i-1}\right), \\ U'' = \sum_{i=1}^{m} i a_i ((i+1)\left(\frac{G'}{G}\right)^{i+2} + (2i+1)\lambda\left(\frac{G'}{G}\right)^{i+1} + i(\lambda^2+2\mu)\left(\frac{G'}{G}\right)^{i} \\ \qquad + (2i-1)\lambda\mu\left(\frac{G'}{G}\right)^{i-1} + (i-1)\mu^2\left(\frac{G'}{G}\right)^{i-2}), \end{cases} \tag{7}$$

and so on, here the prime denotes the derivative with respect to ξ.

Step 4. The positive integer m can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (3) as follows: if we define the degree of $u(\xi)$ as $D[u(\xi)] = m$, then the degree of other expressions is defined by

$$D\left[\frac{d^q u}{d\xi^q}\right] = m + q,$$

$$D\left[u^r\left(\frac{d^q u}{d\xi^q}\right)^s\right] = mr + s(q+m).$$

Therefore, we can get the value of m in Eq. (4).

Step 5. Substituting Eq. (4) into Eq. (3) using general solutions of Eq. (5) and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, then setting each coefficient of this polynomial to zero yield a set of algebraic equations for a_i, c, λ and μ.

Step 6. Substitute a_i, c, λ and μ obtained in step 5 and the general solutions of Eq.(5) into Eq. (4). Next, depending on the sign of discriminant $(\lambda^2 - 4\mu)$, we can obtain the explicit solution of Eq. (1) immediately.

3 Modified dispersive water wave equations

The modified water wave equations(MDWW) are given by [19,20]

$$\begin{aligned} u_t &= -\frac{1}{4}v_{xx}+\frac{1}{2}\left(uv_x+vu_x\right), \\ v_t &= -u_{xx}-2uu_x+\frac{3}{2}vv_x. \end{aligned} \tag{8}$$

Using the wave variable $\xi = x-ct$ in Eq.s (8), then we obtain

$$\begin{aligned} -cU' &= -\frac{1}{4}V''+\frac{1}{2}\left(UV'+U'V\right), \\ -cV' &= -U''-2UU'+\frac{3}{2}VV'. \end{aligned} \tag{9}$$

According to step 4, balancing V'' with UV' and U'' with VV' gives M=1, N=1. Therefore, the solutions of Eq.s (9) can be written in the form

$$\begin{aligned} U(\xi) &= a_0+a_1\left(\frac{G'}{G}\right), a_1\neq 0, \\ V(\xi) &= b_0+b_1\left(\frac{G'}{G}\right), b_1\neq 0, \end{aligned} \tag{10}$$

where a_0, a_1, b_0 and b_1 are constants which are unknowns to be determined later. By Eq. (5) or Eq. (7) we derive

$$\begin{cases} U' = -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) - a_1\mu. \\ U'' = 2a_1\left(\frac{G'}{G}\right)^3 + 3a_1\lambda\left(\frac{G'}{G}\right)^2 + \left(2a_1\mu + a_1\lambda^2\right)\left(\frac{G'}{G}\right) + a_1\lambda\mu, \end{cases}$$
$$\begin{cases} V' = -b_1\left(\frac{G'}{G}\right)^2 - b_1\lambda\left(\frac{G'}{G}\right) - b_1\mu. \\ V'' = 2b_1\left(\frac{G'}{G}\right)^3 + 3b_1\lambda\left(\frac{G'}{G}\right)^2 + \left(2b_1\mu + b_1\lambda^2\right)\left(\frac{G'}{G}\right) + b_1\lambda\mu, \end{cases} \tag{11}$$

Substituting Eq.s (10) and its derivatives Eq.s (11) into Eq.s (9) and equating each coefficient of $\left(\frac{G'}{G}\right)$ to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, b_0 and b_1. Solving this system using Maple, we obtain

Set 1. $a_1 = -\frac{1}{2}, a_0 = -\frac{1}{2}c-\frac{1}{4}\lambda, b_1 = -1, b_0 = -c-\frac{1}{2}\lambda$;

Set 2. $a_1 = -\frac{1}{2}, a_0 = \frac{1}{2}c-\frac{1}{4}\lambda, b_1 = 1, b_0 = -c+\frac{1}{2}\lambda$.

Using these values in Eq.s (10) when $\lambda^2-4\mu>0$, we obtain the hyperbolic solutions respectively:

$$\begin{cases} u_1(x,t) = -\frac{c}{2} - \frac{\sqrt{\lambda^2-4\mu}}{4}\left(\dfrac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right), \\ v_1(x,t) = -c - \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\dfrac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right), \end{cases} \tag{12}$$

$$\begin{cases} u_2(x,t) = \frac{c}{2} - \frac{\sqrt{\lambda^2-4\mu}}{4}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right), \\ v_2(x,t) = -c + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right), \end{cases} \tag{13}$$

where c, λ and μ arbitrary constants and $\xi = x - ct$.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions as follows:

$$\begin{cases} u_3(x,t) = -\frac{c}{2} - \frac{\sqrt{4\mu-\lambda^2}}{4}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right), \\ v_3(x,t) = -c - \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right), \end{cases} \tag{14}$$

$$\begin{cases} u_4(x,t) = \frac{c}{2} - \frac{\sqrt{4\mu-\lambda^2}}{4}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right), \\ v_4(x,t) = -c + \frac{\sqrt{4\mu-\lambda^2}}{4}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right), \end{cases} \tag{15}$$

where c, λ and μ arbitrary constants and $\xi = x - ct$.

When $\lambda^2 - 4\mu = 0$, we obtain the rational solutions respectively:

$$\begin{cases} u_5(x,t) = -\frac{c}{2} - \frac{1}{2}\left(\frac{c_1}{c_1(x-ct)+c_2}\right), \\ v_5(x,t) = -c - \frac{c_1}{c_1(x-ct)+c_2}, \end{cases} \tag{16}$$

$$\begin{cases} u_6(x,t) = \frac{c}{2} - \frac{1}{2}\left(\frac{c_1}{c_1(x-ct)+c_2}\right), \\ v_6(x,t) = -c + \frac{c_1}{c_1(x-ct)+c_2}. \end{cases} \tag{17}$$

In particular, if we take $c_1 \neq 0$ and $c_2^2 < c_1^2$, then the solutions in Eq.s (12)-(15) become respectively:

$$\begin{cases} u_1(x,t) = -\frac{c}{2} - \frac{\sqrt{\lambda^2-4\mu}}{4}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \xi_0\right), \\ v_1(x,t) = -c - \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \xi_0\right), \end{cases} \tag{18}$$

$$\begin{cases} u_2(x,t) = \frac{c}{2} - \frac{\sqrt{\lambda^2-4\mu}}{4} \tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \xi_0\right), \\ v_2(x,t) = -c + \frac{\sqrt{\lambda^2-4\mu}}{2} \tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \xi_0\right), \end{cases} \tag{19}$$

where $\xi_0 = \tanh^{-1}\left(\frac{c_2}{c_1}\right)$,

$$\begin{cases} u_3(x,t) = -\frac{c}{2} + \frac{\sqrt{4\mu-\lambda^2}}{4} \tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi - \xi_0\right), \\ v_3(x,t) = -c + \frac{\sqrt{4\mu-\lambda^2}}{2} \tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi - \xi_0\right), \end{cases} \tag{20}$$

$$\begin{cases} u_4(x,t) = \frac{c}{2} + \frac{\sqrt{4\mu-\lambda^2}}{4} \tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi - \xi_0\right), \\ v_4(x,t) = -c - \frac{\sqrt{4\mu-\lambda^2}}{4} \tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi - \xi_0\right), \end{cases} \tag{21}$$

where $\xi_0 = \tan^{-1}\left(\frac{c_2}{c_1}\right)$ and c, λ and μ arbitrary constants and $\xi = x - ct$. The same manner, if we take $c_1 \neq 0$ and $c_2 = 0$, then the solutions in Eq.s (16)- (17) become respectively:

$$\begin{cases} u_5(x,t) = -\frac{c}{2} - \frac{1}{2(x-ct)}, \\ v_5(x,t) = -c - \frac{1}{x-ct}, \end{cases} \tag{22}$$

$$\begin{cases} u_6(x,t) = \frac{c}{2} - \frac{1}{2(x-ct)}, \\ v_6(x,t) = -c + \frac{1}{x-ct}. \end{cases} \tag{23}$$

4 The Abrahams-Tsuneto reaction diffusion system

In superconductivity, the Abrahams-Tsuneto reaction diffusion system is given by [24, 25].

$$\begin{aligned} u_t &= u_{xx} + \left(1 - u^2 - v^2\right) u, \\ v_t &= v_{xx} + \left(1 - u^2 - v^2\right) v. \end{aligned} \tag{24}$$

Using the wave variable $\xi = x - ct$ carries (24), then we obtain

$$\begin{aligned} -cU' &= U'' + \left(1 - U^2 - V^2\right) U, \\ -cV' &= V'' + \left(1 - U^2 - V^2\right) V. \end{aligned} \tag{25}$$

According to step 4, balancing V^2U with U'' and V'' with V^3 gives N=1,M=1. Therefore, the solutions of Eq.s (25) can be written in the form

$$\begin{aligned} U(\xi) &= a_0 + a_1\left(\frac{G'}{G}\right), a_1 \neq 0, \\ V(\xi) &= b_0 + b_1\left(\frac{G'}{G}\right), b_1 \neq 0 \end{aligned} \tag{26}$$

where a_0, a_1, b_0 and b_1 are constants which are unknowns to be determined later. Substituting Eq. (26) and its derivatives Eq.s (7) into Eq.s (25) and equating each coefficient of $\left(\frac{G'}{G}\right)$ to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, b_0, b_1, λ and c. Solving this system using Maple, we obtain

Set 1. $c = \frac{3\sqrt{2}}{2}, \lambda = \mp\sqrt{4\mu+\frac{1}{2}}, b_1 = k, b_0 = k\left(\frac{\lambda}{2}-\frac{\sqrt{2}}{4}\right), a_1 = \mp\sqrt{2-k^2}$, $a_0 = \mp\frac{2-k^2}{\sqrt{2-k^2}}\left(\frac{\lambda}{2}-\frac{\sqrt{2}}{4}\right)$;

Set 2. $c = -\frac{3\sqrt{2}}{2}, \lambda = \mp\sqrt{4\mu+\frac{1}{2}}, b_1 = k, b_0 = k\left(\frac{\lambda}{2}+\frac{\sqrt{2}}{4}\right), a_1 = \mp\sqrt{2-k^2}$, $a_0 = \mp\frac{2-k^2}{\sqrt{2-k^2}}\left(\frac{\lambda}{2}-\frac{\sqrt{2}}{4}\right)$; where k is arbitrary constant.

Using these values in Eq.s (26) when $\lambda^2-4\mu > 0$, we obtain the hyperbolic solutions:

$$\begin{cases} u_1(x,t) = \mp\frac{\sqrt{2}\sqrt{2-k^2}}{4}\left(-1+\frac{c_1\sinh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1\cosh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right), \\ v_1(x,t) = \frac{\sqrt{2}k}{4}\left(-1+\frac{c_1\sinh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1\cosh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right), \end{cases} \tag{27}$$

where $\xi = x-\frac{3\sqrt{2}}{2}t$ and $k^2 < 2$ is arbitrary constant,

$$\begin{cases} u_2(x,t) = \mp\frac{i\sqrt{2}\sqrt{k^2-2}}{4}\left(-1+\frac{c_1\sinh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1\cosh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right), \\ v_2(x,t) = \frac{\sqrt{2}k}{4}\left(-1+\frac{c_1\sinh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1\cosh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right), \end{cases} \tag{28}$$

where $\xi = x-\frac{3\sqrt{2}}{2}t$ and $k^2 > 2$ is arbitrary constant,

$$\begin{cases} u_3(x,t) = \mp\frac{\sqrt{2}\sqrt{2-k^2}}{4}\left(1+\frac{c_1\sinh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1\cosh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right), \\ v_3(x,t) = \frac{\sqrt{2}k}{4}\left(1+\frac{c_1\sinh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1\cosh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right), \end{cases} \tag{29}$$

where $\xi = x+\frac{3\sqrt{2}}{2}t$ and $k^2 < 2$ is arbitrary constant.

$$\begin{cases} u_4(x,t) = \mp\frac{i\sqrt{2}\sqrt{k^2-2}}{4}\left(1+\frac{c_1\sinh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1\cosh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right), \\ v_4(x,t) = \frac{\sqrt{2}k}{4}\left(1+\frac{c_1\sinh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1\cosh\left(\frac{\sqrt{2}}{4}\xi\right)+c_2\sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right), \end{cases} \tag{30}$$

where $\xi = x+\frac{3\sqrt{2}}{2}t$ and $k^2 > 2$ is arbitrary constant.

In particular, if we take $c_1 \neq 0$ and $c_2^2 < c_1^2$, then the solutions in Eq.s (27)-(30) become respectively:

$$\begin{cases} u_1(x,t) = \mp\frac{\sqrt{2}\sqrt{2-k^2}}{4}\left(-1+\tanh\left(\frac{\sqrt{2}}{4}x-\frac{3}{4}t+\xi_0\right)\right), \\ v_1(x,t) = \frac{\sqrt{2}k}{4}\left(-1+\tanh\left(\frac{\sqrt{2}}{4}x-\frac{3}{4}t+\xi_0\right)\right), \end{cases} \tag{31}$$

where $k^2 < 2$ is arbitrary constant,

$$\begin{cases} u_2(x,t) = \mp \frac{i\sqrt{2}\sqrt{k^2-2}}{4}\left(-1+\tanh\left(\frac{\sqrt{2}}{4}x - \frac{3}{4}t + \xi_0\right)\right), \\ v_2(x,t) = \frac{\sqrt{2}k}{4}\left(-1+\tanh\left(\frac{\sqrt{2}}{4}x - \frac{3}{4}t + \xi_0\right)\right), \end{cases} \tag{32}$$

where $k^2 > 2$ is arbitrary constant,

$$\begin{cases} u_3(x,t) = \mp \frac{\sqrt{2}\sqrt{2-k^2}}{4}\left(1+\tanh\left(\frac{\sqrt{2}}{4}x + \frac{3}{4}t + \xi_0\right)\right), \\ v_3(x,t) = \frac{\sqrt{2}k}{4}\left(1+\tanh\left(\frac{\sqrt{2}}{4}x + \frac{3}{4}t + \xi_0\right)\right), \end{cases} \tag{33}$$

where $k^2 < 2$ is arbitrary constant.

$$\begin{cases} u_4(x,t) = \mp \frac{i\sqrt{2}\sqrt{k^2-2}}{4}\left(1+\tanh\left(\frac{\sqrt{2}}{4}x + \frac{3}{4}t + \xi_0\right)\right), \\ v_4(x,t) = \frac{\sqrt{2}k}{4}\left(1+\tanh\left(\frac{\sqrt{2}}{4}x + \frac{3}{4}t + \xi_0\right)\right), \end{cases} \tag{34}$$

where $k^2 > 2$ is arbitrary constant and $\xi_0 = \tanh^{-1}\left(\frac{c_2}{c_1}\right)$.

5 Reaction diffusion models

Reaction diffusion models appear in many branches of science $[21-23]$. One of the important classes of such reaction diffusion models is

$$u_t = u_{xx} + u^{q+1}(1-u^q), q \in N \tag{35}$$

We will obtain only two families of solutions corresponding to $q = 1$ and $q = 2$.

Using the wave variable $\xi = x - ct$ for $q = 1$ carries (35), then we obtain

$$cU' + U'' + U^2 - U^3 = 0 \tag{36}$$

According to step 4, balancing U^3 and U'' gives N=1. Therefore, the solutions of Eq. (36) can be written in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), a_1 \neq 0, \tag{37}$$

where a_0 and a_1 are constants which are unknowns to be determined later. Substituting Eq. (37) and its derivatives Equations (11). into Equations (36) and equating each coefficient of $\left(\frac{G'}{G}\right)$ to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, λ and c. Solving this system using Maple, we obtain

Set 1. $c = \mp\frac{\sqrt{2}}{2}, \lambda = \mp\sqrt{4\mu + \frac{1}{2}}, a_1 = -2c, a_0 = \frac{4c\lambda - 2 - 12\mu}{6c\lambda - 1}$;

Set 2.$c = \mp\sqrt{2}, \lambda = \mp 2\sqrt{\mu}, a_1 = c, a_0 = \frac{12\mu - c\lambda}{3c\lambda - 2}$.

Using these values in Eq. (37) when $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic solutions :

$$u_1(x,t) = \frac{1}{2}\left(1 - \left(\frac{c_1 \sinh\left(\frac{\sqrt{2}}{4}\xi\right) + c_2 \cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1 \cosh\left(\frac{\sqrt{2}}{4}\xi\right) + c_2 \sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right)\right), \tag{38}$$

where $\xi = x - \frac{\sqrt{2}}{2}t$,

$$u_2(x,t) = \frac{1}{2}\left(1 + \left(\frac{c_1 \sinh\left(\frac{\sqrt{2}}{4}\xi\right) + c_2 \cosh\left(\frac{\sqrt{2}}{4}\xi\right)}{c_1 \cosh\left(\frac{\sqrt{2}}{4}\xi\right) + c_2 \sinh\left(\frac{\sqrt{2}}{4}\xi\right)}\right)\right), \tag{39}$$

where $\xi = x + \frac{\sqrt{2}}{2}t$.

When $\lambda^2 - 4\mu = 0$, we obtain the rational solutions :

$$u_3(x,t) = \frac{\sqrt{2}c_1}{c_1\left(x - \sqrt{2}t\right) + c_2}, \tag{40}$$

$$u_4(x,t) = \frac{-\sqrt{2}c_1}{c_1\left(x + \sqrt{2}t\right) + c_2}. \tag{41}$$

In particular, if we take $c_1 \neq 0$ and $c_2^2 < c_1^2$, then the solutions in Eq.s (38)-(39) become respectively:

$$u_{1,1}(x,t) = \frac{1}{2}\left(1 - \tanh\left(\frac{\sqrt{2}}{4}\left(x - \frac{\sqrt{2}}{2}t\right) + \xi_0\right)\right), \tag{42}$$

$$u_{2,1}(x,t) = \frac{1}{2}\left(1 + \tanh\left(\frac{\sqrt{2}}{4}\left(x + \frac{\sqrt{2}}{2}t\right) + \xi_0\right)\right) \tag{43}$$

where $\xi_0 = \tanh^{-1}\left(\frac{c_2}{c_1}\right)$.

Using the wave variable $\xi = x - ct$ for $q = 2$ carries (35), then we obtain

$$cU' + U'' + U^3 - U^5 = 0 \tag{44}$$

According to step 4, balancing U^5 with U'' gives N=$\frac{1}{2}$. To find travelling wave solutions of equation (44) we use the following transformation:

$$U(\xi) = V^{\frac{1}{2}}(\xi). \tag{45}$$

Then Eq.(44) reduces to

$$cVV' - \frac{1}{2}V'^2 + VV'' + 2V^3 - 2V^4 = 0. \tag{46}$$

Now, balancing V'^2 with V^4 in Eq.(46), we find N=1. So we seek its travelling wave solution of the form

$$V(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), a_1 \neq 0, \tag{47}$$

where a_0 and a_1 are constants which are unknowns to be determined later. Substituting Eq. (47) and its derivatives Eq.s (11) into (46) and equating each coefficient of $\left(\frac{G'}{G}\right)$ to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, λ and c. Solving this system using Maple, we obtain

Set 1. $c=\frac{1}{\sqrt{3}}, \lambda=\mp\sqrt{4\mu+\frac{4}{3}}, a_1=-\frac{\sqrt{3}}{2}, a_0=-\frac{\lambda\sqrt{3}}{4}+\frac{3}{8}+\frac{c\sqrt{3}}{8};$

Set 2. $c=-\frac{1}{\sqrt{3}}, \lambda=\mp\sqrt{4\mu+\frac{4}{3}}, a_1=\frac{\sqrt{3}}{2}, a_0=\frac{\lambda\sqrt{3}}{2}+\frac{3}{8}-\frac{c\sqrt{3}}{8};$

Set 3. $c=-\sqrt{3}, \lambda=\mp 2\sqrt{\mu}, a_1=-\frac{\sqrt{3}}{2}, a_0=-\frac{\lambda\sqrt{3}}{4}+\frac{3}{8}+\frac{c\sqrt{3}}{8};$

Set 4. $c=\sqrt{3}, \lambda=\mp 2\sqrt{\mu}, a_1=\frac{\sqrt{3}}{2}, a_0=\frac{\lambda\sqrt{3}}{4}+\frac{3}{8}-\frac{c\sqrt{3}}{8}.$

Using these values in Eq. (47) and transformation in (45) when $\lambda^2-4\mu>0$, we obtain the hyperbolic solutions :

$$u_1(x,t)=\left(\frac{1}{2}-\frac{1}{2}\left(\frac{c_1\sinh\left(\frac{x}{\sqrt{3}}-\frac{t}{3}\right)+c_2\cosh\left(\frac{x}{\sqrt{3}}-\frac{t}{3}\right)}{c_1\cosh\left(\frac{x}{\sqrt{3}}-\frac{t}{3}\right)+c_2\sinh\left(\frac{x}{\sqrt{3}}-\frac{t}{3}\right)}\right)\right)^{\frac{1}{2}}, \tag{48}$$

$$u_2(x,t)=\left(\frac{1}{2}+\frac{1}{2}\left(\frac{c_1\sinh\left(\frac{x}{\sqrt{3}}+\frac{t}{3}\right)+c_2\cosh\left(\frac{x}{\sqrt{3}}+\frac{t}{3}\right)}{c_1\cosh\left(\frac{x}{\sqrt{3}}+\frac{t}{3}\right)+c_2\sinh\left(\frac{x}{\sqrt{3}}+\frac{t}{3}\right)}\right)\right)^{\frac{1}{2}}. \tag{49}$$

When $\lambda^2-4\mu=0$, we obtain the rational solutions :

$$u_3(x,t)=\left(-\frac{\sqrt{3}}{2}\frac{c_1}{c_1\left(x+\sqrt{3}t\right)+c_2}\right)^{\frac{1}{2}}, \tag{50}$$

$$u_4(x,t)=\left(\frac{\sqrt{3}}{2}\frac{c_1}{c_1\left(x-\sqrt{3}t\right)+c_2}\right)^{\frac{1}{2}}. \tag{51}$$

In particular, if we take $c_1\neq 0$ and $c_2^2<c_1^2$, then the solutions in Eq.s (48) -(49) become respectively:

$$u_1(x,t)=\left(\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{x}{\sqrt{3}}-\frac{t}{3}+\xi_0\right)\right)^{\frac{1}{2}}, \tag{52}$$

$$u_2(x,t)=\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{x}{\sqrt{3}}+\frac{t}{3}+\xi_0\right)\right)^{\frac{1}{2}}, \tag{53}$$

where $\xi_0=\tanh^{-1}\left(\frac{c_2}{c_1}\right)$ .The same manner, if we take $c_1\neq 0$ and $c_2=0$, then the solutions in Eq.s (50)-(51) become respectively:

$$u_3(x,t)=\left(-\frac{\sqrt{3}}{2}\frac{1}{x+\sqrt{3}t}\right)^{\frac{1}{2}}, \tag{54}$$

$$u_4(x,t) = \left(\frac{\sqrt{3}}{2} \frac{1}{x - \sqrt{3}t} \right)^{\frac{1}{2}}. \tag{55}$$

6 Conclusions

In this paper, we implemented the $\left(\frac{G'}{G}\right)$ expansion method to solve for the modified dispersive water wave equations, the Abrahams-Tsuneto reaction diffusion system, and for a class of reaction diffusion models. The method is quite efficient, straightforward, concise and practically well suited for use in finding NLPDEs. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability.Therefore, the $\left(\frac{G'}{G}\right)$ expansion method is applied successfully and reliable to solve other nonlinear partial differential equations. Moreover, our solutions are in more general forms, and many known solutions to these equations are only special cases of them.

We have seen that three types of travelling wave solutions were successfully found, in terms of hyperbolic, trigonometric and rational functions. It will be more important to seek solutions of higher-order nonlinear equations which can be reduced to ODEs of the order greater than 2. We have noted that this method changes the given difficult problems into simple problems which can be solved easily. The method yields a general solution with free parameters which can be identified by the above conditions in section 2. Moreover,some numerical methods like the Adomian decomposition method and homotopy perturbation method depend on the initial conditions and obtain a solution in a series which converges to the exact solution of the problem. However,it is obtained by the $\left(\frac{G'}{G}\right)$ expansion method a general solution without approximation and there is no need to apply the initial and boundary conditions at the outset. The $\left(\frac{G'}{G}\right)$ expansion method is also a standard, direct and computerizable method, which allows us to solve complicated and tedious algebraic calculation. The solution procedure can be easily implemented in Mathematica or Maple.

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