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Constant ratio curves according to parallel transport frame in euclidean 4-space $\mathbb{E}^4$

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Abstract: In this study, we consider a regular curve in Euclidean 4-space $\mathbb{E}^4$ whose position vector is written as a linear combination of its parallel transport frame vectors. We characterize constant ratio curves in terms of their curvature functions. Further, we obtain some results of T-constant type and N-constant type curves according to its Bishop curvatures in $\mathbb{E}^4$.

Keywords: Parallel transport frame, position vector, constant-ratio curves.

1 Introduction

Rectifying curves in Euclidean 3-space $\mathbb{E}^3$ are introduced by B. Y. Chen in [4] as space curves whose position vector (denoted also by x) lies in its rectifying plane, spanned by the tangent and the binormal normal vector fields $T(s)$ and $N_2(s)$ of the curve. In the same paper, B. Y. Chen gave a simple characterization of rectifying curves. In particular, it is shown in [7] that there exists a simple relation between rectifying curves and centrodes, which play an important roles in mechanics kinematics as well as in differential geometry in defining the curves of constant procession. It is also provide that a regular curve is congruent to a non constant linear function of s [5]. Further, in the Minkowski 3-space $\mathbb{E}_1^3$, the rectifying curves are investigated in [8], [12], [13], [14]. In [14] a characterization of the spacelike, the timelike and the null rectifying curves in the Minkowski 3-space in terms of centrodes is given.

In [15], Ilarslan and Nesovic considered the rectifying curve in Euclidean 4-space $\mathbb{E}^4$. They characterized the rectifying curves given by the equation

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s) + \upsilon(s)N_3(s),$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\upsilon(s)$. Further, in the Minkowski 4-space $\mathbb{E}_1^4$, the rectifying curves are investigated in [1], [16], [17].

The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. But, curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In this situation, we need an alternative frame in $\mathbb{E}^3$. Therefore In [2], Bishop defined a new frame for a curve and he called it Bishop frame which is well defined even if the curve has vanishing second derivative in 3-dimensional Euclidean space. In Euclidean 4-space $\mathbb{E}^4$, we have the same problem, that is, one of the $i-th$ $(1<i<4)$ derivative of the curve may vanish. In [10], the autors gave parallel transport frame of a curve and they introduce the relations between the Bishop frame and Frenet frame of the curve in 4-dimensional Euclidean space. They characterized curves whose position vectors lie in their normal,

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rectifying and osculating planes according to the Bishop frame in 4-dimensional Euclidean space $\mathbb{E}^4$.

For a regular curve $x(s)$, the position vector x can be decompose into its tangential and normal components at each point, i.e., $x = x^T + x^N$. A curve $x(s)$ with $\kappa_1(s) > 0$ is said to be of *constant ratio* if the ratio $\left\|x^T\right\| : \left\|x^N\right\|$ is constant on $x(I)$ where $\left\|x^T\right\|$ and $\left\|x^N\right\|$ denote the length of x^T and x^N, respectively [3].

Clearly a curve x in $\mathbb{E}^n$ is of constant ratio if and only if $x^T = 0$ or $\left\|x^T\right\| : \|x\|$ is constant [4]. The distance function $\rho = \|x\|$ satisfies $\|grad\rho\| = c$ for some constant c if and only if we have $\left\|x^T\right\| = c\|x\|$. In particular, if $\|grad\rho\| = c$ then $c \in [0,1]$. In [5], B. Y. Chen gave a classification of constant ratio curves in Euclidean space. A curve in $\mathbb{E}^n$ is called T*-constant* (resp. N*-constant*) if the tangential component x^T (resp. the normal component x^N) of its position vector x is of constant length [3]. Recently in [11] the autors study the constant ratio curves in Euclidean 3-space $\mathbb{E}^3$.

In the present study, we consider a curve in Euclidean 4-space $\mathbb{E}^4$ as a curve whose position vector can be written as linear combination of its parallel transport frame. Then its position vector satisfies the parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)M_1(s) + m_2(s)M_2(s) + m_3(s)M_3(s), \tag{1}$$

for some differentiable functions, $m_i(s)$, $0 \leq i \leq 3$, where $\{T, M_1, M_2, M_3\}$ is its parallel transport frame. We characterize such curves in terms of their curvature functions and give the necessary and sufficient conditions for such curves to become T-constant or N-constant curves in $\mathbb{E}^4$.

2 Basic notations and known results

Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. Let us denote $T(s) = x'(s)$ and call as a unit tangent vector of x at s. We denote the first Serret-Frenet curvature of x by $\kappa(s) = \|x''(s)\|$. If $\kappa(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve x at s is given by $N_1'(s) + \kappa(s)T(s) = \tau(s)N_2(s)$, where τ is the second Serret-Frenet curvatures of x. If $\tau(s) \neq 0$, then the unit second principal normal vector $N_2(s)$ of the curve x at s is given by $N_2'(s) + \tau(s)N_1(s) = \sigma(s)N_3(s)$, where σ is the third Serret-Frenet curvatures of x. Then we have the Serret-Frenet formulae (see, [9]):

$$\begin{aligned} T'(s) &= \kappa(s)N_1(s), \\ N_1'(s) &= -\kappa(s)T(s) + \tau(s)N_2(s), \\ N_2'(s) &= -\tau(s)N_1(s) + \sigma(s)N_3(s), \\ N_3'(s) &= -\sigma(s)N_2(s). \end{aligned} \tag{2}$$

Further, let x be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$ with the tangent vector $T(s)$. One can choose any convenient arbitrary basis which consist of relatively parallel vector fields $M_1(s)$, $M_2(s)$, $M_3(s)$ which are perpendicular to $T(s)$ at each point. The parallel transport frame equations are (see [10]):

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} \tag{3}$$

where k_1, k_2, k_3 are curvature functions (Bishop curvatures) according to parallel transport frame of the curve x and their expressions are:

$$\begin{aligned} k_1(s) =& \kappa(s)\cos\theta(s)\cos\psi(s), \\ k_2(s) =& \kappa(s)(-\cos\theta(s)\sin\psi(s)+\sin\phi(s)\sin\theta(s)\cos\psi(s)), \\ k_3(s) =& \kappa(s)(\sin\phi(s)\sin\psi(s)+\cos\phi(s)\sin\theta(s)\cos\psi(s)), \end{aligned} \tag{4}$$

where

$$\theta' = \frac{\sigma}{\sqrt{\kappa^2+\tau^2}}, \psi' = -\tau - \sigma\frac{\sqrt{\sigma^2-\theta'^2}}{\sqrt{\kappa^2+\tau^2}}, \phi' = -\frac{\sqrt{\sigma^2-\theta'^2}}{\cos\theta}$$

and

$$\kappa = \sqrt{k_1^2+k_2^2+k_3^2},\ \tau = -\psi' + \phi'\sin\theta,\ \sigma = \frac{\theta'}{\sin\psi},\ \phi'\cos\theta + \theta'\cot\psi = 0 \tag{5}$$

hold.

3 Characterization of curves according to parallel transport frame in $\mathbb{E}^4$

In the present section we consider unit speed curves with Bishop curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. Definition of the position vector of the curve satisfies the vectorial equation (1), for some differential functions $m_i(s)$, $0 \le i \le 3$. By taking the derivative of (1) with respect to arclength parameter s and using the parallel transport frame equations (3), we obtain

$$\begin{aligned} x'(s) =& (m_0'(s) - k_1(s)m_1(s) - k_2(s)m_2(s) - k_3(s)m_3(s))T(s) \\ &+ (m_1'(s) + k_1(s)m_0(s))M_1(s) \\ &+ (m_2'(s) + k_2(s)m_0(s))M_2(s) \\ &+ (m_3'(s) + k_3(s)m_0(s))M_3(s). \end{aligned} \tag{6}$$

It follows that

$$\begin{aligned} m_0' - k_1m_1 - k_2m_2 - k_3m_3 =& 1, \\ m_1' + k_1m_0 =& 0, \\ m_2' + k_2m_0 =& 0, \\ m_3' + k_3m_0 =& 0. \end{aligned} \tag{7}$$

Theorem 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ a unit speed curve in $\mathbb{E}^4$ with the vectorial equation (1). If x has constant curvatures ($k_1, k_2, k_3 =$ constant) then the position vector x is given by the curvature functions*

$$\begin{aligned} m_0(s) =& c_1\cos\kappa s + c_2\sin\kappa s, \\ m_1(s) =& -k_1\left(\frac{c_1\sin\kappa s - c_2\cos\kappa s}{\kappa}\right) + c_3, \\ m_2(s) =& -k_2\left(\frac{c_1\sin\kappa s - c_2\cos\kappa s}{\kappa}\right) + c_4, \\ m_3(s) =& -k_3\left(\frac{c_1\sin\kappa s - c_2\cos\kappa s}{\kappa}\right) + c_5. \end{aligned} \tag{8}$$

where $c_i, (0 \leq i \leq 5)$ are integral constants and $\kappa = \sqrt{k_1^2 + k_2^2 + k_3^2}$ is the first Frenet curvature of the curve x.

Proof. Let x be a unit speed curve with the constant Bishop curvatures ($k_1, k_2, k_3 =$ constant). By the use of (7) we get

$$m_0'' = -(k_1^2 + k_2^2 + k_3^2) m_0 \tag{9}$$

One can show that the equation (9) has a non-trivial solution

$$m_0 = c_1 \cos \sqrt{k_1^2 + k_2^2 + k_3^2} s + c_2 \sin \sqrt{k_1^2 + k_2^2 + k_3^2} s$$

Further, substituting this solution in (7) we get the result.

3.1 T-constant curves

Definition 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. If $\left\| x^T \right\|$ is constant then x is called a T-constant curve. For a T-constant curve x, either $\left\| x^T \right\| = 0$ or $\left\| x^T \right\| = \lambda$ for some non-zero smooth function λ (see,[4]). Further, a T-constant curve x is called first kind if $\left\| x^T \right\| = 0$, otherwise second kind.*

As a consequence of (7), we get the following results.

Theorem 2. [10] *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a curve with curvatures k_i $(i = 1, 2, 3)$ in Euclidean 4-space $\mathbb{E}^4$ .Then x lies on a sphere if and only if $ak_1 + bk_2 + ck_3 + 1 = 0$ where a, b, and c are non-zero constants.*

Corollary 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (1).Then x is a T-constant curve of first kind if and only if x lies on a sphere.*

Proof. Let x be a T-constant curve of first kind, then from (7) we get $m_1' = 0$, $m_2' = 0$ and $m_3' = 0$. We have $m_1 = a$, $m_2 = b$, $m_3 = c$ for $a, b, c \in IR$. Substituting this values into the first equation we get $ak_1 + bk_2 + ck_3 + 1 = 0$. From theorem 2 we get the result.

Theorem 3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. x is a T-constant curve of second kind if and only if*

$$k_1 \int k_1 ds + k_2 \int k_2 ds + k_3 \int k_3 ds = \frac{1}{m_0}.$$

Proof. Let x be a T-constant curve of second kind then, from (7) we get

$$k_1 m_1 + k_2 m_2 + k_3 m_3 + 1 = 0 \tag{10}$$

Furher, integrating the second,third and fourth equations in (7) and substituting these values into (10) we get the result.

Corollary 2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. If x is a T-constant curve of second kind, the curvature functions m_i of the curve x satisfy the equation*

$$2m_0 s + c = m_1^2 + m_2^2 + m_3^2 \tag{11}$$

where c is a integral constant.

Proof. Let x be a T-constant curve of second kind, from the equation in (7), we get

$$k_1 = -\frac{m_1'}{m_0}, k_2 = -\frac{m_2'}{m_0}, k_3 = -\frac{m_3'}{m_0}.$$

Substituting this values into first equation in (7), we obtain the differential equation

$$m_1 m_1' + m_2 m_2' + m_3 m_3' = m_0$$

which has the solution (11).

Theorem 4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a T-constant curve of second kind. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \pm\sqrt{2\lambda s + c}. \tag{12}$$

for some real constants c and $\lambda = m_0$.

Proof. Differentiating the squared distance function $\rho^2 = \langle x(s), x(s) \rangle$ and using (1), we get $\rho\rho' = m_0$. If x is a T-constant curve of second kind then by definition, the curvature function $m_0(s)$ of x is constant. It is easy to show that this differential equation has a nontrivial solution (12).

3.2 *N-constant Curves*

Definition 2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. If $\|x^N\|$ is constant then x is called a N-constant curve. For a N-constant curve x, either $\|x^N\| = 0$ or $\|x^N\| = \mu$ for some non-zero smooth function μ (see, [4]). Further, a N-constant curve x is called first kind if $\|x^N\| = 0$, otherwise second kind.*

So, for a N-constant curve x in $\mathbb{E}^4$

$$\|x^N(s)\|^2 = m_1^2(s) + m_2^2(s) + m_3^2(s), \tag{13}$$

becomes a constant function. Therefore, by differentiation

$$m_1 m_1' + m_2 m_2' + m_3 m_3' = 0. \tag{14}$$

For the N-constant curves of first kind we give the following result.

Proposition 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. If x is a N-constant curve of first kind if and only if $x(I)$ is an open part of a straight line.*

Proof. Suppose that x is a N-constant curve of first kind in $\mathbb{E}^4$, then from (13), we have $m_1 = m_2 = m_3 = 0$ which implies that $k_1 = k_2 = k_3 = 0$. Then the first Frenet curvature of the curve x is zero. So x is a part of a straight line.

Further, for the N-constant curves of second kind, we obtain the following results.

Theorem 5. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ and s be a arclength function . If x is a N-constant curve of second kind, then x is a T-constant curve with the parametrization*

$$x(s) = \lambda M_1(s) + \mu M_2(s) + \eta M_3(s), \tag{15}$$

where λ, μ and η are real constants, or the curve has the parametrization

$$\begin{aligned} x(s) =&(s+c)T(s) - \left(\int (s+c)k_1(s)ds \right) M_1(s) - \left(\int (s+c)k_2(s)ds \right) M_2(s) \\ &- \left(\int (s+c)k_3(s)ds \right) M_3(s) \end{aligned}$$

where c is real constant.

Proof. Let x be a N-constant curve of second kind in $\mathbb{E}^4$, then the equation (14) holds. So, we get $m_0(k_1m_1 + k_2m_2 + k_3m_3) = 0$. Hence, there are two possible cases; $m_0 = 0$ or $k_1m_1 + k_2m_2 + k_3m_3 = 0$. The first case with the equation (7) implies that $m_1 = \lambda = const, m_2 = \mu = const$, $m_3 = \eta = const$. So x is a T-constant curve of first kind with the parametrization (15). For the second case by the use of (7), we get

$$\begin{aligned} m_0 =& s+c \\ m_1 =& -\int (s+c)k_1(s)ds \\ m_2 =& -\int (s+c)k_2(s)ds \\ m_3 =& -\int (s+c)k_3(s)ds \end{aligned}$$

which completes the proof of the theorem.

Theorem 6. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a N-constant curve of second kind. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \mp\sqrt{s^2 + 2bs + d} \tag{16}$$

for some constant functions b, d.

Proof. Differentiating the squared distance function $\rho^2 = \langle x(s), x(s)\rangle$ and using (1), we get $\rho\rho' = m_0$. If x is a N-constant curve of second kind then from the previous theorem, $m_0(s) = s+b$. It is easy to show that this differential equation has a nontrivial solution (16).

3.3 Curves of constant-ratio

Definition 3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then the position vector x can be decompose into its tangential and normal components at each point:*

$$x = x^T + x^N.$$

if the ratio $\|x^T\| : \|x^N\|$ is constant on $x(I)$ then x is said to be of constant ratio, or equivalently $\|x^T\| : \|x\| = c =$constant [3].

For a unit speed regular curve x in $\mathbb{E}^n$, the gradient of the distance function $\rho = \|x(s)\|$ is given by

$$grad\rho = \frac{d\rho}{ds}x'(s) = \frac{< x(s), x'(s) >}{\|x(s)\|}T, \tag{17}$$

where T is the tangent vector field of x.

The following results characterize constant-ratio curves.

Theorem 7. [6] *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then x is of constant ratio with $\left\|x^T\right\| : \|x\| = c$ if and only if $\|grad\rho\| = c$ which is constant.*

In particular, for a curve of constant ratio we have $\|grad\rho\| = c \leq 1$.

As a consequence of (17) we obtain the following result.

Theorem 8. [6] *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then $\|grad\rho\| = c$ holds for a constant c if and only if one of the following three cases occurs:*

(i). $\|grad\rho\| = 0 \Longleftrightarrow x(I)$ *is contained in a hypersphere centered at the origin.*
(ii). $\|grad\rho\| = 1 \Longleftrightarrow x(I)$ *is an open portion of a line through the origin.*
(iii). $\|grad\rho\| = c \Longleftrightarrow \rho = \|x(s)\| = cs$, *for* $c \in (0,1)$.

The following result provides some simple characterization of constant ratio curve according to its Bishop curvatures in $\mathbb{E}^4$.

Proposition 2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. Then x is of constant-ratio if and only if*

$$\sum_{i=1}^{3}\left(k_i(s)\int(c^2s+bc)k_i(s)ds\right) = 1-c^2.$$

Proof. Let x be a curve of constant-ratio given with arclength function s. Then, from the previous result the distance function ρ of x satisfies the equality $\rho = \|x(s)\| = cs$ for some real constant c. Further, using (17) we get

$$\|grad\rho\| = \frac{< x(s), x'(s) >}{\|x(s)\|} = c.$$

Since, x is curve of $\mathbb{E}^4$, then it satisfies the equality (1). So, we get $m_0 = c^2s+bc$. Hence, substituting this value into (7) one can get,

$$\begin{aligned} c^2-1 =& k_1m_1+k_2m_2+k_3m_3 \\ m_1 =& -\int(c^2s+bc)k_1(s)ds \\ m_2 =& -\int(c^2s+bc)k_2(s)ds \\ m_3 =& -\int(c^2s+bc)k_3(s)ds \end{aligned}$$

Consequently, we obtain the desired result.

4 Conclusion

Constant-ratio, $T-$constant and $N-$constant curves are first defined by B.Y. Chen. In this paper, according to these definitions, we consider these types of curves with its parallel transport frame in Euclidean 4-space $\mathbb{E}^4$ and we give some results about constant-ratio, $T-$constant and $N-$constant curves.

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