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Numerical and analytical study for integro-differential equations using spectral collocation method

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Received: 25 October 2015, Revised: 1 November 2015, Accepted: 16 November 2015
Published online: 12 December 2015.

Abstract: A numerical method for solving integro-differential equations is presented. This method is based on replacement of the unknown function by truncated series of well-known shifted Legendre expansion of functions. An approximate formula of the integer derivative is introduced. The introduced method converts the proposed equation by means of collocation points to system of algebraic equations with shifted Legendre coefficients. Thus, by solving this system of equations, the shifted Legendre coefficients are obtained. Special attention is given to study the convergence analysis and derive an upper bound of the error of the presented approximate formula. Numerical results are performed in order to illustrate the usefulness and show the efficiency and the accuracy of the present method.

Keywords: Shifted Legendre polynomials, Integro-differential equations.

1 Introduction

The integro-differential equation (IDE) is an equation that involves both integrals and derivatives of an unknown function. Mathematical modeling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro-differential equations, stochastic equations. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in many fields like physics, astronomy, potential theory, fluid dynamics, biological models and chemical kinetics. The integro-differential equations, are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution ([1], [2], [4], [5], [15]). Recently, several numerical methods to solve IDEs have been given, such as variational iteration method [7], homotopy perturbation method ([6], [14]), spline functions expansion ([9], [12]) and collocation method ([8], [10], [16], [17]). Several numerical methods to solve the fourth integro-differential equations have been given such as, Chebyshev cardinal functions [11], variational iteration method [13] and others.

Legendre polynomials occur in the solution of Laplace equation of the potential, $\nabla^2\Phi(x)=0$, in a charge-free region of space, using the method of separation of variables, where the boundary conditions have axial symmetry, the solution for the potential will be

$$\Phi(r,\theta)=\sum_{l=0}^{\infty}[A_l r^l+B_l r^{-(l+1)}]P_l(\cos\theta),$$

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A_l and B_l are to be determined according to the boundary conditions of each problem. They also appear when solving Schrödinger equation in three dimensions for a central force.

In this work, we will derive an approximate formula of the integral derivative $y^{(n)}(x)$ and derive an upper bound of the error of this formula, then we use this formula to solve integro-differential equations:

$$\sum_{r=0}^{N} y^{(r)}(x) = f(x) + \alpha \int_0^x K(x,t)F(y(t))dt, \quad 0 \leq x,t \leq 1, \tag{1}$$

under the initial conditions:

$$y^{(r)}(0) = \lambda_r, \quad r = 0,1,...,N-1, \tag{2}$$

where λ_r are suitable constants; $N = 0,1,2,...,$ $f(x)$ and kernel $k(x,t)$ are given functions.

2 An approximate formula of the integer derivative

The well-known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula [3]

$$L_{k+1}(z) = \frac{2k+1}{k+1} zL_k(z) - \frac{k}{k+1} L_{k-1}(z), \quad k = 1,2,...,$$

where $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $[0,1]$ we define the so called shifted Legendre polynomials by introducing the change of variable $z = 2t-1$. Let the shifted Legendre polynomials $L_k(2t-1)$ be denoted by $L_k^*(t)$. Then $L_k^*(t)$ can be obtained as follows

$$L_{k+1}^*(t) = \frac{(2k+1)(2t-1)}{(k+1)} L_k^*(t) - \frac{k}{k+1} L_{k-1}^*(t), \quad k = 1,2,...,$$

where $L_0^*(t) = 1$ and $L_1^*(t) = 2t-1$. The analytic form of the shifted Legendre polynomials $L_k^*(t)$ of degree k is given by

$$L_k^*(t) = \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!(i!)^2} t^i. \tag{3}$$

Note that $L_k^*(0) = (-1)^k$ and $L_k^*(1) = 1$. The orthogonality condition is

$$\int_0^1 L_i^*(t)L_j^*(t)\,dt = \begin{cases} \frac{1}{2i+1}, & \text{for} \quad i = j; \\ 0, & \text{for} \quad i \neq j. \end{cases}$$

The function $u(t)$, which is a square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$u(t) = \sum_{i=0}^{\infty} c_i L_i^*(t), \tag{4}$$

where the coefficients c_i are given by

$$c_i = (2i+1)\int_0^1 u(t)L_i^*(t)\,dt, \qquad i = 0,1,\ldots. \tag{5}$$

In practice, only the first $(m+1)$-terms of shifted Legendre polynomials are considered. Then we have

$$u_m(t) = \sum_{i=0}^{m} c_i L_i^*(t). \tag{6}$$

The main approximate formula of the integer derivative is given in the following theorem.

Theorem 1. Let $u(t)$ be approximated by shifted Legendre polynomials as (6) then the integer derivative of order n is given by

$$D^{(n)}(u_m(t)) = \sum_{i=n}^{m}\sum_{k=n}^{i} c_i\,\gamma_{i,k}^{(n)}\,t^{k-n}, \tag{7}$$

where $\gamma_{i,k}^{(n)}$ is given by

$$\gamma_{i,k}^{(n)} = \frac{(-1)^{(i+k)}(i+k)}{(i-k)!(k)!(k-n)!}. \tag{8}$$

Proof. Since the differentiation is a linear operation, then from (6) we have

$$D^{(n)}(u_m(t)) = \sum_{i=0}^{m} c_i D^{(n)}(L_i^*(t)). \tag{9}$$

From the formula (3) we have

$$D^{(n)}L_i^*(t) = 0, \qquad i = 0,1,\ldots,n-1. \tag{10}$$

Therefore, for $i = n, n+1, \ldots, m$ and formula (3) we get

$$D^{(n)}L_i^*(t) = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2} D^{(n)}(t^k) = \sum_{k=n}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2} t^{k-n}. \tag{11}$$

A combination of Eqs.(9), (10) and (11) leads to the desired result (7).

Test Example. Consider the function $u(t) = t^2$ and $m = 2$, $n = 1$, the shifted Legendre series of t^2 is

$$t^2 = \frac{1}{3}L_0^*(t) + \frac{1}{2}L_1^*(t) + \frac{1}{6}L_2^*(t).$$

Hence,

$$D^{(1)}t^2 = \sum_{i=1}^{2}\sum_{k=1}^{i} c_i\,\gamma_{i,k}^{(1)}\,t^{k-1}, \text{ where,} \quad \gamma_{1,1}^{(1)} = 2,\ \gamma_{2,1}^{(1)} = -6,\ \gamma_{2,2}^{(1)} = 12,$$

therefore

$$D^{(1)}t^2 = c_1\,\gamma_{1,1}^{(1)} + c_2\,\gamma_{2,1}^{(1)} + c_2\,\gamma_{2,2}^{(1)}t = 2t,$$

which agrees with the exact derivative.

3 Error analysis

In this section, special attention is given to study the convergence analysis and evaluate an upper bound of the error for the proposed approximate formula.

Theorem 2. (Legendre truncation theorem) [3] The error in approximating $u(t)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If

$$u_m(t) = \sum_{k=0}^{m} c_k L_k(t), \tag{12}$$

then

$$E_T(m) \equiv |u(t) - u_m(t)| \leq \sum_{k=m+1}^{\infty} |c_k|, \tag{13}$$

for all $u(t)$, all m, and all $t \in [-1,1]$.

Theorem 3. The integer derivative of order n for the shifted Legendre polynomials can be expressed in terms of the shifted Legendre polynomials themselves in the following form

$$D^{(n)}(L_i^*(t)) = \sum_{k=n}^{i} \sum_{j=0}^{k-n} \Theta_{i,j,k} L_j^*(t), \tag{14}$$

where

$$\Theta_{i,j,k} = \frac{(-1)^{i+k}(i+k)!(2j+1)}{(i-k)!(k)!\Gamma(k-n+1)} \times \sum_{r=0}^{j} \frac{(-1)^{j+r}(j+r)!}{(j-r)!(r!)^2(k-n+r+1)}, \quad j = 0,1,\ldots. \tag{15}$$

Proof. Using the properties of the shifted Legendre polynomials [3], then t^{k-n} in (11) can be expanded in the following form

$$t^{k-n} = \sum_{j=0}^{k-n} c_{kj} L_j^*(t), \tag{16}$$

where c_{kj} can be obtained using (5) such that $u(t) = t^{k-n}$, then we can claim the following

$$c_{kj} = (2j+1) \int_0^1 t^{k-n} L_j^*(t) dt, \quad j = 0,1,\ldots.$$

$$\text{But at } j=0 \text{ we have,} \quad c_{k0} = \int_0^1 t^{k-n} dt = \frac{1}{k-n+1},$$

also, for any j, and using the formula (3), we can claim

$$c_{kj} = (2j+1) \sum_{r=0}^{j} (-1)^{j+r} \frac{(j+r)!}{(j-r)!(r!)^2(k-n+r+1)}, \quad j = 1,2,\ldots,$$

employing Eqs.(11) and (16) gives

$$D^{(n)}(L_i^*(t)) = \sum_{k=n}^{i} \sum_{j=0}^{k-n} \Theta_{i,j,k} L_j^*(t), \quad i = n, n+1, \ldots,$$

where $\Theta_{i,j,k}$ is defined in (15) and this completes the proof of the theorem.

Theorem 4. The error $|E_T(m)| = |D^{(n)}u(t) - D^{(n)}u_m(t)|$ in approximating $D^{(n)}u(t)$ by $D^{(n)}u_m(t)$ is bounded by

$$|E_T(m)| \leq \Big| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=n}^{i} \sum_{j=0}^{k-n} \Theta_{i,j,k} \Big) \Big|. \tag{17}$$

Proof. A combination of Eqs.(4), (6) and (14) leads to

$$|E_T(m)| = \Big| D^{(n)}u(t) - D^{(n)}u_m(t) \Big| = \Big| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=n}^{i} \sum_{j=0}^{k-n} \Theta_{i,j,k} L_j^*(t) \Big) \Big|,$$

but $|L_j^*(t)| \leq 1$, so, we can obtain

$$|E_T(m)| \leq \Big| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=n}^{i} \sum_{j=0}^{k-n} \Theta_{i,j,k} \Big) \Big|,$$

and subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds completes the proof of the theorem.

4 Procedure solution for the integro-differential equation

In this section, we present the proposed method to solve numerically the integro-differential equation of the form (1). The unknown function $y(x)$ may be expanded by finite series of shifted Legendre polynomials as the following approximation:

$$y_m(x) = \sum_{n=0}^{m} c_n L_n^*(x), \tag{18}$$

From Eqs.(1), (18) and Theorem 1 we have

$$\sum_{r=0}^{N} \sum_{i=r}^{m} \sum_{k=r}^{i} c_i \gamma_{i,k}^{(r)} x^{k-r} = f(x) + \alpha \int_0^x K(x,t) F\Big(\sum_{n=0}^{m} c_n L_n^*(t)\Big) dt, \quad 0 \leq x, t \leq 1. \tag{19}$$

We now collocate Eq.(19) at $(m-1+N)$ points x_s, $s = 0, 1, \ldots, m-N$ as:

$$\sum_{r=0}^{N} \sum_{i=r}^{m} \sum_{k=r}^{i} c_i \gamma_{i,k}^{(r)} x_s^{k-r} = f(x_s) + \alpha \int_0^{x_s} K(x_s,t) F\Big(\sum_{n=0}^{m} c_n L_n^*(t)\Big) dt. \tag{20}$$

For suitable collocation points, we use roots of shifted Legendre polynomial $L_{m+1-N}^*(x)$. The integral terms in Eq.(20) can be found using composite trapezoidal integration technique as:

$$\int_0^{x_s} K(x_s,t) F\Big(\sum_{n=0}^{m} c_n L_n^*(t)\Big) dt \cong \frac{h_s}{2} \Big(\Omega(t_0) + \Omega(t_l) + 2 \sum_{k=1}^{l-1} \Omega(t_k) \Big), \tag{21}$$

where $\Omega(t) = K(x_s,t)F(\sum_{n=0}^{m} c_n L_n^*(t))dt$, $h_s = \frac{x_s}{l}$, for an arbitrary integer l, $t_{j+1} = t_j + h_s,\ s = 0,1,...,m-N,\ j = 0,1,...,l$. So, by using Eqs.(21) and (20) we obtain

$$\sum_{r=0}^{N}\sum_{i=r}^{m}\sum_{k=r}^{i} c_i\,\gamma_{i,k}^{(r)}\,x_s^{k-r} = f(x_s) + \alpha\frac{h_s}{2}(\Omega(t_0)+\Omega(t_l)+2\sum_{k=1}^{l-1}\Omega(t_k)). \tag{22}$$

Also, by substituting Eqs.(18) in the initial conditions (2) we can obtain $N-1$ equations as follows:

$$\sum_{r=1}^{N-1}\sum_{i=r}^{m} c_i L_i^{*(r)}(0) = \lambda_r. \tag{23}$$

Eq.(22), together with $N-1$ equations of the initial conditions (23), give system of $(m+1)$ algebraic equations which can be solved, for the unknowns $c_n, n = 0,1,...,m$, using conjugate gradient method or Newton iteration method.

5 Numerical results

In this section, to achive the validity, the accuracy and support our theoretical discussion of the proposed method, we give some computational results of numerical examples.

Example 1. Consider the linear integro-differential equation as in Eq.(1) and (2) with $N = 1$, $f(x) = 2x + \frac{3x^2}{2} - \frac{7x^4}{12}$, $\alpha = 1$, $K(x,t) = x+t$, $F(y(t)) = y(t)$, then the integro-differential equation will be

$$y'(x) = 2x + \frac{3x^2}{2} - \frac{7x^4}{12} + \int_0^x (x+t)y(t)dt, \qquad 0 \le x \le 1, \tag{24}$$

subject to the initial condition

$$y(0) = -1. \tag{25}$$

The exact solution of this problem is $y(x) = x^2 - 1$.

We apply the suggested method with $m = 3$, and approximate the solution $y(x)$ as follows

$$y_3(x) = \sum_{n=0}^{3} c_n L_n^*(x). \tag{26}$$

From Eqs.(24), (26) and Theorem 1 we have

$$\sum_{i=1}^{3}\sum_{k=1}^{i} c_i\,\gamma_{i,k}^{(1)}\,x^{k-1} = 2x + \frac{3x^2}{2} - \frac{7x^4}{12} + \int_0^x (x+t)\sum_{n=0}^{3} c_n L_n^*(t)dt. \tag{27}$$

We now collocate Eq.(27) at points $x_s,\ \ s = 0,1$ as:

$$\sum_{i=1}^{3}\sum_{k=1}^{i} c_i\,\gamma_{i,k}^{(1)}\,x_s^{k-1} = 2x_s + \frac{3x_s^2}{2} - \frac{7x_s^4}{12} + \int_0^{x_s} (x_s+t)\sum_{n=0}^{3} c_n L_n^*(t)dt. \tag{28}$$

For suitable collocation points we use roots of shifted Legendre polynomial $L_2^*(x)$. The integral terms in Eq.(28) can be found using composite trapezoidal integration technique as:

$$\int_0^{x_s}(x_s+t)\sum_{n=0}^{3}c_nL_n^*(t)dt=\frac{h_s}{2}(\Omega(t_0)+\Omega(t_l)+2\sum_{k=1}^{l-1}\Omega(t_k)), \tag{29}$$

where $\Omega(t)=(x_s+t)\sum_{n=0}^{3}c_nL_n^*(t)$, $h_s=\frac{x_s}{l}$, for an arbitrary integer l, $t_{j+1}=t_j+h_s,\ s=0,1,\ j=0,1,...,l$. So by using Eqs.(29) and (28) we obtain

$$\sum_{i=1}^{3}\sum_{k=1}^{i}c_i\,\gamma_{i,k}^{(1)}\,x_s^{k-1}=2x_s+\frac{3x_s^2}{2}-\frac{7x_s^4}{12}+\frac{h_s}{2}(\Omega(t_0)+\Omega(t_\ell)+2\sum_{k=1}^{\ell-1}\Omega(t_k)). \tag{30}$$

Also, by substituting Eqs.(26) in the initial condition (25) we can obtain fourth equation as follows:

$$c_0-c_1+c_2-c_3=-1. \tag{31}$$

Eq.(30), together with the equation of the initial condition (31), represent a linear system of four algebraic equations in the coefficients c_n, by solving it using the conjugate iteration method, we obtain:

$$c_0=-0.6667,\quad c_1=0.500,\quad c_2=0.1667,\quad c_3=5.4433\times10^{-18}.$$

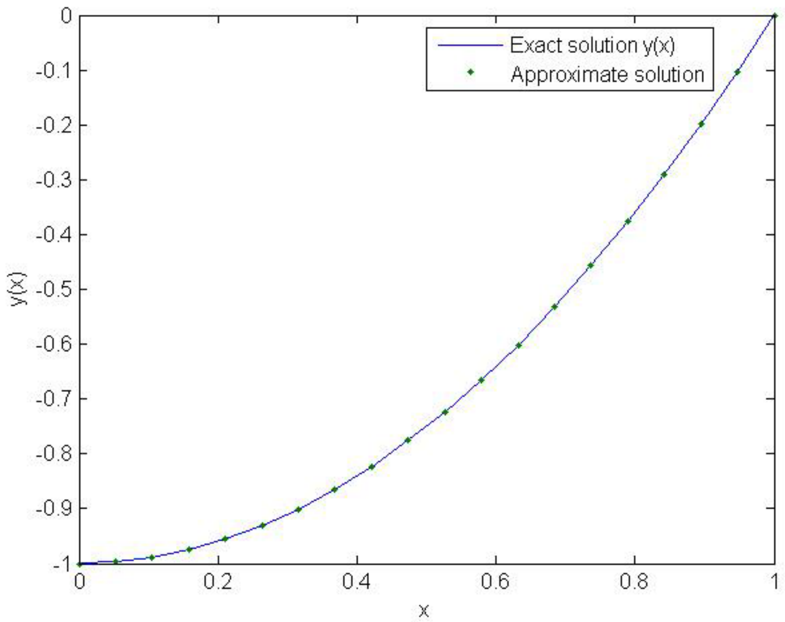

Fig. 1: The behavior of the exact solution and the approximate solution at $m=3$.

The behavior of the approximate solution using the proposed method with $m=3$ and the exact solution are presented in Figure 1. From Figure 1, it is clear that the proposed method can be considered as an efficient method to solve the linear integro-differential equations.

Example 2. Consider the non-linear integro-differential equation as in Eq.(1) and (2) with $N=2$, $f(x)=2+x^2-\frac{11x^6}{30}$, $\alpha=1$, $K(x,t)=x+t$, $F(y(t))=(y(t))^2$, then the integro-differential equation will be

$$y''(x)+y(x)=2+x^2-\frac{11x^6}{30}+\int_0^x (x+t)(y(t))^2dt, \qquad 0\leq x\leq 1, \tag{32}$$

subject to the initial conditions

$$y(0)=0, \quad y'(0)=0. \tag{33}$$

The exact solution of this problem is $y(x)=x^2$.

We apply the suggested method with $m=4$, and approximate the solution $y(x)$ as follows

$$y_4(x)=\sum_{n=0}^{4} c_n L_n^*(x). \tag{34}$$

From Eqs.(32), (34) and Theorem 1 we have

$$\sum_{i=2}^{4}\sum_{k=2}^{i} c_i\gamma_{i,k}^{(2)} x^{k-2}+\sum_{n=0}^{4} c_n L_n^*(x)=2+x^2-\frac{11x^6}{30}+\int_0^x (x+t)(\sum_{n=0}^{4} c_n L_n^*(t))^2dt. \tag{35}$$

We now collocate Eq.(35) at points $x_s,\ s=0,1,2$ as:

$$\sum_{i=2}^{4}\sum_{k=2}^{i} c_i\gamma_{i,k}^{(2)} x_s^{k-2}+\sum_{n=0}^{4} c_n L_n^*(x_s)=2+x_s^2-\frac{11x_s^6}{30}+\int_0^{x_s} (x_s+t)(\sum_{n=0}^{4} c_n L_n^*(t))^2dt. \tag{36}$$

For suitable collocation points we use roots of shifted Legendre polynomial $L_3^*(x)$. The integral terms in Eq.(36) can be found using composite trapezoidal integration technique as:

$$\int_0^{x_s} (x_s+t)\left(\sum_{n=0}^{4} c_n L_n^*(t)\right)^2 dt=\frac{h_s}{2}(\Omega(t_0)+\Omega(t_l)+2\sum_{k=1}^{l-1}\Omega(t_k)), \tag{37}$$

where $\Omega(t)=(x_s+t)\left(\sum_{n=0}^{4} c_n L_n^*(t)\right)^2$, $h_s=\frac{x_s}{l}$, for an arbitrary integer l, $t_{j+1}=t_j+h_s,\ s=0,1,2,\ j=0,1,...,l$. So by using Eqs.(37) and (36) we obtain

$$\begin{aligned}&\sum_{i=2}^{4}\sum_{k=2}^{i} c_i\gamma_{i,k}^{(2)} x_s^{k-2}+\sum_{n=0}^{4} c_n L_n^*(x_s)\\&=2x_s+\frac{3x_s^2}{2}-\frac{7x_s^4}{12}+\frac{h_s}{2}(\Omega(t_0)+\Omega(t_l)+2\sum_{k=1}^{l-1}\Omega(t_k)).\end{aligned} \tag{38}$$

Also, by substituting Eq.(34) in the initial conditions (33) we can obtain two equations as follows:

$$\begin{aligned}&c_0-c_1+c_2-c_3+c_4=0,\\&c_0L_0^{*'}(0)+c_1L_1^{*'}(0)+c_2L_2^{*'}(0)+c_3L_3^{*'}(0)+c_4L_4^{*'}(0)=0.\end{aligned} \tag{39}$$

Eq.(38), together with two equations of the initial conditions (39), represent a non-linear system of five algebraic equations in the coefficients c_n, by solving it using the Newton iteration method, we obtain:

$$c_0 = 0.375,\;\; c_1 = 0.500,\;\; c_2 = 0.125,\;\; c_3 = -2.533 \times 10^{-17},\;\; c_4 = -3.86428 * 10^{-18}.$$

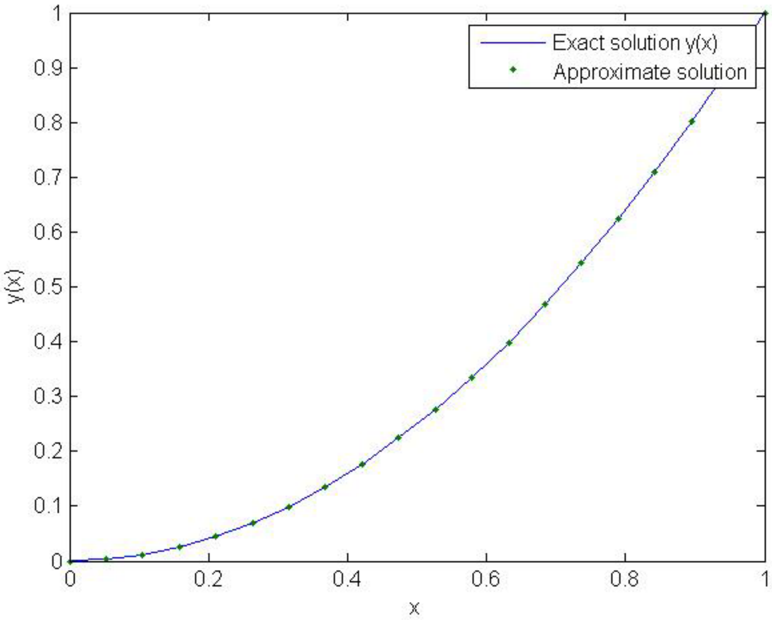

Fig. 2: The behavior of the exact solution and the approximate solution at $m = 4$.

The behavior of the approximate solution using the proposed method with $m = 4$ and the exact solution are presented in Figure 2. From Figure 2, it is clear that the proposed method can be considered as an efficient method to solve the non-linear integro-differential equations.

6 Conclusion and discussion

Integro-differential equations are usually difficult to solve analytically. so, it is required to obtain the approximate solution. In this paper, we proposed the spectral collocation method using shifted Legendre polynomials for solving the integro-differential equations. The proposed method is useful both for acquiring the general solution and particular solution as demonstrated in examples. Special attention is given to study the converges analysis and derive an upper bound of the error of the derived approximate formula. From our obtained results, we can conclude that the proposed method gives the solutions in excellent agreement with the exact solution and better than the other methods. An interesting feature of this method is that when an integral system has linearly independent polynomial solution of degree m or less than m, the method can be used for finding the analytical solution. All computational are done using Matlab 8.

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