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New results involving airy polynomials, fractional calculus and solution to generalized heat equation

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Abstract: The main object of this paper is to demonstrate how we can make significant progress in treating a variety of problems in the theory of partial fractional differential equations by combining theory of special functions and operational methods. In this article, it is shown that the combined use of integral transforms and special functions provides a powerful tool to solve certain type of fractional PDEs and generalized heat equation. Constructive examples are also provided.

Keywords: Fractional partial differential equations, Heat equations, Kd.V equations, Airy polynomials, Riemann-Liouville fractional derivative.

1 Introduction and preliminaries

We present a general method of operational nature to obtain solutions for several types of partial differential equations.

Definition 1. Laplace transform of function $f(t)$ is as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt := F(s).$$

If $L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\}$ is given by [3],[5],[7].

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s)\,ds$$

where$F(s)$ is analytic in the region $\mathrm{Re}(s) > c$.

Definition 2. If $f(t) \in C([a,b])$ and $a < t < b$ then [10].

$$I_{a^+}{}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(\xi)}{(t-\xi)^{1-\alpha}}\,d\xi$$

when $\alpha \in R^+$, is called Riemann-Liouville fractional integral of order α. In the same fashion for $\alpha \in]0,1[$,we let

$$D_{a^+}{}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{f(\xi)}{(t-\xi)^{\alpha}}\,d\xi.$$

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which is called Riemann – Liouville fractional derivative of order α. It follows then $D_{a^+}{}^{\alpha} f(t)$ exists for all $f(t) \in C^1([a,b])$,and $t \in]a,b]$.

A very useful fact about the R-L operators is that they satisfy semi group properties of fractional integrals. The special case of fractional derivative when $\alpha = 0.5$ is called semi-derivative.

Lemma 1. Let us assume that $L\{\, f(\,t\,)\,\} = F(\,s\,)L\{\, g(\,t\,)\,\} = G(\,s\,)$ then one has

$$\int_0^t f(t-x)g(x)dx = L^{-1}\{F(s)G(s)\}.$$

Proof. See [1].

Lemma 2. Let us assume that$L\{\, f(\,t\,)\,\} = F(\,s\,)$,the following relation holds true

(1). $F(s^{\beta}) = L\{\int_{\circ}^{+\infty} f(\tau)\left(\int\limits_0^{+\infty} e^{-tr-(\tau\cos\pi\beta)r^{\beta}}\sin(\tau r^{\beta}\sin\pi\beta)dr\right)d\tau\}.$
(2). $F(\sqrt{s}) = L\{\int_{\circ}^{+\infty} \frac{\tau}{2t\sqrt{\pi t}}\exp(-\frac{\tau^2}{4t})f(\tau)d\tau\}.$
(3). $e^{-\lambda\sqrt{s}} = \frac{\lambda}{2\sqrt{\pi}}\int_{\circ}^{+\infty}\exp(-\frac{\lambda^2+4st^2}{4t})\,dt\}.$

Proof. See [2].

1.1 Preliminary considerations on exponential operators

In this section, we establish the rules relevant to the action of an exponential operators on a given function.

Lemma 3. The following exponential operational rules hold true

(1). $\exp\left(\pm\,\kappa\frac{d}{dt}\right)\, f(t) = f(t\pm\kappa),$
(2). $\exp\left(\pm\,\kappa t\frac{d}{dt}\right)\, f(t) = f(e^{\pm\,\kappa}t),$
(3). $\exp\left(\kappa t^2\frac{d}{dt}\right)\, f(t) = f(\frac{t}{1-\kappa t}), \quad |t| < \frac{1}{\kappa},$
(4). $\exp\left(\kappa t^n\frac{d}{dt}\right)\, f(t) = f(\frac{t}{\sqrt[n-1]{1-\kappa(n-1)t^{n-1}}}), \quad |t| < \sqrt[n-1]{\frac{1}{\kappa(n-1)}},$
(5). $\exp\left(\kappa q(t)\frac{d}{dt}\right)\, f(t) = f\{Q^{-1}(\kappa+Q(t))\},$

where $Q(t)$is primitive of $q(t)$,$Q^{-1}(t)$ is inverse of$Q(t)$.

Proof. See [8], [9]ş

Example 1. The following relations hold true

(1). $\exp\left(\kappa\sqrt{a^2-b^2t^2}\frac{d}{dt}\right)\, f(t) = f\left(\frac{b}{a}sin(b\kappa + Arcsin\frac{bt}{a})\right),$
(2). $\exp\left(\kappa\sqrt{2at-t^2}\frac{d}{dt}\right)\, f(t) = f\left(a(1+sin(\kappa + Arcsin\frac{t-a}{a}))\right).$

Let us introduce a change of variable $t = \frac{a}{b}\sin\xi$, then we have the following

$$\frac{d}{dt} = \frac{b}{\sqrt{a^2-b^2t^2}}\frac{d}{d\xi},$$

or

$$\frac{d}{dt} = \kappa b \frac{d}{d\xi}.$$

Substitution of these values in the above relation, leads to

$$\exp\left(\kappa\sqrt{a^2 - b^2t^2}\,\tfrac{d}{dt}\right)\, f(t) = exp\left((\kappa b)\tfrac{d}{d\xi}\right)\, f(\tfrac{a}{b}\sin\xi) = f(\tfrac{a}{b}\sin(b\kappa+\xi)) = f(\tfrac{a}{b}\sin(b\kappa + Arc\sin\tfrac{b}{a}t).$$

Note. In special case $a = b = 1$, one gets

$$\exp\left(\kappa\sqrt{1-t^2}\frac{d}{dt}\right)\, f(t) = f\left(sin(\kappa + Arcsint)\right).$$

Left hand side may be rewritten as follows

$$\exp\left(\kappa\sqrt{2at - t^2}\frac{d}{dt}\right)\, f(t) = \exp\left(\kappa\sqrt{a^2 - (t-a)^2}\frac{d}{dt}\right)\, f(t).$$

Let us introduce a change of variable $t - a = a\sin\xi$, then we have the following

$$\frac{d}{dt} = \frac{1}{\sqrt{2at - t^2}}\frac{d}{d\xi},$$

or

$$\frac{d}{dt} = \frac{1}{a\cos\xi}\frac{d}{d\xi}.$$

Substitution of these values in the above relation, leads to

$$\exp\left(\kappa\sqrt{a^2 - (t-a)^2}\,\tfrac{d}{dt}\right)\, f(t) = \exp\left(\kappa\tfrac{d}{d\xi}\right) f(a + a\, sin\xi) = f(a + a\, sin(\kappa+\xi)) = f\left(a(1 +\, sin(\kappa + Arc\sin\tfrac{t-a}{a}))\right).$$

Example 2. Let us consider the following space- fractional PDE of orderα,with fractionalderivative in the Riemann-Liouville sense

$$\lambda D_x{}^{\alpha}u + t^{-2}u_t + 3\,\kappa u = 0, \quad \lambda,\kappa \in R, \tag{1}$$

$$u(x,0) = f(x), \tag{2}$$

then the above boundary value problem has the following formal solution

$$u(x,t) = \frac{(\lambda t)^3 e^{-\kappa t^3}}{2\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-\frac{\lambda^2 t^6}{36\xi}}}{\xi\sqrt{\xi}}\, f(x-\xi)d\xi.$$

The PDE may be rewritten in the following form

$$u_t = -t^2(\lambda D_x{}^{\alpha} - 3\,\kappa)u.$$

In order to obtain a solution for equation (1), first by solving the first order PDE with respect to t, treating the derivative operator as a generic constant and applying the initial condition (2),we get the following

$$u(x,t) = \exp\left(\kappa t^3 - \lambda t^2 \frac{\partial^\alpha}{\partial x^\alpha}\right) f(x) = exp\left(\kappa t^3\right).\exp\left(-\lambda t^2 \frac{\partial^\alpha}{\partial x^\alpha}\right) f(x). \tag{3}$$

At this point, we consider two cases

(1). $\alpha = 1$, in this case, by using part 1 of lemma 3, leads to the following formal solution

$$u(x,t) = exp\left(\kappa t^3\right).\exp\left(-\lambda t^2 \frac{\partial}{\partial x}\right) f(x) = exp\left(\kappa t^3\right) f(x - \lambda t^2).$$

(2). $\alpha = 0.5$ (not integer), in this case one gets the following

$$u(x,t) = exp\left(\kappa t^3\right).\exp\left(-\lambda t^2 \frac{\partial^{0.5}}{\partial x^{0.5}}\right) f(x)$$

By replacing s with $\frac{\partial}{\partial x}$, in part 3 of lemma (3) and by proceeding as before, one gets the following formal solution

$$u(x,t) = exp\left(\kappa t^3\right).\exp\left(-\lambda t^2 \frac{\partial^{0.5}}{\partial x^{0.5}}\right) f(x) = \frac{\lambda t^2 exp\left(\kappa t^3\right)}{2\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-\frac{\lambda^2 t^4}{4\xi}}}{\xi\sqrt{\xi}} f(x-\xi).$$

Lemma 4. The following second order exponential operator relation holds true.

$$e^{\lambda \frac{\partial^2}{\partial x^2}} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-\frac{\xi^2}{2\lambda}} d\xi \, (e^{\xi \frac{\partial}{\partial x}} + e^{-\xi \frac{\partial}{\partial x}}), \lambda > 0 \tag{4}$$

Proof. Let us consider the following elementary integral

$$I = \int_0^{+\infty} e^{-\frac{\xi^2}{4\lambda}} \cosh \beta \, \xi d\xi$$

By integration by parts one can easily find the value of the integral and after some algebra we obtain

$$I = \int_0^{+\infty} e^{-\frac{\xi^2}{4\lambda}} \cosh \beta \, \xi d\xi = \frac{\sqrt{\pi\lambda} e^{\lambda\beta^2}}{2},$$

or

$$e^{\lambda\beta^2} = \frac{1}{\sqrt{\pi\lambda}} \int_0^{+\infty} e^{-\frac{\xi^2}{4\lambda}} d\xi (e^{\xi\beta} + e^{-\xi\beta}).$$

Letting $\beta = \frac{\partial}{\partial x}$in the above relation, we get the desired identity

$$e^{\lambda \frac{\partial^2}{\partial x^2}} = \frac{1}{\sqrt{\pi\lambda}} \int_0^{+\infty} e^{-\frac{\xi^2}{4\lambda}} d\xi \, (e^{\xi \frac{\partial}{\partial x}} + e^{-\xi \frac{\partial}{\partial x}}).$$

As an application of the above operational identity, let us solve the heat equation.

Example 3. Solving the following heat equation

$$u_t = \kappa u_{xx},$$
$$u(x,0) = \exp(-\beta x).$$

Direct use of operational identity (4) followed by lemma 4, leads us to the formal solution

$$\begin{aligned} u(x,t) = e^{\kappa t \frac{\partial^2}{\partial x^2}} e^{-\beta x} &= \frac{1}{\sqrt{\pi \kappa t}} \int_0^{+\infty} e^{-\frac{\xi^2}{4\kappa t}} d\xi \, (e^{\xi \frac{\partial}{\partial x}} + e^{-\xi \frac{\partial}{\partial x}}) e^{-\beta x} \\ &= \frac{1}{\sqrt{\pi \kappa t}} \int_0^{+\infty} e^{-\frac{\xi^2}{4\kappa t}} \left(e^{-\beta(x+\xi)} + e^{-\beta(x-\xi)}\right) d\xi \; = \frac{2e^{-\beta x}}{\sqrt{\pi \kappa t}} \int_0^{+\infty} e^{-\frac{\xi^2}{4\kappa t}} \cosh \beta \xi \, d\xi \\ &= \frac{2e^{-\beta x}}{\sqrt{\pi \beta \kappa t}} \left(\frac{\sqrt{\pi \kappa t}}{2} e^{\kappa t}\right) = \frac{1}{\sqrt{\beta}} e^{-\beta x + \beta^2 \kappa t} \end{aligned}$$

Lemma 5. The following exponential operator relations hold true.

(1).

$$\frac{4}{\sqrt{\pi}} \int_0^{+\infty} \xi^{\mu-1} d\xi \, e^{-\xi(\beta + \delta \frac{\partial}{\partial x})} = (\beta + \delta \frac{\partial}{\partial x})^{-\mu}, \tag{5}$$

(2).

$$(\beta + \delta \frac{\partial}{\partial x})^{-\mu} f(x) = \frac{4}{\Gamma(\mu)} \int_0^{+\infty} \xi^{\mu-1} e^{-\xi \beta} f(x - \delta \xi) d\xi. \tag{6}$$

Proof. Let us consider the following elementary integral

$$\int_0^{+\infty} e^{-(\beta+\gamma)\xi} \xi^{\mu-1} d\xi = \frac{\Gamma(\mu)}{4(\beta+\gamma)^{\mu}}$$

or

$$(\beta+\gamma)^{-\mu} = \frac{4}{\Gamma(\mu)} \int_0^{+\infty} \xi^{\mu-1} d\xi e^{-\xi(\beta+\gamma)}$$

In the above relation, we set $\gamma = \delta \frac{\partial}{\partial x}$, to obtain

$$(\beta + \delta \frac{\partial}{\partial x})^{-\mu} = \frac{4}{\Gamma(\mu)} \int_0^{+\infty} \xi^{\mu-1} d\xi e^{-\xi(\beta + \delta \frac{\partial}{\partial x})}.$$

Now let us find the action of this operator on certain function $f(x)$,

$$(\beta + \delta \frac{\partial}{\partial x})^{-\mu} f(x) = \frac{4}{\Gamma(\mu)} \int_0^{+\infty} \xi^{\mu-1} d\xi e^{-\xi(\beta + \delta \frac{\partial}{\partial x})} f(x) = \frac{4}{\Gamma(\mu)} \int_0^{+\infty} \xi^{\mu-1} e^{-\xi \beta} f(x - \delta \xi) d\xi.$$

Theorem 1. The following identity holds true for $e^{\lambda^n s^n} = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} e^{s\xi} \mathrm{A}_n(\xi,\lambda)d\xi, \quad n=2,3,4$ where $\mathrm{A}_n(\xi,\lambda)$is as following

(1). $\mathrm{A}_n(\xi,\lambda) = \int\limits_0^{+\infty} \cos(r\xi + (-1)^{\frac{n+1}{2}} \lambda^n r^n)dr, \quad n=2k+1.$

(2). $\mathrm{A}_n(\xi,\lambda) = \frac{1}{n\lambda}\int\limits_0^{+\infty} \frac{e^{-r}\cos(\lambda^{-1}\sqrt[n]{r}\xi)}{r^{1-\frac{1}{n}}}dr, \quad n=4k+2.$

(3). $\mathrm{A}_n(\xi,\lambda) = \frac{1}{n\lambda}\int\limits_0^{+\infty} \frac{e^{r}\cos(\lambda^{-1}\sqrt[n]{r}\xi)}{r^{1-\frac{1}{n}}}dr, \quad n=4k.$

Proof. See [4].

2 Solution to homogenous linear Kd.V

The KdV equations are attracting many researchers, and a great deal of works has already been done in some of these equations. In this section, we will implement the method of exponential to construct exact solution for a variant of the KdV equation.

Example 4. Solving the following homogenous linear KdV.

$$u_t + \alpha u + \beta u_x + \gamma u_{xxx} = 0, \tag{7}$$

$$u(x,0) = f(x). \tag{8}$$

The PDE may be rewritten in the following form

$$u_t = -\left(\alpha + \beta\frac{\partial}{\partial x} + \gamma\frac{\partial^3}{\partial x^3}\right)u.$$

In order to obtain a solution for equation (7) first by solving the first order PDE with respect to t, and applying the initial condition (8), we get,

$$u(x,t) = \exp\left(-\alpha t - \beta t\frac{\partial}{\partial x} - \gamma t\frac{\partial^3}{\partial x^3}\right)f(x) = exp\,(-\alpha t)\,.\exp\left(-\beta t\frac{\partial}{\partial x} - \gamma t\frac{\partial^3}{\partial x^3}\right)f(x).$$

The above relation can be written as follows

$$u(x,t) = exp\,(-\alpha t)\,.exp\left(-\gamma t\frac{\partial^3}{\partial x^3}\right)\left(\exp\left(-\beta t\frac{\partial}{\partial x}\right)f(x)\right) = exp\,(-\alpha t)\,.\{exp\left(-\gamma t\frac{\partial^3}{\partial x^3}\right)f(x-\beta t)\}.$$

At this point, in order to find the result of the action of third order operator, we use the following well- known Identity for Airy function [12],

$$e^{\kappa\omega^3} = \int_{-\infty}^{+\infty} \exp(\sqrt[3]{3\kappa}\omega\,\phi)Ai(\phi)\,d\phi.$$

Therefore

$$u(x,t) = exp\,(-\alpha t)\int_{-\infty}^{+\infty} Ai(\phi)\left(\exp(\sqrt[3]{-3\kappa t}\,\phi\frac{\partial}{\partial x})f(x-\beta t)\right)d\phi.$$

By using part one of lemma 1, one gets the following

$$u(x,t) = exp\,(-\alpha t)\int_{-\infty}^{+\infty} Ai(\phi)f(x-\beta t + \sqrt[3]{-3\kappa t}\,\phi)\,d\phi.$$

Special case. $\alpha=\beta=0, f(x)=\exp(-x^2)$, and a change of variable $x+\sqrt[3]{-3\kappa t}\,\phi=\zeta$ leads to the standard solution for Kd.V

$$u(x,t)=\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-\zeta^2}}{\sqrt[3]{-3\kappa t}}Ai\left(\frac{\zeta-x}{\sqrt[3]{-3\kappa t}}\right)d\phi.$$

3 Airy polynomials and its applications

We define an Airy polynomials of degree n, $Pi_n(t)$, using the Airy transform of t^n as following

$$Pi_n(t)=\int_0^{+\infty}\xi^n Ai(t-\xi)d\xi.$$

As an interesting property of Airy polynomials, can be mentioned, for instance generating-function [6], [11], $e^{-\frac{x^3}{3}+xt}=\sum_{n=0}^{+\infty}\frac{x^n}{n!}Pi_n(t)$,

Lemma 6. The following exponential operator relation holds true.

$$e^{-\lambda\frac{\partial^3}{\partial x^3}}\Phi(x)=\sum_{n=0}^{+\infty}\frac{(3\lambda)^{\frac{n}{3}}Pi_n(t)\Phi_\xi{}^{(n)}(x-\sqrt[3]{3\lambda}t)}{n!},$$

Proof. The generating function of Airy polynomials can be rewritten as follows

$$e^{-\frac{x^3}{3}}=\sum_{n=0}^{+\infty}Pi_n(t)\frac{e^{-tx}x^n}{n!}.$$

At this point, let us introduce a change of variable $x=\sqrt[3]{3\lambda}\,\omega$, then we get the following

$$e^{-\lambda\omega^3}=\sum_{n=0}^{+\infty}\frac{Pi_n(t)}{n!}\,e^{-t\left(\sqrt[3]{3\lambda}\,\omega\right)}\left(\sqrt[3]{3\lambda}\,\omega\right)^n.$$

Letting $\omega=\frac{\partial}{\partial\xi}$, one gets the following identity for third order exponential operator

$$e^{-\lambda\frac{\partial^3}{\partial\xi^3}}=\sum_{n=0}^{+\infty}\frac{Pi_n(t)}{n!}\,e^{-t\left(\sqrt[3]{3\lambda}\,\frac{\partial}{\partial\xi}\right)}\left(\sqrt[3]{3\lambda}\,\frac{\partial}{\partial\xi}\right)^n,$$

Applying the above operator on function $\Phi(\xi)$ and using lemma 3. leads to

$$e^{-\lambda\frac{\partial^3}{\partial\xi^3}}\Phi(\xi)=\sum_{n=0}^{+\infty}\frac{Pi_n(t)}{n!}\left(\sqrt[3]{3\lambda}\,\frac{\partial}{\partial\xi}\right)^n e^{-\left(\sqrt[3]{3\lambda}\,t\frac{\partial}{\partial\xi}\right)}\Phi(\xi)=\sum_{n=0}^{+\infty}\frac{Pi_n(t)}{n!}\left(\sqrt[3]{3\lambda}\,\frac{\partial}{\partial\xi}\right)^n\Phi(\xi-\sqrt[3]{3\lambda}t)$$
$$=\sum_{n=0}^{+\infty}\frac{(3\lambda)^{\frac{n}{3}}Pi_n(t)\Phi_\xi{}^{(n)}(\xi-\sqrt[3]{3\lambda}t)}{n!}$$

Example 5. Let us solve the linear Kd.V

$$u_t+\kappa u_{xxx}=0, \tag{9}$$
$$u(x,0)=\sin x. \tag{10}$$

Direct use of example 5, followed by lemma 3, leads to the following solution

$$u(x,t)=\sum_{n=0}^{+\infty}\frac{(3\kappa)^{\frac{n}{3}}Pi_n(t)sin(x-\sqrt[3]{3\kappa}t+\frac{n\pi}{2})}{n!}.$$

4 Main results

Let us consider the following system of space-fractional PDEs of order α, with fractional derivatives in the Riemann-Liouville sense

$$\lambda {D_x}^{\alpha}u+t^{-2}u_t-3\,\kappa v=0, \tag{11}$$

$$\lambda {D_x}^{\alpha}v+t^{-2}v_t+3\,\kappa u=0,\lambda,\kappa\in R \tag{12}$$

$$u(x,0)=f(x), \tag{13}$$

$$v(x,0)=g(x), \tag{14}$$

then the above system of PDEs ($\alpha=0.5$) has the following formal solution

$$u(x,t)=\frac{\lambda t^2 cos\kappa t^3}{2\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-\frac{\lambda^2t^4}{4\xi}}}{\xi\sqrt{\xi}}f(x-\xi)d\xi-\frac{\lambda t^2\sin\kappa t^3}{2\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-\frac{\lambda^2t^4}{4\xi}}}{\xi\sqrt{\xi}}g(x-\xi)d\xi,$$

$$v(x,t)=\frac{\lambda t^2 cos\kappa t^3}{2\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-\frac{\lambda^2t^4}{4\xi}}}{\xi\sqrt{\xi}}g(x-\xi)d\xi+\frac{\lambda t^2\sin\kappa t^3}{2\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-\frac{\lambda^2t^4}{4\xi}}}{\xi\sqrt{\xi}}f(x-\xi)d\xi.$$

Let us define $u(x,t)+iv(x,t)=w(x,t),\quad f(x)+ig(x)=h(x)$then the above system can be written in the following form

$$\lambda {D_x}^{\alpha}w+t^{-2}w_t+3\,i\kappa w=0, \tag{15}$$

$$w(x,0)=h(x), \tag{16}$$

The PDE (5) may be rewritten in the following form

$$w_t=-t^2(\lambda {D_x}^{\alpha}-3\,i\kappa)w.$$

In order to obtain a solution for equation (1), first by solving the first order PDE with respect to t and treating the derivative operator as a generic constant and applying the initial condition (6), we get the following

$$w(x,t)=\exp\left(i\kappa t^3-\lambda t^2\frac{\partial^\alpha}{\partial x^\alpha}\right)h(x)=exp\left(i\kappa t^3\right).\exp\left(-\lambda t^2\frac{\partial^\alpha}{\partial x^\alpha}\right)h(x). \tag{17}$$

At this point, we study two cases:

Case 1. $\alpha=1$, in this case, by using part 1 of lemma 3,leads to the following formal solution

$$w(x,t)=u(x,t)+iv(x,t)=exp\left(i\kappa t^3\right).\exp\left(-\lambda t^2\frac{\partial}{\partial x}\right)\,h(x)=exp\left(i\kappa t^3\right)h(x-\lambda t^2).$$

We get the solutions to the system as following $u(x,t) = cos\kappa t^3 f(x-\lambda t^2) - sin\kappa t^3 g(x-\lambda t^2)$;$v(x,t) = cos\kappa t^3 g(x-\lambda t^2) + sin\kappa t^3 f(x-\lambda t^2)$.

Case 2. $\alpha = 0.5$ (Semi derivative), in this case one gets the following

$$w(x,t) = exp\left(i\kappa t^3\right).\exp\left(-\lambda t^2 \frac{\partial^{0.5}}{\partial x^{0.5}}\right) h(x).$$

By replacing s with $\frac{\partial}{\partial x}$,in part 3 of lemma 3, and by proceeding as before, one gets the following formal solution to (5),(6).

$$w(x,t) = exp\left(i\kappa t^3\right).\exp\left(-\lambda t^2 \frac{\partial^{0.5}}{\partial x^{0.5}}\right) h(x) = \frac{\lambda t^2 exp\left(i\kappa t^3\right)}{2\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-\frac{\lambda^2 t^4}{4\xi}}}{\xi\sqrt{\xi}} h(x-\xi),$$

or

$$u(x,t) + iv(x,t) = \frac{\lambda t^2 exp\left(i\kappa t^3\right)}{2\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-\frac{\lambda^2 t^4}{4\xi}}}{\xi\sqrt{\xi}} \left(f(x-\xi) + ig(x-\xi)\right).$$

Finally, one gets the formal solution to the system as below

$$u(x,t) = \frac{\lambda t^2 cos\kappa t^3}{2\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-\frac{\lambda^2 t^4}{4\xi}}}{\xi\sqrt{\xi}} f(x-\xi)d\xi - \frac{\lambda t^2 \sin\kappa t^3}{2\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-\frac{\lambda^2 t^4}{4\xi}}}{\xi\sqrt{\xi}} g(x-\xi)d\xi,$$

$$v(x,t) = \frac{\lambda t^2 cos\kappa t^3}{2\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-\frac{\lambda^2 t^4}{4\xi}}}{\xi\sqrt{\xi}} g(x-\xi)d\xi + \frac{\lambda t^2 \sin\kappa t^3}{2\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-\frac{\lambda^2 t^4}{4\xi}}}{\xi\sqrt{\xi}} f(x-\xi)d\xi.$$

Example 6. Let us consider the following generalized linear Heat equation of order $2q$.

$$\frac{\partial u}{\partial t} = \alpha u + \beta \frac{\partial u}{\partial x} + (-1)^{q+1} \frac{\partial^{2q} u}{\partial x^{2q}}, q = 2m+1, m = 0,1,2,3,.. \tag{18}$$

$$u(x,0) = \Phi(x). \tag{19}$$

In order to obtain a solution for equation (11) first by solving the first order PDE with respect to t, and applying the initial condition (12), we get the following

$$u(x,t) = \exp\left(\alpha t + \beta t \frac{\partial}{\partial x} + (-1)^{q+1} t \frac{\partial^{2q}}{\partial x^{2q}}\right) \Phi(x) = exp\left(\alpha t\right).\exp\left(\beta t \frac{\partial}{\partial x} + (-1)^{q+1} t \frac{\partial^{2q}}{\partial x^{2q}}\right) \Phi(x)$$

or,

$$\begin{aligned} u(x,t) &= exp\left(\alpha t\right).exp\left((-1)^{q+1} t \tfrac{\partial^{2q}}{\partial x^{2q}}\right)\left(\exp\left(\beta t \tfrac{\partial}{\partial x}\right)\Phi(x)\right) = \\ &= exp\left(\alpha t\right) exp\left((-1)^{q+1} t \tfrac{\partial^{2q}}{\partial x^{2q}}\right)\Phi(x+\beta t) = exp\left(\alpha t\right) exp\left((-1)^{q+1} t \tfrac{\partial^{2q}}{\partial x^{2q}}\right)\Phi(x+\beta t). \end{aligned}$$

At this point, in order to evaluate the last expression, we will make use of part 2 of theorem 1, by setting $\lambda = \sqrt[n]{t}, s = \frac{\partial}{\partial x}, n = 2q$, we get the following formal solution to generalized Heat equation of order $2q$,

$$u(x,t) = exp\left(\alpha t\right) exp\left((-1)^{q+1} t \frac{\partial^{2q}}{\partial x^{2q}}\right)\Phi(x+\beta t) = \frac{exp\left(\alpha t\right)}{\pi} \int_{-\infty}^{+\infty} \mathrm{A}_{2q}(\xi,\lambda) d\xi \left(e^{\xi \frac{\partial}{\partial x}} \Phi(x+\beta t)\right)$$

or

$$u(x,t) = \frac{exp(\alpha t)}{\pi}\int_{-\infty}^{+\infty} \mathrm{A}_{2q}(\xi,\lambda)\Phi(x+\xi+\beta t)d\xi,$$

$$u(x,t) = \frac{exp(\alpha t)}{2q\lambda\pi}\int_{-\infty}^{+\infty} \Phi(x+\xi+\beta t)\int_{0}^{+\infty}\frac{e^{-r}\cos(\lambda^{-1}\sqrt[2q]{r}\xi)}{r^{1-\frac{1}{2q}}}drd\xi.$$

Special case. Let us consider the special case $\alpha=\beta=0, q=1$, we arrive at

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{20}$$

$$u(x,0) = \Phi(x). \tag{21}$$

Then we get the solution to standard heat equation as below

$$u(x,t) = \frac{1}{2\pi\sqrt{t}}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\Phi(x+\xi)\frac{e^{-r}\cos(\xi\sqrt{\frac{r}{t}})}{\sqrt{r}}drd\xi = \int_{-\infty}^{+\infty}\Phi(x+\xi)\left(\frac{1}{\pi}\int_{0}^{+\infty}e^{-tu^2}\cos(\xi u)du\right)d\xi$$

But the value of inner integral is $\frac{1}{2}\sqrt{\frac{\pi}{t}}e^{-\frac{\xi^2}{4t}}$, so that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\Phi(x+\xi)e^{-\frac{\xi^2}{4t}}d\xi = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\Phi(\varsigma)e^{-\frac{(\varsigma-x)^2}{4t}}d\varsigma.$$

In order to show that $u(x,0)=\Phi(x)$, we have the following

$$u(x,0) = \int_{-\infty}^{+\infty}\Phi(\varsigma)\left(\lim_{t\to 0^+}\frac{1}{2\sqrt{\pi t}}e^{-\frac{(\varsigma-x)^2}{4t}}\right)d\varsigma = \int_{-\infty}^{+\infty}\Phi(\varsigma)\delta(\varsigma-x)d\varsigma = \Phi(x).$$

5 Conclusion

The paper is devoted to study exponential operators and their applications in solving certain boundary value problems. The author also discussed Airy polynomials as well. The operational method is a powerful method for analyzing fractional linear systems. The main purpose of this work is to develop methods for solving a variant of homogenous KdV equation, certain system of space fractional PDEs and solution to a generalized heat equation of order 2q+1.

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