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# Accelerating convergence for backward Euler and trapezoid time discretization schemes

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**Abstract:** In this study, we introduce two algorithms to numerically solve any initial value problem (IVP). These algorithms depend on time relaxation model (TRM) which is obtained adding a time relaxation term into IVP. Discretizing TRM by using backward Euler (BE) method gives the first algorithm. Similarly, the second algorithm is followed by using trapezoid (TR) time stepping scheme . Under some conditions, the first algorithm increases the order of convergence from one to two and the second one increases the order from two to three. Thus, more accurate results can be obtained. To verify the accuracy of the methods, they are applied to some numerical examples. Numerical results overlap with the theoretical results.

**Keywords:** Relaxation model, backward Euler method, accelerating convergence,trapezoid time discretization.

## 1 Introduction and preliminaries

It is well-known that the oldest and simplest numerical approximation method to numerically solve

$$y_t = f(t,y),\ t \geq t_0,\ y(t_0) = y_0. \tag{1}$$

is the Euler or tangent line method. This method is given by the equation

$$y^{n+1} = y^n + (t^{n+1} - t^n) f(t^n, y^n). \tag{2}$$

If the step size is the same for all intervals, then (2) can be given as

$$y^{n+1} = y^n + \Delta t f(t^n, y^n). \tag{3}$$

where $\Delta t = t^{n+1} - t^n$. Euler's method works with repeatedly evaluating (2) or (3), using the result of each step to compute the next step. In this way, it is found a sequence of values $\{y^i : i \in \mathbb{N}\}$ that approximate the values of the exact solution $y$ at the points $\{t^i : i \in \mathbb{N}\}$. A variation on the Euler formula, BE formula, can be obtained by writing backward difference quotient instead of forward difference quotient. That is,

$$y^{n+1} = y^n + \Delta t f(t^{n+1}, y^{n+1}). \tag{4}$$

BE method is of order one and A-stable. This method is an implicit method since the new approximation appears on both sides of the equation, and thus the method needs to solve an algebraic equation for $y^{n+1}$. Hence, for many problems, it needs more computation or more time to get accurate results. We will use the form for BE time discretization which is equivalent to (4)

$$\frac{y^{n+1} - y^n}{\Delta t} = f(t^{n+1}, y^{n+1}). \tag{5}$$

Another implicit method is the 'implicit trapezoidal rule',

$$\frac{y^{n+1} - y^n}{\Delta t} = \frac{f(t^{n+1}, y^{n+1}) + f(t^n, y^n)}{2} \tag{6}$$

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which is A-stable method and is of order two.

Time relaxation term is a regularization term used in computational fluid dynamics to regularize the flow as reported in Adams and Stolz [1], Layton and Neda [4], Neda [6], Ervin, Layton and Neda [2]. Also, Layton, Pruett and Rebholz [5] regularized temporally the flow by adding the term

$$\text{"} + \kappa(y - \bar{y})\text{"} \tag{7}$$

to the Navier-Stokes equations. Here, $\bar{y}$ is defined as follows: Select a local temporal filtering operator associated with a time scale $\delta$. For a function $y$, its time filtered analog $\bar{y}$ is the unique solution of

$$\bar{y}_t = \frac{y - \bar{y}}{\delta},\ t > 0,\ \bar{y}(x,0) = y(x,0) \tag{8}$$

which is the temporal differential filter of [7]. Note that if $\bar{y}$ is independent of $t$, then $y = \bar{y}$ and the relaxation term is equal zero. Isik [3] investigated spin up problem and accelerating convergence to steady state for Navier-Stokes time relaxation models by using the BE time discretization.

In this paper, we consider the time relaxation model (TRM), adding the time relaxation term (7) to (1),

$$\begin{cases} y_t + \kappa\bar{y}_t = f(t,y), \\ \bar{y}_t = \dfrac{y - \bar{y}}{\delta} \\ \bar{y}(0) = y(0) = y_0, \end{cases}$$

where $\delta > 0$. By using Backward-Euler time discretization for TRM, we first analyze the following algorithm.

**Algorithm 1**. *Given a time step $\Delta t > 0, \kappa, \delta$, the method computes $y^1, y^2, \ldots, \bar{y}^1, \bar{y}^2, \ldots$ where $y^i = y(t^i)$, $\bar{y}^i = \bar{y}(t^i)$ and $t^i = i\Delta t$. At every time-step $n \geq 0$ the following system is solved. Given $y^n, \bar{y}^n$ find $y^{n+1}, \bar{y}^{n+1}$ satisfying*

$$\begin{aligned} \frac{y^{n+1} - y^n}{\Delta t} + \kappa\left(\frac{\bar{y}^{n+1} - \bar{y}^n}{\Delta t}\right) &= f(t^{n+1}, y^{n+1}) \\ \frac{\bar{y}^{n+1} - \bar{y}^n}{\Delta t} &= \frac{y^{n+1} - \bar{y}^{n+1}}{\delta} \\ y(0) = \bar{y}(0) &= y_0. \end{aligned} \tag{9}$$

After analyzing the error

$$\frac{\Delta t}{2} y_{tt}(t^n) - \kappa\bar{y}_t(t^n) - \frac{\kappa\Delta t}{2}\bar{y}_{tt}(t^n) + O(\Delta t^2), \tag{10}$$

a better approximate formula is obtained for $\delta = \dfrac{\Delta t}{2}$ and $\kappa = \dfrac{\Delta t}{2}\dfrac{y_{tt}(t^n)}{y_t(t^n)}$, $y_t(t^n) \neq 0$. Algorithm (1) does gives an improvement over the BE formula because the local truncation error for Algorithm (1) is proportional to $\Delta t^2$, while for the Euler method it is proportional to $\Delta t$. Thus, Algorithm (1) is a second order method.

As a second algorithm, by using TR time stepping scheme for TRM, we consider the following algorithm.

**Algorithm 2**. *Given a time step $\Delta t > 0, \kappa, \delta$, the method computes $y^1$, $y^2$, $\ldots, \bar{y}^1$, $\bar{y}^2, \ldots$ where $y^i = y(t^i)$, $\bar{y}^i = \bar{y}(t^i)$ and $t^i = i\Delta t$. At every time-step $n \geq 0$ the following system is solved. Given $y^n, \bar{y}^n$ find $y^{n+1}, \bar{y}^{n+1}$ satisfying*

$$\frac{y^{n+1} - y^n}{\Delta t} + \kappa\left(\frac{\bar{y}^{n+1} - \bar{y}^n}{\Delta t}\right) = \frac{f(t^{n+1}, y^{n+1}) + f(t^n, y^n)}{2} \tag{11}$$

$$\frac{\bar{y}^{n+1} - \bar{y}^n}{\Delta t} = \frac{y^{n+1} + y^n - \bar{y}^{n+1} - \bar{y}^n}{2\delta}$$

$$y(0) = \bar{y}(0) = y_0.$$

By making a similar analysis to the first algorithm, the error is obtained as

$$\frac{\Delta t^2}{12}y_{ttt}(t^n)-\kappa\bar{y}_t(t^n)-\frac{\kappa\Delta t}{2}\bar{y}_{tt}(t^n)-\frac{\kappa\Delta t^2}{6}\bar{y}_{ttt}(t^n)+O(\Delta t^3). \tag{12}$$

Then, Algorithm 2 is of order three for $\delta=\frac{\Delta t}{2}$ and $\kappa=\frac{\Delta t^2 y_{ttt}(t^n)}{12y_t(t^n)+2\Delta t^2 y_{ttt}(t^n)}$, $12y_t(t^n)+2\Delta t^2 y_{ttt}(t^n)\neq 0$.

That is, the local truncation error is proportional to $\Delta t^3$.

We call Algorithm 1 and Algorithm 2 as BE-TR and TRA-TR methods respectively.

This paper is organized as follows: In Section 2, we analyze (10) to find the local truncation error for the BE-TR method. Similarly, the local truncation error of the TRA-TR method is examined in Section 3. Some examples are given in Section 4 to verify the accuracy of the methods. In the last section, the results are summarized.

## 2 Local truncation error for the BE-TR method

To find the local error for the BE-TR method, subtracting (1) from (1) for $t=t^{n+1}$ yields the truncation error

$$y_t(t^{n+1})-\frac{y^{n+1}-y^n}{\Delta t}-\kappa\frac{\bar{y}^{n+1}-\bar{y}^n}{\Delta t}. \tag{13}$$

By simplifying (13) with Taylor theorem, we obtain the truncation error as in (10). If the first three terms in (10) are cancelled each other, then the order of convergence will be upgrade to $O(\Delta t^2)$.

**Theorem 3.** *For $\delta=\frac{\Delta t}{2}$ and $\kappa=\frac{\Delta t}{2}\frac{y_{tt}(t^n)}{y_t(t^n)}$, the order of convergence of BE-TR method is $O(\Delta t^2)$ for all IVP where* $y_t(t^n)\neq 0$.

*Proof.* It is enough to show that the first three terms in (10) are cancelled each other. For $\delta=\frac{\Delta t}{2}$, by using the definition of $\bar{y}$, the following is obtained:

$$\bar{y}_t(t^n)+\frac{\Delta t}{2}\bar{y}_{tt}(t^n)=y_t(t^n).$$

Hence,

$$\frac{\Delta t}{2}y_{tt}(t^n)-\kappa\bar{y}_t(t^n)-\frac{\kappa\Delta t}{2}\bar{y}_{tt}(t^n)=\frac{\Delta t}{2}y_{tt}(t^n)-\kappa y_t(t^n)=\frac{\Delta t}{2}y_{tt}(t^n)-\frac{\Delta t}{2}\frac{y_{tt}(t^n)}{y_t(t^n)}y_t(t^n)=0$$

and the order is $O(\Delta t^2)$.

## 3 Local truncation error for the TRA-TR method

Subtracting (11) from (1) for $t=t^{n+1}$ gives the truncation error as

$$\frac{y_t(t^{n+1})}{2}+\frac{y_t(t^n)}{2}-\frac{y^{n+1}-y^n}{\Delta t}-\kappa\frac{\bar{y}^{n+1}-\bar{y}^n}{\Delta t}. \tag{14}$$

By using Taylor theorem, (14) is obtained as in (12). If the first four terms in (12) are vanished each other, then the order of convergence will be improved to $O(\Delta t^3)$.

**Theorem 4.** *For $\delta=\frac{\Delta t}{2}$ and $\kappa=\frac{\Delta t^2 y_{ttt}(t^n)}{12y_t(t^n)+2\Delta t^2 y_{ttt}(t^n)}$,*

*the order of convergence of TRA-TR method is $O(\Delta t^3)$ for all IVP where $12y_t(t^n)+2\Delta t^2 y_{ttt}(t^n)\neq 0$.*

*Proof.* For $\delta = \frac{\Delta t}{2}$, from $\bar{y}_t = \frac{y-\bar{y}}{\delta}$ we have followings;

$$y = \bar{y} + \frac{\Delta t}{2}\bar{y}_t$$
$$y_t = \bar{y}_t + \frac{\Delta t}{2}\bar{y}_{tt} \tag{15}$$
$$y_{tt} = \bar{y}_{tt} + \frac{\Delta t}{2}\bar{y}_{ttt}$$
$$y_{ttt} = \bar{y}_{ttt} + \frac{\Delta t}{2}\bar{y}_{tttt}. \tag{16}$$

Recall that the local truncation error is

$$\frac{\Delta t^2}{12}y_{ttt}(t^n) - \kappa\bar{y}_t(t^n) - \frac{\kappa\Delta t}{2}\bar{y}_{tt}(t^n) - \frac{\kappa\Delta t^2}{6}\bar{y}_{ttt}(t^n) + O(\Delta t^3)$$
$$= \frac{\Delta t^2}{12}y_{ttt}(t^n) - \kappa\left(\bar{y}_t(t^n) - \frac{\Delta t}{2}\bar{y}_{tt}(t^n)\right) - \frac{\kappa\Delta t^2}{6}\bar{y}_{ttt}(t^n) + O(\Delta t^3).$$

With the help of (15) and (16), the error is obtained as

$$\frac{\Delta t^2}{12}y_{ttt}(t^n) - \kappa y_t - \frac{\kappa\Delta t^2}{6}\left(y_{ttt} - \frac{\Delta t}{2}\bar{y}_{tttt}\right) + O(\Delta t^3)$$
$$= \frac{\Delta t^2}{12}y_{ttt}(t^n) - \kappa y_t - \frac{\kappa\Delta t^2}{6}y_{ttt} + \frac{\kappa\Delta t^3}{12}\bar{y}_{tttt} + O(\Delta t^3)$$
$$= \frac{\Delta t^2}{12}y_{ttt}(t^n) - \kappa\left(y_t - \frac{\Delta t^2}{6}y_{ttt}\right) + O(\Delta t^3). \tag{17}$$

If we put $\kappa = \frac{\Delta t^2 y_{ttt}(t^n)}{12y_t(t^n) + 2\Delta t^2 y_{ttt}(t^n)}$ into (17), then we have

$$\frac{\Delta t^2}{12}y_{ttt}(t^n) - \left(\frac{\Delta t^2 y_{ttt}(t^n)}{12y_t(t^n) + 2\Delta t^2 y_{ttt}(t^n)}\right)\left(y_t - \frac{\Delta t^2}{6}y_{ttt}\right) = 0$$

and the order is $O(\Delta t^3)$.

## 4 Examples

In this section, we give some numerical examples that including a nonlinear IVP. For the BE-TR method, $\bar{y}^{n+1}$ can be calculated in each time step with $O(\Delta t^2)$ error as follows:

$$\bar{y}^{n+1} = \frac{y^{n+1} + \frac{\delta\bar{y}^n}{\Delta t}}{\left(1 + \frac{\delta}{\Delta t}\right)}.$$

When we rearrange the terms in the BE-TR method for the values of $\delta$ and $\kappa$, we have

$$\frac{y^{n+1} - y^n}{\Delta t} + \kappa\left(\frac{y^{n+1} - \bar{y}^n}{3}\right) = f(t^{n+1}, y^{n+1})$$
$$\bar{y}^{n+1} = \frac{2y^{n+1} + \bar{y}^n}{3}$$
$$\kappa = \frac{\Delta t}{2}\frac{y_{tt}(t^n)}{y_t(t^n)}$$
$$y(0) = \bar{y}(0) = y_0$$

Similarly, for the TRA-TR method, $\bar{y}^{n+1}$ can be calculated in each time step with $O(\Delta t^3)$ error as follows:

$$\bar{y}^{n+1} = \frac{\Delta t\left(y^{n+1} + y^n - \bar{y}^n\right) - 2\delta\bar{y}^n}{(2\delta + \Delta t)}.$$

If we rearrange the terms in the TRA-TR method for the values of $\delta$ and $\kappa$, then we have

$$\frac{y^{n+1} - y^n}{\Delta t} + \kappa\left(\frac{\bar{y}^{n+1} - \bar{y}^n}{\Delta t}\right) = \frac{f(t^{n+1}, y^{n+1}) + f(t^n, y^n)}{2}$$
$$\bar{y}^{n+1} = \frac{\left(y^n + y^{n+1}\right)}{2}$$
$$\kappa = \frac{\Delta t^2 y_{ttt}(t^n)}{12 y_t(t^n) + 2\Delta t^2 y_{ttt}(t^n)}$$
$$y(0) = \bar{y}(0) = y_0.$$

All calculations are done in Maple 15.

*Example 1.* Let us consider the following IVP

$$y_t = y - t^2 + 2t, \ y(0) = 1 \tag{18}$$

which has the exact solution $f(t) = t^2 + e^t$. Let us apply the BE-TR method and the TRA-TR method to (18). Since the conditions of Theorem 3 and Theorem 4 are satisfied, we expect the order of converge as two and three for the BE-TR method and the TRA-TR method, respectively. The results are given in Table 1 and Table 2. As seen from the Table 1 and Table 2, the BE-TR method has order 2 and the TRA-TR method has order 3. Thus, the numerical results coincide with the theoretical results.

**Table 1:** Local errors for the BE and the BE-TR methods for Example 1

| $\Delta t$ | BE method | BE-TR method | Rates for BE-TR method |
|---|---|---|---|
| 0.1 | 0.3364 | $0.1926e-1$ | |
| 0.05 | 0.1607 | $0.5898e-2$ | 1.7072 |
| 0.025 | $0.7860e-1$ | $0.1641e-2$ | 1.8456 |
| 0.0125 | $0.3887e-1$ | $0.4339e-3$ | 1.9191 |
| 0.00625 | $0.1933e-1$ | $0.1116e-3$ | 1.9582 |
| 0.003125 | $0.9642e-2$ | $0.2833e-4$ | 1.9787 |

**Table 2:** Local errors for the TRA and the TRA-TR methods for Example 1

| $\Delta t$ | TRA method | TRA-TR method | Rates for TRA-TR method |
|---|---|---|---|
| 0.1 | $0.2269e-2$ | $0.2027e-3$ | |
| 0.05 | $0.5665e-3$ | $0.2679e-4$ | 2.9199 |
| 0.025 | $0.1415e-3$ | $0.3443e-5$ | 2.9601 |
| 0.0125 | $0.3539e-4$ | $0.4363e-6$ | 2.9801 |
| 0.00625 | $0.8848e-5$ | $0.5492e-7$ | 2.9900 |
| 0.003125 | $0.2212e-5$ | $0.6889e-8$ | 2.9950 |

*Example 2.* Let us apply the methods to the following IVP

$$y_t = y - y^2, \ y(0) = 0.5$$

which admits $f(t)=\dfrac{1}{1+e^{-t}}$ as the exact solution. The results for the BE-TR method and the TRA-TR method are given Table 3 and Table 4, respectively. The results reveal that the convergence orders of the BE-TR method and the TRA-TR method are 2 and 3, respectively. Therefore, the numerical results overlap with the theoretical results.

**Table 3:** Local errors for the BE and the BE-TR method for Example 2

| $\Delta t$ | BE method | BE-TR method | Rates for BE-TR method |
|---|---|---|---|
| 0.1 | $0.2375e-2$ | $0.2719e-3$ | |
| 0.05 | $0.1184e-2$ | $0.6565e-4$ | 2.0505 |
| 0.025 | $0.5913e-3$ | $0.1611e-4$ | 2.0260 |
| 0.0125 | $0.2954e-3$ | $0.3993e-5$ | 2.0132 |
| 0.00625 | $0.1476e-3$ | $0.9936e-6$ | 2.0066 |
| 0.003125 | $0.7381e-4$ | $0.2478e-6$ | 2.033 |

**Table 4:** Local errors for the TRA and the TRA-TR methods for Example 2

| $\Delta t$ | TRA method | TRA-TR method | Rates for TRA-TR method |
|---|---|---|---|
| 0.1 | $0.6330e-4$ | $0.9021e-6$ | |
| 0.05 | $0.1582e-4$ | $0.1072e-6$ | 3.0723 |
| 0.025 | $0.3956e-5$ | $0.1307e-7$ | 3.0364 |
| 0.0125 | $0.9890e-6$ | $0.1613e-8$ | 3.0182 |
| 0.00625 | $0.2472e-6$ | $0.2004e-9$ | 3.0091 |
| 0.003125 | $0.6181e-7$ | $0.2497e-10$ | 3.0045 |

## 5 Conclusions

The BE method sometimes needs more computations to satisfy given tolerance condition. Moreover, it has low order accuracy. In this paper, first we seek the order of convergence for the BE-TR method. Under the condition $y_t(t^n)\neq 0$, $y$ is the exact solution, the BE-TR method increases the accuracy one order for $\delta=\dfrac{\Delta t}{2}$ and

$$\kappa=\frac{\Delta t}{2}\frac{y_{tt}(t^n)}{y_t(t^n)}.$$

Next, we analyze the TRA-TR method and find similar results that the accuracy increases one order for $\delta=\dfrac{\Delta t}{2}$ and

$$\kappa=\frac{\Delta t^2 y_{ttt}(t^n)}{12y_t(t^n)+2\Delta t^2 y_{ttt}(t^n)}$$

under the condition $12y_t(t^n)+2\Delta t^2 y_{ttt}(t^n)\neq 0$. Hence the BE-TR method and the TRA-TR method are of order 2 and order 3, respectively. These results are verified on some numerical examples. As seen from the examples, the BE-TR method and the TRA-TR method give more accurate results than the BE and TRA methods, respectively. Moreover, theoretical and numerical results are consistent.

## References


[1] N.A. Adams and S. Stolz, Deconvolution methods for subgrid-scale approximation in LES, Modern Simulation Strategies for Turbulent Flow, R. T. Edwards, 2001.

[2] V.J. Ervin, W. Layton and M. Neda, Numerical analysis of a higher order time relaxation model of fluids, Int. J. Numer. Anal. Mod., 4 (2007) 648–670.

[3] O.R. Isik, Spin up problem and accelerating convergence to steady state, Appl. Math. Modell. (2013) 3242-3253.
[4] W. Layton and M. Neda, Truncation of scales by time relaxation, J. Math. Anal. Appl,325 ( 2007) 788–807.
[5] W. Layton, C. David Pruett and L. G. Rebholz, Temporally regularized direct numerical simulation, Appl. Math. Comp. 216 (2010) 3728–3738.
[6] M. Neda, Discontinuous Time Relaxation Method for the Time-Dependent Navier-Stokes Equations, Adv. Numer. Anal., Volume 2010, doi:10.1155/2010/419021.
[7] C.D. Pruett, T.B. Gatski, C.E. Grosch and W.D. Thacker, The temporally filtered Navier–Stokes equations: properties of the residual-stress, Phys. Fluids, 15 (2003) 2127–2140.