New Trends in Mathematical Sciences
http://www.ntmsci.com

A numerical method for solutions of pantograph type differential equations with variable coefficients using Bernstein polynomials

Salih Yalcinbas, Huseyin Hilmi Sorkun and Mehmet Sezer

Department of Mathematics, Faculty of Science, Celal Bayar University, Manisa, Turkey

Received: 12 May 2015, Revised: 20 May 2015, Accepted: 20 May 2015
Published online: 17 December 2015.

Abstract: In this paper, a new numerical method based on the Bernstein polynomials is introduced for the approximate solution of pantograph type differential equations with retarded case or advanced case. Either the approximate solutions that are converging to the exact solutions or exact solutions of the problems are obtained by using this presented process. In addition, some numerical examples are presented to show the properties of the given method and the results are discussed.

Keywords: Differential equations, pantograph equations, collocation solutions, delay differential equations, Bernstein polynomials.

1 Introduction

Functional-differential equations with proportional delays are usually referred to as pantograph equations or generalized pantograph equations. The name *pantograph* originated from the work of Ockendon and Tayler on the collection of current by the pantograph head of an electric locomotive [1,2].

From the beginning of 1990s, there has been a growing interest in the numerical treatment of pantograph equations of the retarded and advanced type. A special feature of this type is the existence of compactly supported solutions [3]. This phenomenon was studied in [4] and has direct applications to approximation theory and to wavelets [5]. Pantograph equations are characterized by the presence of a linear functional argument and play an important role in explaining many different phenomena. In particular they turnout to be fundamental when ODEs-based model fail. These equations arise in industrial applications [6,7] and in studies based on number theory, electrodynamics, astro-physics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics, economy, control theory and cell growth, among others [8,9].

In recent years, the Taylor method has been used to find the approximate solutions of differential, difference, integral and integro-differential-difference equations [7,11-20]. The basic motivation of this work is to apply the Bernstein method to the non-homogenous pantograph equation with variable coefficients, which is extended of the multi-pantograph equation

* Corresponding author e-mail: salih.yalcinbas@cbu.edu.tr

given in [6,10,31]. Our purpose in this study is to develop and to apply mentioned methods to the pantograph equation

$$u'(x) = \sum_{j=0}^{J} P_j(x)\,u(\lambda_j x + \mu_j) + f(x),\ \ 0 \le x \le 1 \tag{1}$$

with initial conditions

$$u(0) = \gamma \in R \tag{2}$$

where $u(x)$ is the unknown function, the known functions $P_j(x)$ and $f(x)$ are defined on the domain which we are analytical functions and also λ_j and μ_j are real or complex constants.

2 Preliminaries and notations

We will now generalize the method in [11-20] in order to solve the pantograph equations. Let us first consider the equation (1) as

$$u'(x) = \sum_{j=0}^{J} P_j(x)\,u(\lambda_j x + \mu_j) + f(x)\ \ 0 \le x \le 1 \tag{3}$$

with initial conditions

$$u(0) = \gamma \in R \tag{4}$$

where $u(x)$ is the unknown function, the known functions $P_j(x)$ and $f(x)$ are defined on the domain which we are analytical functions and also λ_j and μ_j are real or complex constants.

We take for granted that the solution is expressed in general form of the Bernstein polynomials in [21] of nth-degree are defined by

$$B_{n,N}(x) = \binom{N}{n} \frac{x^n (R-x)^{N-n}}{R^N},\ \ 0 \le n \le N \tag{5}$$

for $n = 0, 1, \ldots, N$, where the binomial coefficients $B_{n,N}(x) = \binom{N}{n} \frac{x^n (R-x)^{N-n}}{R^N}$, are given by

$$\binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{6}$$

and

$$(R-x)^{N-n} = \sum_{k=0}^{N-n} \binom{N-n}{k} (-1)^k R^{N-n-k} x^k. \tag{7}$$

From (5) and (7), it is clear that

$$B_{n,N}(x) = \sum_{k=0}^{N-n} \binom{N}{n} \binom{N-n}{k} \frac{(-1)^k}{R^{n-k}} x^{n+k} \tag{8}$$

where R is the maximum range of the interval $[0, R]$ over which the polynomials are defined to form a complete basis. There are $n+1$ nth- degree polynomials. For laborsaving, we set $B_{n,N}(x) = 0$, if $n < 0$ or $n > N$. It can be easily shown that any given polynomial of degree n can be expanded in terms of a linear combination of the basis functions

$$u(x)=\sum_{n=0}^{N}u_nB_{n,N}(x) \tag{9}$$

We can write to begin with $B_{n,N}(x)$ in the matrix form as follows:

$$B_N^T(x)=S^TX^T(x)\Leftrightarrow B_N(x)=X(x)S \tag{10}$$

where

$$B_N(x)=\begin{bmatrix}B_{0,N}(x) & B_{1,N}(x) & \cdots & B_{N,N}(x)\end{bmatrix},\ X(x)=\begin{bmatrix}1 & x & x^2 & \cdots & x^N\end{bmatrix},$$

$$S^T=\begin{bmatrix}
1\binom{N}{0}\binom{N}{1}\frac{(-1)^1}{R} & \binom{N}{0}\binom{N}{2}\frac{(-1)^2}{R^2} & \cdots & \binom{N}{0}\frac{(-1)^N}{R^N}\\
0 \quad \binom{N}{1}\frac{1}{R} & \binom{N}{1}\binom{N-1}{1}\frac{(-1)^1}{R^2} & \cdots & \binom{N}{1}\frac{(-1)^{N-1}}{R^N}\\
0 \quad 0 & \binom{N}{2}\frac{1}{R^2} & \cdots & \binom{N}{2}\frac{(-1)^{N-2}}{R^N}\\
\vdots \quad \vdots & \vdots & \ddots & \vdots\\
0 \quad 0 & 0 & \cdots & \binom{N}{N-1}\frac{(-1)^1}{R^N}\\
0 \quad 0 & 0 & \cdots & \binom{N}{N}\frac{1}{R^N}
\end{bmatrix}.$$

3 Basic matrix representations

We first consider the desired solution $u(x)$ of (1) defined by the Bernstein polynomials in (10). Then we can put (10) in the matrix form

$$[u(x)]=B_N(x)\,\mathbf{U}$$

or from (10)

$$[u(x)]=\mathbf{X}(x)\,SU. \tag{11}$$

It is clearly seen that the relation [30] between the matrix $X(x)$ and its derivative $X^{(1)}(x)$ is

$$X^{(1)}(x)=X(x)T \tag{12}$$

where

$$\mathrm{T}=\begin{bmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & N\\ 0 & 0 & 0 & \dots & 0\end{bmatrix}.$$

On the other hand, the solution expressed by (9) and its derivative can be written in the matrix forms using the relation (12)

$$[u(x)] = \mathrm{X}(x)\,\mathbf{S}\mathrm{U} \text{ and } [u^{(1)}(x)] = \mathrm{X}(x)\,\mathbf{TSU} \tag{13}$$

where

$$\mathrm{X}(x) = [\,1 \;\; x \;\; x^2 \;\; \cdots \;\; x^N\,],\; \mathrm{U} = [\,u_0 \;\; u_1 \;\; u_2 \;\; \cdots \;\; u_N\,]^T,$$

and for the collocation points $x = x_i$; $(i = 0,1,...,N)$, the matrices system

$$[u(x_i)] = \mathrm{X}(x_i)\,\mathbf{S}\mathrm{U} \text{ and } [u^{(1)}(x_i)] = \mathrm{X}(x_i)\,\mathrm{TSU}, \tag{14}$$

where

$$X = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \cdots & x_0^N \\ 1 & x_0 & \cdots & x_0^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N^N \end{bmatrix}, \mathrm{U} = [\,u_0 \;\; u_1 \;\; u_2 \;\; \cdots \;\; u_N\,]^T.$$

We can write the expression $u(\lambda x + \mu)$ as

$$u(\lambda x + \mu) = \sum_{n=0}^{N} u_n B_{n,N} (\lambda x + \mu)^n \tag{15}$$

Hence (15) we have matrix form

$$[\,u(\lambda x + \mu)\,] = \mathrm{X}(x)\,\mathrm{B}(\lambda,\,\mu)\,\mathbf{S}\mathrm{U} \tag{16}$$

so that

$$\mathrm{X}(x) = [\,1 \;\; x \;\; x^2 \;\; \cdots \;\; x^N\,], \mathrm{U} = [\,u_0 \;\; u_1 \;\; u_2 \;\; \cdots \;\; u_N\,]^T,$$

$$\mathrm{B}(\lambda,\,\mu) = \begin{bmatrix} 1 & \binom{1}{0}\mu & \binom{2}{0}\mu^2 & \cdots & \binom{N}{0}\mu^N \\ 0 & \lambda & \binom{2}{1}\lambda\,\mu & \cdots & \binom{N}{1}\lambda\,\mu^{N-1} \\ 0 & 0 & \lambda^2 & \cdots & \binom{N}{2}\lambda^2\mu^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^N \end{bmatrix},$$

and for $x = x_i$; $(i = 0,1,...,N)$ the matrices system

$$[\,u(\lambda x_i + \mu)\,] = \mathrm{X}(x_i)\,\mathrm{B}(\lambda,\,\mu)\,\mathbf{S}\mathrm{U} \tag{17}$$

where

$$X = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \cdots & x_0^N \\ 1 & x_1 & \cdots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N^N \end{bmatrix}.$$

4 The process of the method by using the matrix representations

We are now ready to construct the fundamental matrix equation corresponding to (1). For this intention, we substitute the matrix relation (13) and (16) into (1) and obtain the matrix equation

$$X(x)\mathbf{TSU} = \sum_{j=0}^{J} P_j(x)\mathbf{X}(x)B(\lambda_j,\mu_j)\mathbf{SU} + f(x) \tag{18}$$

By using in (18) collocation points x_i defined by

$$x_i = \frac{i}{N}\ , i = 0,1,\ldots,N,$$

the system of matrix equations is obtained as

$$X(x_i)\mathbf{TSU} = \sum_{j=0}^{J} P_j(x_i)\mathbf{X}(x_i)B(\lambda_j,\mu_j)\mathbf{SU} + f(x_i),\ i = 0,1,\ldots,N,$$

or in short the fundamental matrix equation

$$\left\{ X\mathbf{TS} - \sum_{j=0}^{J} P_j\mathbf{X}B(\lambda_j,\mu_j)\mathbf{S} \right\} \mathbf{U} = \mathbf{F} \tag{19}$$

where

$$P_j = \begin{bmatrix} P_j(x_0) & 0 & 0 & \cdots & 0 \\ 0 & P_j(x_1) & 0 & \cdots & 0 \\ 0 & 0 & P_j(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_j(x_N) \end{bmatrix}, F = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix}, \mathrm{X} = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \cdots & x_0^N \\ 1 & x_1 & \cdots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N^N \end{bmatrix},$$

and we can also write (19) in the form

$$WU = F \text{ or } [W;\mathbf{F}],\ \text{where } W = X\mathbf{TS} - \sum_{j=0}^{J} P_j\mathbf{X}B(\lambda_j,\mu_j)\mathbf{S} \tag{20}$$

that corresponds to a system of $(N+1)$ algebraic equations with the unknown Bernstein coefficients.

We can gain the matrix form corresponding to the condition $u(0) = \gamma$ as, from the relation (13),

$$[\,1\ \ 0\ \ \cdots\ 0\,]\mathbf{SU} = [\gamma] \tag{21}$$

or

$$[\,1\;0\cdots 0\;;\;\gamma].$$

To obtain the solution of (1) subject to (2), now we have the new augmented matrix by replacing the row matrix (21) by the last row of matrix (20)

$$[\tilde{\mathrm{W}};\tilde{\mathrm{F}}]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ;\,0 \\ w_{01} & w_{11} & \cdots & w_{1N} & ;\,0 \\ \vdots & \vdots & \ddots & \vdots & ;\,\vdots \\ w_{0,N-1} & w_{N-1,1} & \cdots & w_{N-1,N} & ;\,0 \\ u_0 & u_1 & \cdots & u_N & ;\,\gamma \end{bmatrix} \tag{22}$$

where for condition $u(0)=\gamma$; $u_0=1,\ u_1=\cdots=u_N=0$ [20-28].

If $rank\,\tilde{\mathrm{W}}=rank\,[\tilde{\mathrm{W}};\tilde{\mathrm{F}}]=N+1$ in (17), then we can write

$$\mathbf{U}=(\tilde{\mathbf{W}})^{-1}\,\tilde{\mathbf{F}} \tag{23}$$

where it can be uniquely determined. If $\det(\tilde{\mathrm{W}})=0$, in that case there is no solution and the method cannot be used or we may obtain the particular solutions by means of the system. Besides, by means of systems we may obtain the particular solutions.

5 Error analysis

Theorem 1. *For a function $f(x)$ bounded on $[0,1]$, the relation*

$$\lim_{n\to\infty} B_{n.N}(x)=f(x) \tag{24}$$

holds at each point of continuity x of f; and the relation holds uniformly on $[0,1]$ if $f(x)$ is continuous on this interval [22].

Proof. We shall compute the value of

$$T=\sum_{n=0}^{N}(n-Nx)^2 B_{n,N}(x)=\sum_{n=0}^{N}\{n(n-1)-(2Nx-1)n+N^2x^2\}B_{n,N}(x) \tag{25}$$

Clearly $\sum\limits_{n=0}^{N} B_{n,N}(x)=1$; moreover, we have

$$\sum_{n=0}^{N} nB_{n,N}(x)=Nx\sum_{\mu=0}^{N-1}\binom{N-1}{\mu}x^{\mu}(1-x)^{N-\mu-1}=Nx,$$

$$\sum_{n=0}^{N} n(n-1)B_{n,N}(x)=N(N-1)x^2\sum_{\mu=0}^{N-2}\binom{N-2}{\mu}x^{\mu}(1-x)^{N-\mu-2}=N(N-1)x^2,$$

and therefore

$$T = N^2x^2 - (2Nx-1)Nx + N(N-1)x^2 = Nx(1-x) \tag{26}$$

Since $x(1-x) \le \frac{1}{4}$ on $[0,1]$, we obtain the inequality

$$\sum_{\left|\frac{n}{N}-x\right|\ge\delta} B_{n,N}(x) \le \frac{1}{\delta^2}\sum_{\left|\frac{n}{N}-x\right|\ge\delta}\left(\frac{n}{N}-x\right)^2 B_{n,N}(x) \le \frac{1}{N^2\delta^2}T = \frac{x(1-x)}{N\delta^2} \le \frac{1}{4N\delta^2} \tag{27}$$

If now the function $f$is bounded, say $|f(u)| \le M$ in $0 \le u \le 1$ and x a point of continuity, for a given $\varepsilon > 0$ we can find a $\delta > 0$ such that $|x-x'| < \delta$ implies $|f(x)-f(x')| < \varepsilon$. We have

$$|f(x) - B_{n,N}(x)| = \left|\sum_{n=0}^{N}\left\{f(x) - f\left(\begin{matrix} n \\ N \end{matrix}\right)\right\} B_{n,N}(x)\right| \tag{28}$$

$$\le \sum_{\left|\frac{n}{N}-x\right|<\delta}\left|f(x) - f\left(\begin{matrix} n \\ N \end{matrix}\right)\right| B_{n,N}(x) + \sum_{\left|\frac{n}{N}-x\right|\ge\delta}$$

The first sum is $\le \varepsilon \sum B_{n,N}(x) = \varepsilon$, the second is, by (22), $\le 2M(4N\delta^2)^{-1}$. Therefore

$$|f(x) - B_{n,N}(x)| \le \varepsilon + M(2N\delta^2)^{-1}, \tag{29}$$

and if N is sufficiently large, $|f(x) - B_{n,N}(x)| \le 2\varepsilon$. Finally, if $f(x)$ is continuous in the whole interval $[0,1]$ then (35) holds with a δ independent of x, so that $B_{n,N}(x) \to f(x)$ uniformly. This completes the proof.

The Bernstein polynomials are not orthogonal. However, these can be expressed in terms of some orthogonal polynomials, such as the Chebychev polynomials $U_n(x)$ of second kind [23]. It can be shown that

$$B_{n,N}(x) = \frac{1}{2^N}\binom{N}{n}\sum_{s=0}^{N} d_s^{n,N}\frac{1}{2^N}\sum_{m=0}^{\left[\frac{s}{2}\right]}\left\{\binom{s}{m} - \binom{s}{m+1}\right\}U_{s-2m}(x) \tag{30}$$

with

$$d_s^{n,N} = \sum_k (-1)^{s-k}\binom{n}{k}\binom{N-n}{s-k} \tag{31}$$

the summation over k being taken as follows:

For $n < N < N-n$,

(i) $k = 0$ to s for $s \le n$,
(ii) $k = 0$ to n for $n < s \le N-n$,
(iii) $k = s-(N-n)$ to $N-n$ for $N-n < s \le N$ while for $n = N-n$ (N being an even integer);
  (a) $k = 0$ to s for $s \le n$,
  (b) $k = s-n$ to n for $n < s \le N-n$

and $N-n$ above are to be interchanged.

Thus, an approximation $u_n(x)$ of the function $u(x)$ in terms of the Bernstein polynomials in the form

$$u(x) \cong u_N(x) = \sum_{n=0}^{N} a_n B_{n,N(x)} \tag{32}$$

is eventually expressed as

$$u_N(x) = \sum_{j=0}^{N} b_j U_j(x) \tag{33}$$

where b_j, $(j = 0, 1, ..., N)$ can be expressed in terms of a_n, $(n = 0, 1, ..., N)$ and vice-versa. If $u_j(x) = \sqrt{\frac{2}{\pi}} U_j(x)$, then $u_j(x)$, $(j = 0, 1, ..., N)$ form an orthonormal polynomial basis in $[-1, 1]$ with respect to the weight function $w(x) = (1 - x^2)^{1/2}$. Then

$$\|u - u_n\|_w < \sqrt{\frac{\pi}{2}} b_0 n^{-r}, \ \ r > 0$$

where $\| l \|_w \equiv \int_{-1}^{1} \{l(x)\}^2 w(x)dx$ and b_0 is some constant. Thus, the convergence is very fast if r is large. In our method on pantograph equations, both $P_j(x)$ and $f(x)$ are C^∞-functions, and such as, the method converges rapidly. This is also reflected in the numerical computations.

On the other hand, we can easily check the accuracy of the obtain solutions as follows [24]. Since the obtained polynomial expansion is an approximate solution, when the function $u(x)$ are the derive $u'(x)$ are substituted in (1), the resulting equation must be satisfied approximately: that is, for $x = x_i \in [a, b]$, $i = 0, 1, ..., N$

$$E(x_i) = \left| u'(x_i) - \sum_{j=0}^{J} P_j(x_i) u(\lambda_j(x_i) + \mu_j) + f(x_i) \right| \cong 0$$

or

$$E(x_i) \leq 10^{k_i}, \ \ (k_i \text{ is any positive integer}).$$

If $\max(10^{k_i}) = 10^{-k}$, (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of points x_i becomes smaller than the prescribed 10^{-k}.

6 Illustrations

In this section, five numerical examples are given to illustrate the accuracy and efficiency of the presented method.

Example 1. (Z.-H. Yu, [25]). Consider the multi-pantograph equation

$$\begin{cases} u'(x) = -\frac{5}{6} u(x) + 4u\left(\frac{x}{2}\right) + 9u\left(\frac{x}{3}\right) + x^2 - 1, \ \ 0 < x \leq 1 \\ u(0) = 1. \end{cases} \tag{34}$$

We assume that the problem has a Bernstein polynomial solution in the form

$$u(x) = \sum_{n=0}^{N} u_n B_{n,N}(x)$$

where $N=4$, $P_0(x)=-5/6$, $P_1(x)=4$, $P_2(x)=9$, $f(x)=x^2-1$ and $\lambda_0=1$, $\mu_0=0$, $\lambda_1=1/2$, $\mu_1=0$, $\lambda_2=1/3$, $\mu_2=0$.

Following the procedure in Section 3, collocation points are computed as

$$\left\{x_0=0, x_1=\frac{1}{4}, x_2=\frac{1}{2}, x_3=\frac{3}{4}, x_4=1\right\}$$

and from (19), the fundamental matrix equation of the problem is

$$\left\{X\mathbf{T}\mathbf{S}-\sum_{j=0}^{J}P_j\mathbf{X}B(\lambda_j,\mu_j)\mathbf{S}\right\}\mathbf{U}=\mathbf{F}$$

$$\{X\mathbf{T}\mathbf{S}-P_0XB(\lambda_0,\mu_0)\mathbf{S}-P_1XB(\lambda_1,\mu_1)\mathbf{S}-P_2XB(\lambda_2,\mu_2)\mathbf{S}\}U=F$$

where

$$\mathbf{S}=\begin{bmatrix}1&0&0&0&0\\-4&4&0&0&0\\6&-12&6&0&0\\-4&12&-12&4&0\\1&-4&6&-4&1\end{bmatrix},\ P_0=\begin{bmatrix}-5/6&0&0&0&0\\0&-5/6&0&0&0\\0&0&-5/6&0&0\\0&0&0&-5/6&0\\0&0&0&0&-5/6\end{bmatrix},\ P_1=\begin{bmatrix}4&0&0&0&0\\0&4&0&0&0\\0&0&4&0&0\\0&0&0&4&0\\0&0&0&0&4\end{bmatrix},$$

$$P_2=\begin{bmatrix}9&0&0&0&0\\0&9&0&0&0\\0&0&9&0&0\\0&0&0&9&0\\0&0&0&0&9\end{bmatrix},\ \mathbf{T}=\begin{bmatrix}0&1&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&4\\0&0&0&0&0\end{bmatrix},\ X=\begin{bmatrix}1&0&0&0&0\\1&1/4&1/16&1/64&1/256\\1&1/2&1/4&1/8&1/16\\1&3/4&9/16&27/64&81/256\\1&1&1&1&1\end{bmatrix},$$

$$F=\begin{bmatrix}-1\\-15/16\\-3/4\\-7/16\\0\end{bmatrix},\ B(1,0)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\ B(1/2,0)=\begin{bmatrix}1&0&0&0&0\\0&1/2&0&0&0\\0&0&1/4&0&0\\0&0&0&1/8&0\\0&0&0&0&1/16\end{bmatrix},$$

$$B(1/3,0)=\begin{bmatrix}1&0&0&0&0\\0&1/3&0&0&0\\0&0&1/9&0&0\\0&0&0&1/27&0\\0&0&0&0&1/81\end{bmatrix},$$

The augmented matrix for this fundamental matrix equation is

$$[W;F]=\left[\begin{array}{ccccccc} -97/6 & 4 & 0 & 0 & 0 & ; & -1 \\ -93295/9216 & -7601/2304 & 1073/1536 & 1135/2304 & 593/9216 & ; & -15/16 \\ -3487/576 & -857/144 & -151/96 & 127/144 & 305/576 & ; & -3/4 \\ -10805/3072 & -1465/256 & -2133/512 & -153/256 & 1881/1024 & ; & -7/16 \\ -73/36 & -41/9 & -25/6 & -53/9 & 161/36 & ; & 0 \end{array}\right]$$

From (21), the matrix forms for initial condition is

$$[\,1\;0\cdots 0\;;\;\gamma]=\left[1\;0\;0\;0\;0\;;\;1\right]$$

From system (24), the new augmented matrix based on conditions can be obtained as follows

$$[\tilde{W};\tilde{F}]=\left[\begin{array}{ccccccc} -97/6 & 4 & 0 & 0 & 0 & ; & -1 \\ -93295/9216 & -7601/2304 & 1073/1536 & 1135/2304 & 593/9216 & ; & -15/16 \\ -3487/576 & -857/144 & -151/96 & 127/144 & 305/576 & ; & -3/4 \\ -10805/3072 & -1465/256 & -2133/512 & -153/256 & 1881/1024 & ; & -7/16 \\ 1 & 0 & 0 & 0 & 0 & ; & 1 \end{array}\right]$$

solving this system, Bernstein coefficients matrix are obtained as

$$U=\left[1\;\;91/24\;\;4519/432\;\;121057/5184\;\;58075/1296\right]^{T}.$$

We, therefore, obtain the solution of the problem for $N=4$ become

$$u(x)=\frac{12157}{1296}x^3+\frac{1675}{72}x^2+\frac{67}{6}x+1$$

which is the also exact solution.

Example 2. Consider the linear delay differential equation of first order

$$\begin{cases} u'(x)=-u(0.8x)-u(x), \\ u(0)=1. \end{cases} \tag{35}$$

We assume that the problem has a Bernstein polynomial solution in the form

$$u(x)=\sum_{n=0}^{N}a_nB_{n,N}(x)$$

From (19), the fundamental matrix equation of the problem is

$$\{X\mathbf{T}\mathbf{S}-P_0XB(\lambda_0,\mu_0)\mathbf{S}-P_1XB(\lambda_1,\mu_1)\mathbf{S}\}U=F$$

where $P_0(x) = -1, \quad P_1(x) = -1, \quad f(x) = 0, \quad \lambda_0 = 0.8, \quad \mu_0 = 0, \quad \lambda_1 = 1, \quad \mu_1 = 0.$
Following the procedure in Section 3, we find the solutions of the problem for $N = 8$,

$$y(x) = 1.7999995x^2 - 2.0x - 0.98399264x^3 + 0.37190582x^4 - 0.10470307x^5 + 0.02288761x^6 - 0.003808588x^7 + 0.0003815305x^8 + 1 .$$

Table 1 shows solutions of (41) with $N = 8$and10 by presented method. The previous results of Rao and Palanisamy by Walsh series approach [26], Hwang and Shih by Laguerre series approach [24], and Sezer and Akyüz by Taylor series approach [14] are also given in Table 1 for comparison. The Bernstein method seems more rapidly convergent than Taylor method. The error functions are seen in Table 2.

Table 1. Numerical results of Example 2 for the approximate solution $u_N(x)$, $\quad N = 8, 10$ and the other methods.

x_i	Walsh series method	Laguerre series method	Taylor series method		Bernstein method	
		$n = 20$	$N = 8$	$N = 11$	$N = 8$	$N = 10$
0	1.000000	0.999971	1.000000	1.000000	1.000000000000000	1.000000000000000
0.2	0.665621	0.664703	0.664691	0.664691	0.664691000243432	0.664691000946436
0.4	0.432426	0.433555	0.433561	0.433561	0.433560778295009	0.433560778879920
0.6	0.275140	0.276471	0.276483	0.276482	0.276482329799246	0.276482330288006
0.8	0.170320	0.171482	0.171484	0.171484	0.171484111252193	0.171484112022145
1	0.100856	0.102679	0.102744	0.102670	0.1026701625	0.102670125200000

Table 2. Comparison of errors of the approximate solutions for (35).

	Taylor series method	Present method
x_i	$N = 11$, $E(x_i)$	$N = 10$, $E(x_i)$
0	4.75 E-15	0
0.2	5.24 E-10	3.489 E-10
0.4	3.29 E-10	2.631 E-11
0.6	7.20 E-10	3.764 E-12
0.8	1.69 E-11	2.455 E-11
1	1.48 E-12	2.226 E-09

Example 3. Consider the following problem:

$$\begin{cases} u'(x) = \frac{1}{2}e^{x/2}u\left(\frac{x}{2}\right) + \frac{1}{2}u(x), \ 0 \le x \le 1, \\ u(0) = 1, \end{cases} \tag{36}$$

which has exact solution $u(x) = \exp(x)$.

From (19), the fundamental matrix equation of the problem is

$$\{X\mathbf{T}\mathbf{S} - P_0XB(\lambda_0, \mu_0)\mathbf{S} - P_1XB(\lambda_1, \mu_1)\mathbf{S}\}U = F$$

where $P_0(x) = \frac{1}{2}e^{x/2}$, $P_1(x) = \frac{1}{2}$, $f(x) = 0$, $\lambda_0 = \frac{1}{2}$, $\mu_0 = 0$, $\lambda_1 = 1$, $\mu_1 = 0$.

Following the procedure in Section 3, we find the solution of the problem for $N = 10$.

$$u(x) = 1.0x + 0.50000018x^2 + 0.16666396x^3 + 0.04168504x^4 + 0.00826252x^5$$
$$+0.0015564x^6 - 0.00004968x^7 + 0.00024932x^8 - 0.00011084x^9 + 0.000024933x^{10} + 1.0 \ .$$

The absolute error and estimation error of solution obtained by presented method for $N = 12$ are compared with the absolute errors of solutions given by other methods [3,14,25,27,28,30] in the Table 3 and Figure 1. The Present method has better results than the Variational iteration method, Taylor method, Spline Function Approximation and the Spline method. However, the absolute errors of Adomian and the present methods seem like each other.

Table 3. Comparison of the absolute and approximate errors for the present method and other methods for Example 3.

	Spline method, $h = 0.001$		Adomain method with 13 terms [8] $E(x_i)$	Taylor method	Variational method	Bernstein method $N = 9$, $E(x_i)$	
x_i	$m = 3$ [7-23], $E(x_i)$			$N = 9$, $E(x_i)$		Ext.err	Est.err
0.2	0.198 E-7	1.37 E-11	0.00	0.705 E-14	2.44 E-05	3.479 E-11	7.614 E-11
0.4	0.473 E-7	3.27 E-11	2.22 E-16	0.106 E-10	2.28 E-04	3.989 E-11	1.756 E-11
0.6	0.847 E-7	5.86 E-11	2.22 E-16	0.294 E-9	9.00 E-04	5.521 E-11	7.094 E-11
0.8	0.135 E-6	9.54 E-11	1.33 E-15	0.386 E-8	2.50 E-03	8.580 E-11	1.650 E-10
1	0.201 E-6	1.43 E-10	4.88 E-15	0.290 E-7	5.71 E-03	4.540 E-10	1.389 E-7

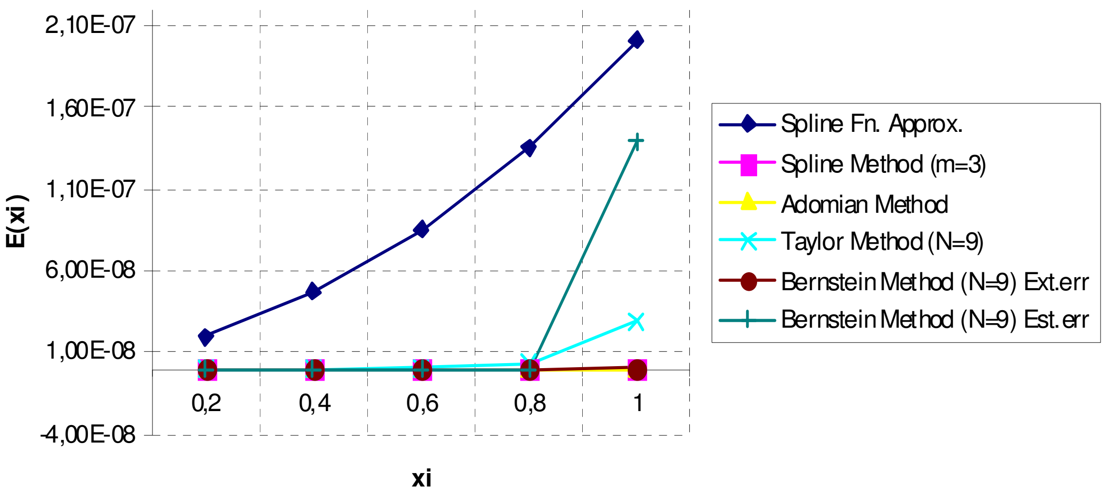

Fig. 1. Comparison of the absolute errors and estimate of errors $E(x_i)$ for $N = 9$ of an approximated solution by the methods

Example 4. Consider the pantograph equation of

$$\begin{cases} u'(x) = -u(x) + \frac{q}{2}u(qx) - \frac{q}{2}e^{-qx}, \ 0 \le x \le 1, \\ u(0) = 1, \end{cases} \tag{37}$$

which has exact solution $u(x) = e^{-x}$.

From (19), the fundamental matrix equation of the problem is

$$\{X\mathbf{TS} - P_0XB(\lambda_0,\mu_0)\mathbf{S} - P_1XB(\lambda_1,\mu_1)\mathbf{S}\}U = F$$

where $P_0(x) = -1, \quad P_1(x) = \frac{q}{2}, \quad f(x) = -\frac{q}{2}e^{-qx}, \quad \lambda_0 = 1, \quad \mu_0 = 0, \quad \lambda_1 = q, \quad \mu_1 = 0.$

Following the procedure in Section 3, we find the solution of the problem for $N = 13$ and $q = 1,\ q = 0.2$ respectively,

$$\begin{aligned} u(x) &= 1 - x + 0.500000000000328x^2 - 0.166666666674781x^3 + 0.416666667587229E-1x^4 \\ &- 0.833333394476986E-2x^5 + 0.138889150582022E-2x^6 - 0.198420261618556E-3x^7 \\ &+ 0.248166609118034E-4x^8 - 0.277648561972118E-5x^9 + 0.294952748666358E-6x^{10} \\ &- 0.367508039050922E-7x^{11} + 0.618440621168329E-8x^{12} - 0.773904241676987E-9x^{13}, \end{aligned}$$

$$\begin{aligned} u(x) &= 1 - x + 0.50000000000022x^2 - 0.16666666667236x^3 + 0.41666666733808E-1x^4 \\ &- 0.833333379392323E-2x^5 + 0.138889091302536E-2x^6 - 0.19841867419705E-3x^7 \\ &+ 0.248136977517014E-4x^8 - 0.277261736288716E-5x^9 + 0.29148068206839E-6x^{10} \\ &- 0.347045064964356E-7x^{11} + 0.547021422781133E-8x^{12} - 0.661910550152943E-9x^{13}. \end{aligned}$$

Table 4 compares the results of the present method, the collocation method [29] and the Taylor method [14]. Also, the absolute errors are compared for $q = 0.2$ and $q = 1$ in Figure 2(a)-(b). Note that $q = 1$ is not a pantograph equation, is a linear differential equation. In any case, the Bernstein method has far better results than collocation method. In any case, the absolute errors of the Taylor method and the present method seem like each other.

Table 4. Comparison of the absolute errors for $q = 0.2$ and $q = 1$ of (37).

x_i	$q=1$			$q=0.2$		
	Collocation Method	Taylor method	Bernstein method	Collocation Method	Taylor method	Bernstein method
	$m=2$	$N=6$	$N=6$	$m=2$	$N=6$	$N=6$
2^{-1}	5.005 E-06	1.458 E-06	1.002 E-08	2.719 E-05	1.458 E-06	4.6788 E-07
2^{-2}	1.877 E-07	1.174 E-08	2.4306 E-09	1.080 E-06	1.174 E-08	6.5210 E-08
2^{-3}	6.434 E-09	9.315 E-11	5.0626 E-10	3.817 E-08	9.315 E-11	6.1038 E-09
2^{-4}	2.106 E-10	7.334 E-13	8.4669 E-11	1.269 E-09	7.334 E-13	2.6234 E-10
2^{-5}	6.700 E-12	5.662 E-15	1.2309 E-11	4.090 E-11	5.662 E-15	8.3965 E-10
2^{-6}	2.100 E-13	0.000	1.6608 E-12	1.200 E-12	0.000	2.3625 E-11

Table 5. Numerical Results of Example 4 for $N = 5$,$N = 7$ and $N = 9$.

x_i	Exact Solution $u(x) = \exp(-x)$	$N = 5$	$N = 7$	$N = 9$
2^{-1}	0.6065306597	0.6065301918	0.6065306589	0.6065306597
2^{-2}	0.7788007831	0.778800131	0.7788007823	0.7788007831
2^{-3}	0.8824969026	0.8824962922	0.8824969014	0.8824969026
2^{-4}	0.9394130628	0.9394128005	0.9394130622	0.9394130628
2^{-5}	0.9692332345	0.9692331505	0.9692332343	0.9692332345
2^{-6}	0.984496437	0.9844964134	0.984496437	0.984496437

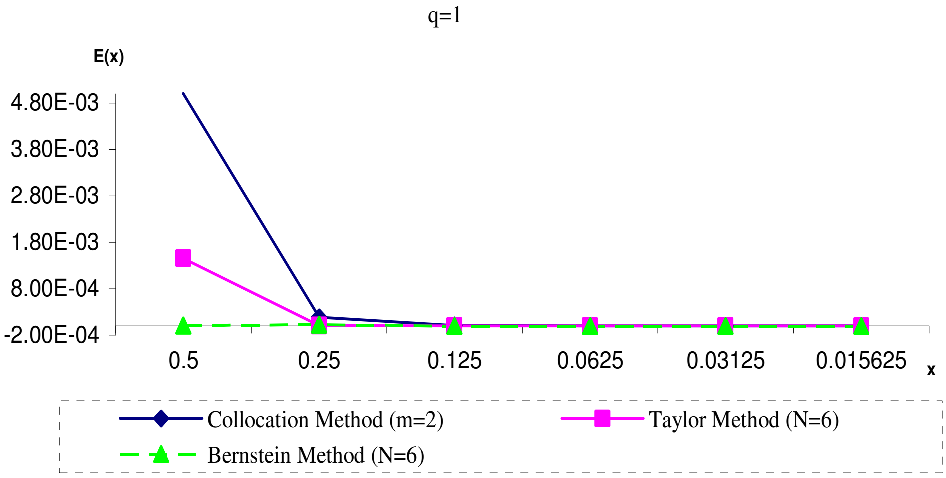

Fig. 2(a). Comparison of the absolute errors$E(x_i)$ for $q = 1$.

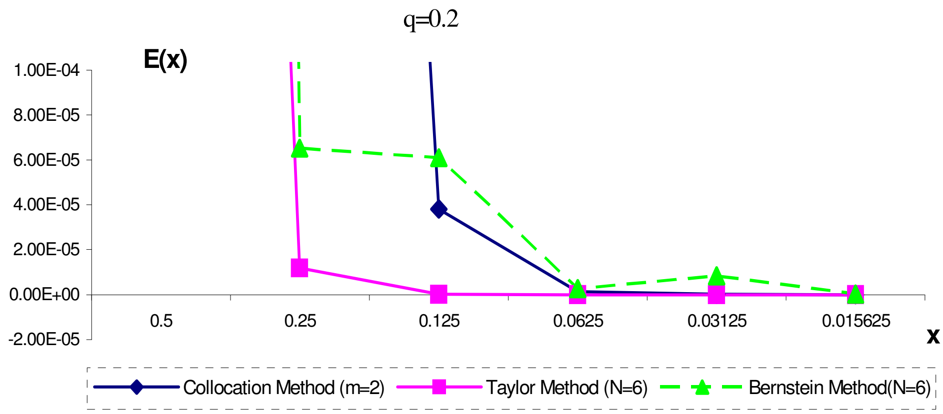

Fig. 2(b). Comparison of the absolute errors$E(x_i)$ for $q = 0.2$.

Example 5. Consider the pantograph equation with variable coefficients

$$\begin{cases} u'(x) = -u(x) + \mu_1(x)u(x/2) + \mu_2(x)u(x/4), 0 \le x \le 1, \\ u(0) = 1. \end{cases} \tag{38}$$

Here$\mu_1(x) = -e^{-0.5x}\sin(0.5x)$, $\mu_2(x) = -2e^{-0.75x}\cos(0.5x)\sin(0.25x)$. It can be seen that the exact solution of this problem is $u(x) = e^{-x}\cos(x)$ [17].

From (19), the fundamental matrix equation of the problem is

$$\{X\mathbf{T}\mathbf{S} - P_0XB(\lambda_0,\mu_0)\mathbf{S} - P_1XB(\lambda_1,\mu_1)\mathbf{S} - P_2XB(\lambda_2,\mu_2)\mathbf{S}\}U = F$$

where $P_0(x) = -1$, $P_1(x) = \mu_1(x)$, $P_2(x) = \mu_2(x)$,$\lambda_0 = 1$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$, $\mu_0 = \mu_1 = \mu_2 = f(x) = 0$.

Following the procedure in Section 3, we find the solution of the problem for $N = 7$,

$$u(x) = 0.333403165x^3 - 0.000005916x^2 - 0.999999999x - 0.16701361x^4 \\ +0.034243842x^5 - 0.001333669x^6 - 0.0005281649x^7 + 1.0 \ .$$

For $N = 7$ in Figure 3 and Table 6, the absolute error and solution obtained by the Bernstein method is compared the absolute error and solution of Taylor method given in [23]. It is seen from Figure 4 and Table 5 that the results obtained by the present method is very superior to that obtained by Taylor method.

Table 6. Comparison of the solutions and the absolute errors of (37).

x_i	Exact Solution $u(x) = e^{-x}\cos(x)$	$N = 5$	$N = 7$	$N = 9$	$N = 9, E(x_i)$
0	1.0	1.0	1.0	1.0	0
0.2	0.8024106474	0.8024125376	0.802410633	0.8024106474	5.74800 E-11
0.4	0.6174056479	0.6174062718	0.6174056369	0.6174056479	1.64602 E-12
0.6	0.4529537892	0.4529551434	0.4529537829	0.4529537892	5.47500 E-11
0.8	0.313050504	0.3130489942	0.313050505	0.3130505041	9.55200 E-11
1.0	0.1987661104	0.1988138463	0.1987656481	0.1987661083	2.04641 E-09

	Exact Solution	Taylor method		Bernstein method	
x_i	$u(x) = e^{-x}\cos(x)$	$N = 7, u(x)$	$N = 7, E(x_i)$	$N = 7, u(x)$	$N = 7, E(x_i)$
0	1	1	0	1	0
0.2	8.0241 E-01	8.0241 E-01	9.9331 E-10	8.0241 E-01	1.4522 E-08
0.4	6.1741 E-01	6.1741 E-01	2.4854 E-07	6.1741 E-01	1.1150 E-08
0.6	4.5295 E-01	4.5295 E-01	6.2234 E-06	4.5295 E-01	6.3522 E-09
0.8	3.1305 E-01	3.1299 E-01	6.0719 E-05	3.1305 E-01	8.8968 E-10
1	1.9877 E-01	1.9841 E-01	3.5341 E-04	1.9877 E-01	4.6165 E-07

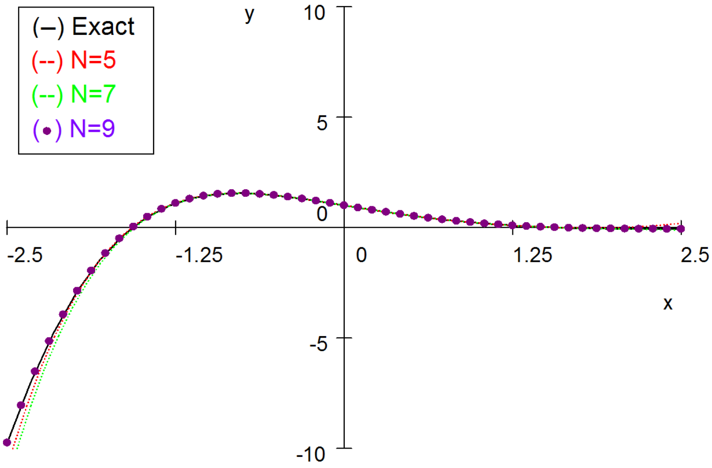

Fig. 3. Results for $N = 5, 7, 9$ and exact solution.

7 Conclusions

A new technique, using the Bernstein polynomials, to numerically solve the pantograph equations is presented. Comparison of the results obtained by the present method with and that other method reveals that the present method is very effective and convenient. The numerical results show that the accuracy improves when N is increased. Tables and figures indicate that as N increases, the errors decrease more rapidly; hence for better results, using large number N is recommended. Another considerable advantage of the method is that Bernstein coefficients of the solution are found very easily by using the computer programs.

References

[1] J. R. Ockendon, A. B. Tayler, The dynamic of a current collection system for an electric locomotive, Proc. Roy. Soc. London Ser. A 322 (1971) 447-468.

[2] A. B. Tayler, Mathematical Models in Applied Mathematics, Clarendon Pres, Oxford, 1986, see pp. 40-53.

[3] A. El-Safty, S.M. Abo-Hasha, On the application of spline function to initial value problems with retarded argument, Int. J. Comput. Math. 32 (1990) 173-179.

[4] G. Derfel, On compactly supported solutions of a class of functional-differential equations, in: Modern Problems of Functions Theory and Functional Analysis, Karaganda University Press, 1980 (In Russian).

[5] G. Derfel, N. Dyn, D. Levin, Generalized refinement equation and subdivision process, J. Approx. Theory 80 (1995) 272-297.

[6] G. Derfel, A. Iserles, The pantograph equation in the complex plane, J. Math. Anal. Appl. 213 (1997) 117-132.

[7] M. Gülsu, M. Sezer, The approximate solution of high-order linear difference equation with variable coefficients in terms of Taylor polynomials, Appl. Math. Comp. 168 (2005) 76-88.

[8] W.G. Ajello, H.I. Freedman, J. Wu, A model of stage structured population growth with density depended time delay, SIAM J. Appl. Math. 52 (1992) 855-869.

[9] M.D. Buhmann, A. Iserles, Stability of the discretized pantograph differential equation, Math. Comput. 60 (1993) 575-589.

[10] M.Z. Liu, D. Li, Properties of analytic solution and numerical solution and multi-pantograph equation, Appl. Math. Comput. 155 (2004) 853-871.

[11] G. A. Derfel, A. Iserles, The pantograph equation in the complex plane, J. Math. Anal. App. 213 (1997) 117-132.

[12] M. Gülsu, M. Sezer, A Taylor polynomial approach for solving differential-difference equations, J. Comput. Appl. Math. 186 (2006) 349-364.

[13] M. Gülsu, M. Sezer, A method for the approximate solution of the high-order linear difference equations in terms of Taylor polynomial, Int. J. Comput. Math. 82 (5) (2005) 629-642.

[14] M. Sezer, A. Akyüz-Daşçıoğlu, A Taylor method for numerical solution of generalized pantograph equations with linear functional argument, J. Comp. and Appl. Math. 200 (2007) 217-225.

[15] Ş. Nas, S. Yalçınbaş, M. Sezer, A Taylor polynomial approach for solving high-order linear Fredholm integro-differential equations, Int. J. Math. Ed. Sci. Tecnol. 31 (2) (2000) 213-225.

[16] M. Sezer, A method for approximate solution of the second order linear differential equation in terms of Taylor polynomials, Int. J. Math. Ed. Sci. Tecnol. 27 (6) (1996) 821-834.

[17] M. Sezer, M. Gülsu, A new polynomial approach for solving difference and Fredholm integro-difference equations with mixed argument, Appl. Math. Comp. 171 (1) (2005) 332-344.

[18] S. Yalçınbaş, M. Sezer, The approximate solution of high-order linear Volterra-Fredholm integro-differential equation in terms of Taylor polynomials, Appl. Math. Comp. 112 (2000) 291-308.

[19] M. Sezer, S. Yalçınbaş, N. Şahin, Approximate solution of multi-pantograph equation with variable coefficients, J. Comp. and Appl. Math. 214 (2008) 406-416.

[20] N. Kurt, M. Sezer, Polynomial solution of high-order linear Fredholm integro-differential equations with constant coefficients, Journal of the Franklin Institute. 345 (2008) 839-850.

[21] M. Idrees Bhatti, P. Bracken, Solutions of differential equations in a Bernstein polynomial basis, J. Comput. and Appl. Math. 205 (2007) 272 – 280.

[22] Bernstein S., Demonstration du theoreme de Weierstrass, fondee sur le calcul des probabilities, Commun. Soc. Math. Kharkow (2) 13 (1912-13), 1-2.

[23] M. A. Snyder, Chebychev Methods in Numerical Approximation, Prentice-Hall, Engle-wood Cliffs, NJ, 1996.

[24] C. Hwang, Y.-P. Shih, Laguerre series solution of a functional differential equation, Int. J. Syst. Sci. 13 (7) (1982) 783-788.

[25] Zhan-Hua Yu, Variational iteration method for solving the multi-pantograph delay equation, Physics Letters A 372 (2008) 6475-6479.

[26] G.P. Rao, K.R. Palanisamy, Walsh stretch matrices and functional differential equation, IEEE Trans. Autom. Control 27 (1982) 272-276.

[27] A. El-Safty, M.S. Salim, M.A. El-Khatib, Convergence of the spline function for delay dynamic system, Int. J. Comput. Math. 80 (4) (2003) 509-518.

[28] D.J. Evans, K.R. Raslan, The Adomain decomposition method for solving delay differential equation, Int. J. Comput. Math. 82 (1) (2005) 49-54.

[29] Y. Muroya, E. Ishiwata, H. Brunner, On the attainable order of collocation methods for pantograph integro-differential equations, J. Comput. Appl. Math. 152 (2003) 347-366.

[30] M. Shadia, Numerical solution of delay differential and neutral differential equations using spline methods, Ph.D. Thesis, Assuit University, 1992.

[31] S. Yalçınbaş, M. Aynigül, M. Sezer, A collocation method using Hermite polynomials for approximate solution of Pantograph equations series, Journal of the Franklin Institute, 348 (6) (2011) 1128-1139.